INTERNATIONAL ASSOCIATION FOR HYDRAULIC RESEARCH

ASSOCIATION INTERNATIONALE DE RECHERCHES HYDRAULIQUES

TENTH CONGRESS LONDON 1963

DIXIEME CONGRES LONDRES 1963

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VOLUME 4

Modern developments in hydraulic machinery and equipment

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Modern developments in hydraulic machinery and equipment

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GENERAL LECTURE

"WAVES IN THIN SHEETS OF WATER"

by

PROFESSOR SIR GEOFFREY TAYLOR, F.R.S.

CHAIRMAN: PROFESSOR A.T. IPPEN.
PRESIDENT OF THE ASSOCIATION

Meeting in the Great Hall

of

The Institution of Civil Engineers

on

Wednesday, 4 September, 1963 at 12:00. Noon

Professor Ippen said that it was his particular pleasure and privilege to introduce for their General Lecture, the outstanding scientist of the world in the field of fluid mechanics, Professor Sir Geoffrey Taylor, Fellow of Trinity College Cambridge since 1910. Certainly there was no one in the room who had not on many occasions and on many problems had reference to his profound and lasting contributions made over a period of half a century.

The breadth of Sir Geoffrey's writings had been extraordinary. As well as all his scientific and professional experiences, whether it was mathematics, metrology, aeronautics or naval architecture, his ideas and lucid developments had had the most profound influence, and it had been evident also during the present Congress, that the progress in their more restricted sphere of hydraulics and of its engineering applications had been shaped in many ways by his genius.

They were most grateful, therefore, that he had accepted their invitation to be with them that day and to speak to them on "Waves in thin sheets of water". They were greatly privileged to hear during the meeting their great scientist and their most honoured and beloved teacher and friend, Sir Geoffrey Taylor.

Professor Sir Geoffrey Taylor:

When your President asked me to give this address he knew that I am not a person who knows much about the technical side of Hydraulic Engineering, and so I must assume that he was asking me to offer entertainment rather than useful information. For this reason I thought it appropriate to describe some work which has entertained me during the last few years even though it has, I fear, little usefulness to engineers. Interest in the dynamics of thin sheets of fluid goes back a long way. In 1833 M. Felix Savart published a description of experiments in which he produced radially expanding sheets by making a jet of water strike a flat disc, or by making two equal collinear but oppositely directed jets strike one another. No analysis of Savart's experiments appeared until 1869, when Boussinesq gave the equations which represent the motion of the sheet under the action of gravity and also of a difference in pressure between the two sides of the sheet. It was only recently that interest in thin sheets revived with the experimental study of atomisers and drop formation in sprays. Most recent work has been directed to acquiring knowledge about how thin sheets of fluid break up into drops. My own studies were not directed to this aspect of the subject. They were those of a mathematician, who can only hope to give an analytical description of very simple phenomena, and who has no feeling that he ought to produce something that may have practical value.

For me the subject has been interesting because all the results which I shall give were predicted beforehand using very simple mathematics; and the apparatus was designed to illustrate the results of the analysis.

The two most striking things about a thin sheet of fluid, that is to say, a sheet which has two surfaces, not one which is just water running down a flat sheet - are the waves one can produce on the sheet and the nature of its edge. In a moving sheet, both waves and edges can be produced which will remain at rest in space, just as the waves produced in a moving stream of water by a fixed obstacle remain stationary. These waves are the ones which can be most easily observed, and so I will confine my attention to them.

Waves on the surface of water are controlled by a combination of gravity and surface tension. Such waves are always dispersive in the sense that different wave-lengths travel with different velocities. The effect of gravity on a thin sheet projected horizontally is merely to make it assume a parabolic form, each particle describing the ordinary free parabola, but otherwise gravity has no effect on the waves in the sheet. These are controlled by surface tension and, thinking only of waves of small amplitude, the mathematician need concern himself only with waves of one wave-length because such waves are superposable. Confining attention to plane waves of a given wave-length, all disturbances on deep water with one free surface can be regarded as a combination of two waves of different amplitudes travelling in opposite directions. In progressive waves, for instance only one of these exists, and in standing waves there are two of equal amplitude going in opposite directions. In general, four numbers are required to specify a disturbance of a given wave-length on deep water because for each of the two component waves going in opposite directions there have to be two numbers - an amplitude and a phase.

The same kind of thing applies to waves on a thin sheet, only in that case, instead of two waves, four must combine to produce the most general disturbance of a given wave-length, and eight numbers are, therefore, required to describe the most general wave of a given length in a thin sheet. These waves turn out to be of two kinds - antisymmetrical and symmetrical.

Fig. 1.- With antisymmetrical waves the amplitudes on both surfaces are the same and in the same phase. In symmetrical waves, the amplitudes are the same but in opposite phase. When the wave-length is long compared with the thickness of the sheet, the velocity of antisymmetrical waves is simply $W_a = \left(\frac{2T}{\rho h_o}\right)^{\frac{1}{2}}$ where T is surface tension, ρ density and h_o thickness, while that of symmetrical waves is $W_s = \frac{\pi h_o}{\lambda} W_a$, where λ is wave-length. The first of those formulae is the same as that for waves in a stretched string. The 2 T appears instead of T because there are two surfaces. I will call that W_a, the velocity of antisymmetrical waves.

The symmetrical waves in thin sheet are propagated much more slowly than the antisymmetrical waves.

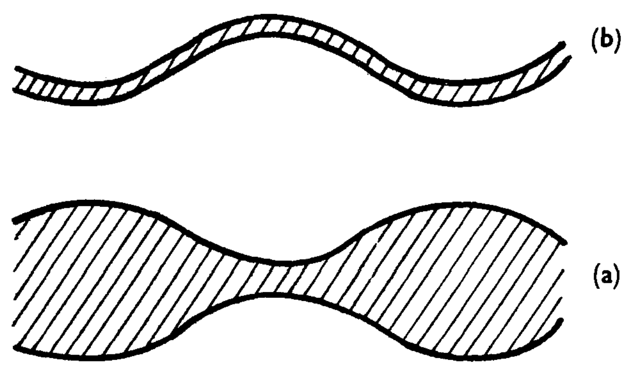

FIG. I

Sketch (a) symmetrical waves.
Sketch (b) antisymmetrical waves.

Another point of particular interest is that the antisymmetrical waves are entirely non-dispersive because all wave-lengths are propagated at the same speed. You see that λ does not appear in the expression for the velocity of antisymmetrical waves, but appears for the symmetrical waves. All that I am going to say now really is concerned with these two formulae.

The other interesting feature of a thin sheet that I mentioned is the nature of the edge.

This is conditioned by the fact that the surface tension acts on one side of it, and there is nothing to prevent the edge being sucked in by the surface tension. Thus the edge moves relative to the fluid at right angles to itself and accumulates in a more or less cylindrical volume of turbulent fluid of continually increasing volume: calling W_e the velocity with which the edge moves into the sheet, the balance of momentum requres that $2T = W_e$ times (rate of increase in mass of the edge). This rate of increase is $\rho h_o W_e$ so that $W_e^2 = \frac{2T}{\rho h_o}$. You will notice that $W_e = W_a$ so that antisymmetrical waves are propagated at the same speed, relative to the water, as the edge.

If the velocity of the fluid is U, it is very little retarded by the air, and so you can take it as constant for the sheet. It appears therefore that either an edge or an antisymmetrical wave can lie at rest at angle ψ to direction of the stream provided that $\sin\psi = \frac{W_a}{U}$.

In order to illustrate those formulae, I constructed first of all, apparatus for making thin sheets of water which were not diverging. All I did there was to make a little chamber into which water could be supplied in a non-turbulent condition, and I cut a slit about 0.005 in. from the top to the bottom of the chamber in order to produce a sheet of uniform thickness.

Fig. 2.- shows a picture that I got by reflection from a line source of light. On the left is the chamber from which the sheet came. In order to make a wave, I simply blew, through a little tube, a jet of air on to it. This produced a displacement in the sheet and gave rise to waves. You will see in that picture, on the extreme left, the little tube through which I blew the air. The reflections from the points of light situated on the line-source reach the camera only when they are reflected at a point in the wave where the slope of the waves to the undeflected sheet has a given value. The doubling of the lines is due to the fact that the maximum slope in the wave was slightly greater than the angle at which light is reflected into the camera lens and the fact that the two lines of each pair are so parallel is conclusive evidence that the wave is non-dispersive to a high degree of accuracy.

When the photograph (Fig. 2) was taken, the top and bottom of the sheet were held by two glass tubes which the water wetted. The other lines are waves produced by disturbances at the top and bottom of the slit orifice along the left side of the figure.

Fig. 3.- I have said that the edges and the antisymmetrical waves are propagated at the same speed and so they lie at the same angle in a moving sheet. On one occasion (Fig. 3) a bubble stuck in the slit from which the water came. While the wave was being photographed it parted the sheet and produced two edges. You can see the two edges and will notice that they are parallel to the waves.

If you take away the glass tubes at the top and the bottom, the whole sheet becomes a triangle with vertex angle twice that whose sine is $\frac{W_a}{U}$.

So far I have dealt only with sheets of uniform thickness. With expanding sheets, which are like the ones produced by Savart, the sheet varies in thickness as you go from the middle, and so, of course, the velocity of antisymmetrical waves varies. That is to say, the angle to the direction of motion at which the wave can lie at rest is varying.

It is an easy matter to calculate what shape this non-dispersive wave should take in an expanding sheet. First of all, there is the formulae which I have already given you which relates the angle ψ between the direction of flow (which is radial) and the wave to the U and h, the local thickness of the sheet. This is $\sin^2\psi = \frac{2T}{\rho U^2 h}$. Since U is constant and the total flow Q is also constant $Q = 2\ \ r U h$ where r is the radial co-ordinate of a point in the expanding sheet. It is convenient to define a length $R = \frac{\rho U Q}{4\pi T}$. The equation to the wave is then $\sin^2\psi = \frac{r}{R}$ and in polar co-ordinates this is $\frac{2r}{R} = 1 - \cos(\theta - \theta_0)$ where

θ_o is merely an angle which determines the angular position of the point of origin of the wave. The curve represented by this equation is called a cardioid. It is heart shaped, and touches at its outer edges the circle $r = R$. It is the curve which would be produced by a point on the circumference of a circular disc of radius $\frac{1}{4}$ R when it rolls round another disc also of radius $\frac{1}{4}$ R. For this reason, it can be produced mechanically and Fig, 4 shows a set of 8 such cardioids drawn with θ_o increasing by 45° for each successive curve. This pattern was chosen so that the waves produced by 8 initiating sources of disturbance might produce an easily recognisable pattern for comparison with theory.

Fig. 5.- I will show you first of all a picture of the expanding sheet. Underneath you see the source box from which the jet comes through a sharp-edged orifice. On top you see the impacter - the little disc, held by a vertical rod, onto which the jet is projected. You will see that the sheet goes out getting thinner and thinner until it gets to the radius R at which the edge can establish itself and then drops fall off. The energy of the stream is not absorbed in producing the surface of drops. It is used in producing turbulence in the edge. You will see that there are drops of all sorts of shapes. If there was no turbulence in them they would be round.

Fig, 6.- This shows the photograph which I took of the flat sheet made by projecting a jet onto a disc with eight little nicks in it. You can see comparison with Fig. 4 how very close the wave-pattern is to the predicted one. This picture was taken by putting the camera slightly off the axis and then simply moving an ordinary electric light bulb at random in order that reflections from all parts of the sheet might enter the camera lens. I moved it at random for about 20 seconds and got that picture. If you put a non-wetting wire through the film you would get two edges, which you might think at first would be parallel with the cardioids, but they are not, and the reason for this is that the surface tension has not only to stop the component of velocity in the sheet at right angles to the edge at any point but it has also to deflect the fluid which has been accumulating in the edge from the obstruction up to the point in question.

Like sound waves, antisymmetrical waves are non-dispersive but unlike them the existance of a small wave only alters the direction of motion of the water particles but not the wave velocity. A consequence of this is that small deflections can be superposed on parallel lines which may be as close to one another as one likes. In other words, a finite deflection should be possible at a straight line in a uniform sheet or at a cardioid in an expanding sheet.

Fig. 7.- is a diagram showing the apparatus I used for testing this surprising theoretical prediction. The plan shows the initial position of the circular edge of a horizontal expanding sheet when it is obstructed. This edge is drawn slightly wavy because the falling drops (see Fig. 6) prevent the ideal circular edge from being formed. The vertical jet rising from below and the impacter descending from above the sheet are shown in the elevation. To obtain a finite deflection of the sheet at a cardioid edge, an obstruction in the form of a vertical metal sheet with two downward-sloping edges was placed in the path of the water sheet. The water which struck the obstruction spread out, wetting its surface and so preventing the formation of an edge, just as the glass tubes did which held the top and bottom of the uniform sheet shown in Figs 2 and 3. The result was the formation of a sharp edge at cardioid lines on the water sheet.

Fig. 8. -shows the two photographs of the sheet. The upper photograph was taken with a camera aimed downwards at 26° to the horizontal and illuminated by a source of light at the reflecting angle so as to show the plane part of the sheet. The cardioid edge shows clearly but, owing to the method of illumination, the outline of the deflected edge shows only faintly. The lower photograph was taken with a camera aimed horizontally with its centre lines in the plane of the unobstructed sheet. The illumination was by means of a flash bulb in the centre line of the camera, but direct light was cut off by the obstruction to the flow. It will be seen that the unobstructed part of the fluid sheet shows as a thin nearly-horizontal line and the outline of the portion of the sheet which has made a sudden sharp bend at the cardioid shows very clearly. In this case the sharp angle of the sheet was about 90° but I have even obtained acute angles.

Now I would like to describe experiments with symmetrical waves. These are quite different in character from antisymmetrical waves because they are dispersive and a different technique is required for revealing them. The antisymmetrical waves can be much larger than symmetrical waves because the amplitude of the latter is limited to half the thickness of the sheet. In order to show up the symmetrical waves, therefore, I had to use a property which is not possessed by the antisymmetrical waves, and so I observed them by transmitted light. Transmitted rays will be deflected owing to the variation of sheet thickness when they pass through symmetrical waves but are not appreciably deflected on passing through antisymmetrical waves. For this reason the Schlieren method of illumination is ideal for revealing the presence of symmetrical waves even in the presence of antisymmetrical waves of much larger amplitude.

Fig. 9.- is a Schlieren picture of the disturbance produced in a uniform sheet by the same airjet which produced the reflection photograph in Fig. 2. Though both types of wave are present in the disturbance produced by the airjet, only the antisymmetrical waves appear in Fig. 2, and only the symmetrical ones in Fig. 9.

The calculated form of the symmetric wave system in a sheet of uniform thickness consists of a set of parabolas and in fact the curves of Fig. 9 are very closely parabolic, moreover, they are spaced at the distance apart predicted theoretically.

Similar results are obtained with an expanding sheet.

Fig. 10.- is a reflection photograph of the disturbance produced by a needle point which just enters an expanding sheet. The almost complete absence of dispersion is shown by the narrowness of the cardioid waves and no symmetric waves are visible.

Fig. 11.- shows the area covered by the ring drawn on Fig. 10 when observed by transmitted light using the Schlieren method. The symmetrical waves which are invisible in Fig. 10 now appear strongly, but the antisymmetrical waves are invisable. In order to show where the antisymmetrical waves are I have marked Fig. 11 with a broken line to show their position.

I think I have described the whole course of these experiments. They were undertaken as an entertaining exercise in making mathematical predictions from well understood physical principles and then verifying them experimentally. I do not believe that an experimentor who looked at Figs. 10 and 11 would have believed he was looking at the same physical situation unless he had already made a theoretical analysis of it.

oOo

ILLUSTRATIONS

Fig. 1 - (a) Symmetrical waves
(b) Antisymmetrical waves.

Fig. 2 - Antisymmetrical waves produced by airjet in uniform sheet.

Fig. 3 - Edge produced by bubble in orifice slit.

Fig. 4 - Pattern of cardioids expected from 8 disturbance centres at equally spaced intervals.

Fig. 5 - Expanding sheet with edge at R=10 cm. approximately.

Fig. 6 - Photograph of disturbance for comparison with Fig. 4.

Fig. 7 - Arrangement for producing a sheet with a sharp edge.

Fig. 8 - Sheet with cardioid edge. - Top: Camera aiming downward at 26° to horizontal.
- Bottom: Camera horizontal and in the plane of the undeflected part of the sheet.

Fig. 9 - Schlieren photograph of disturbance produced in uniform sheet by air jet.

Fig.10 - Antisymmetrical wave produced by fine needle partially penetrating an expanding sheet for which R=21.1 cm. Broken line circle shows position of condenser used for the Schlieren photograph of Fig. 11.

Fig.11 - Schlieren photograph of same jet as that of Fig. 11.

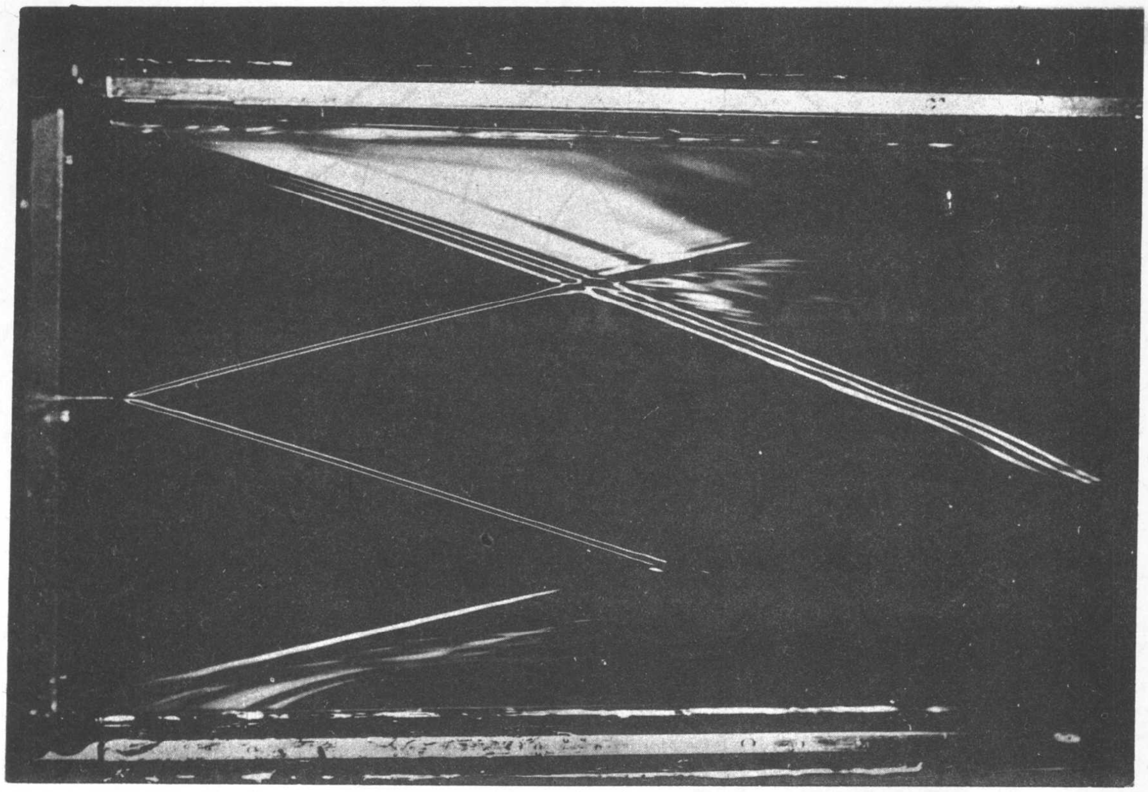

FIG. 2

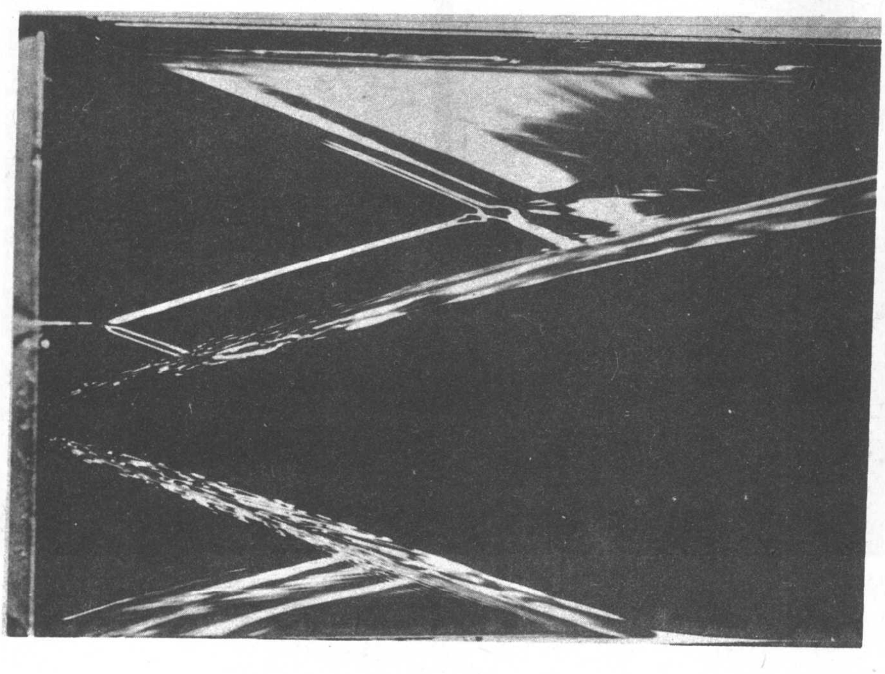

FIG. 3

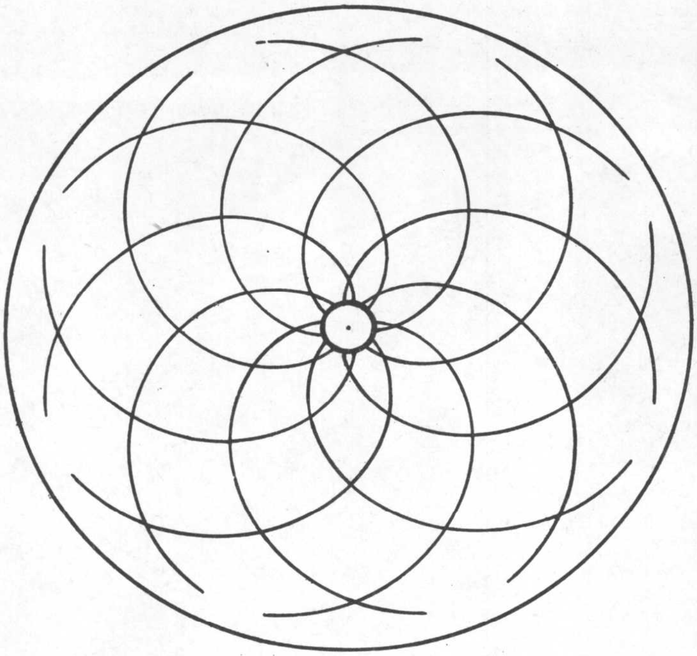

FIG. 4

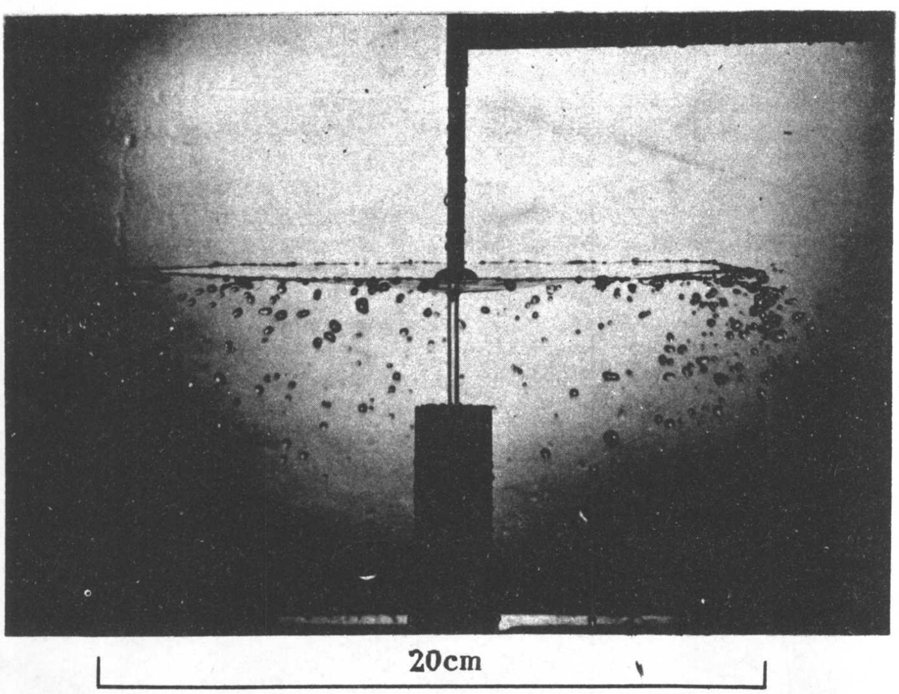

FIG. 5

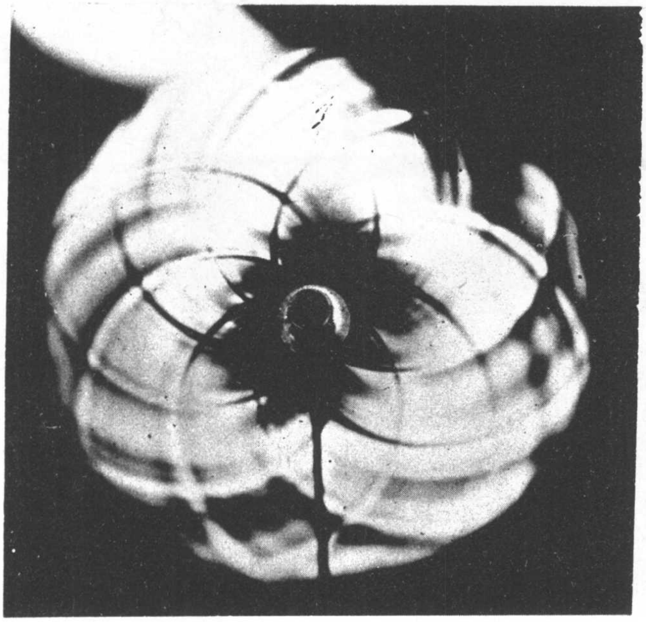

FIG. 6

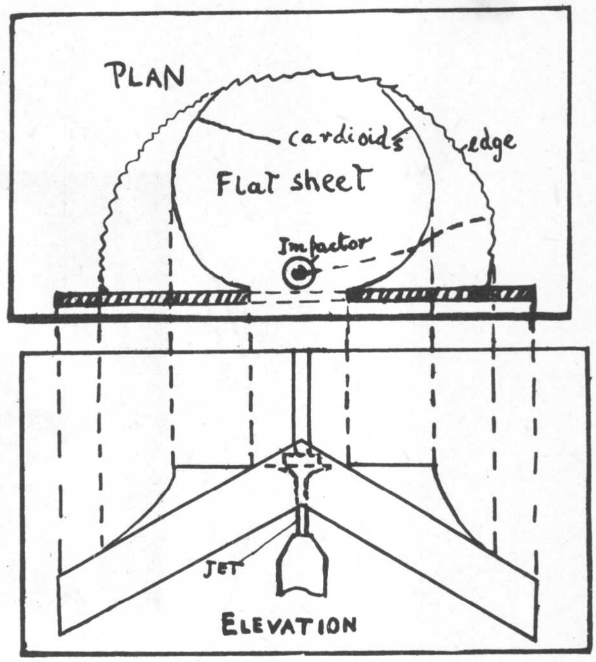

FIG. 7

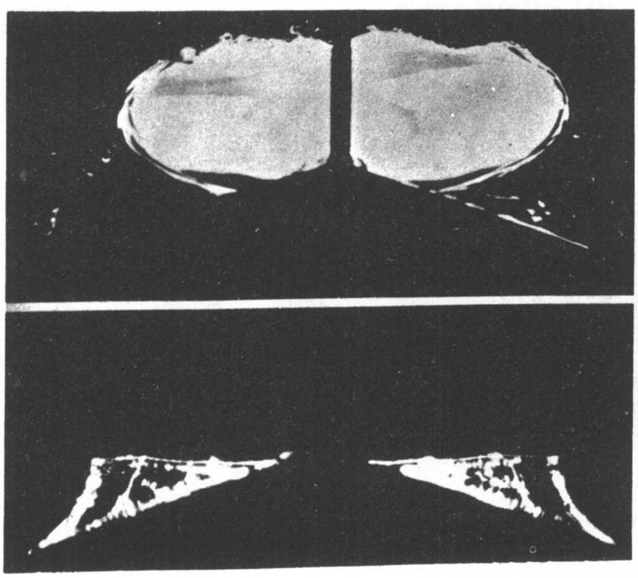

FIG. 8

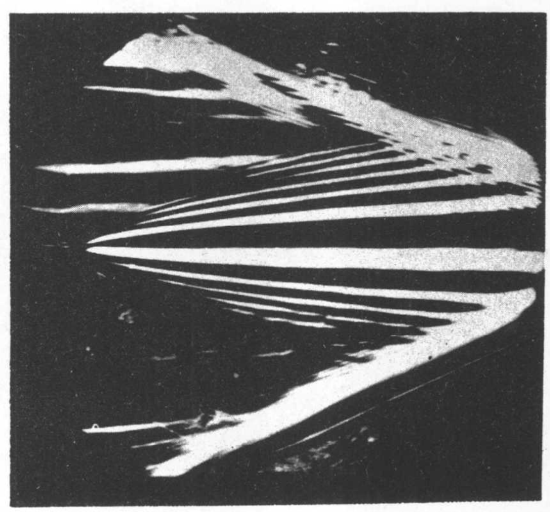

FIG. 9

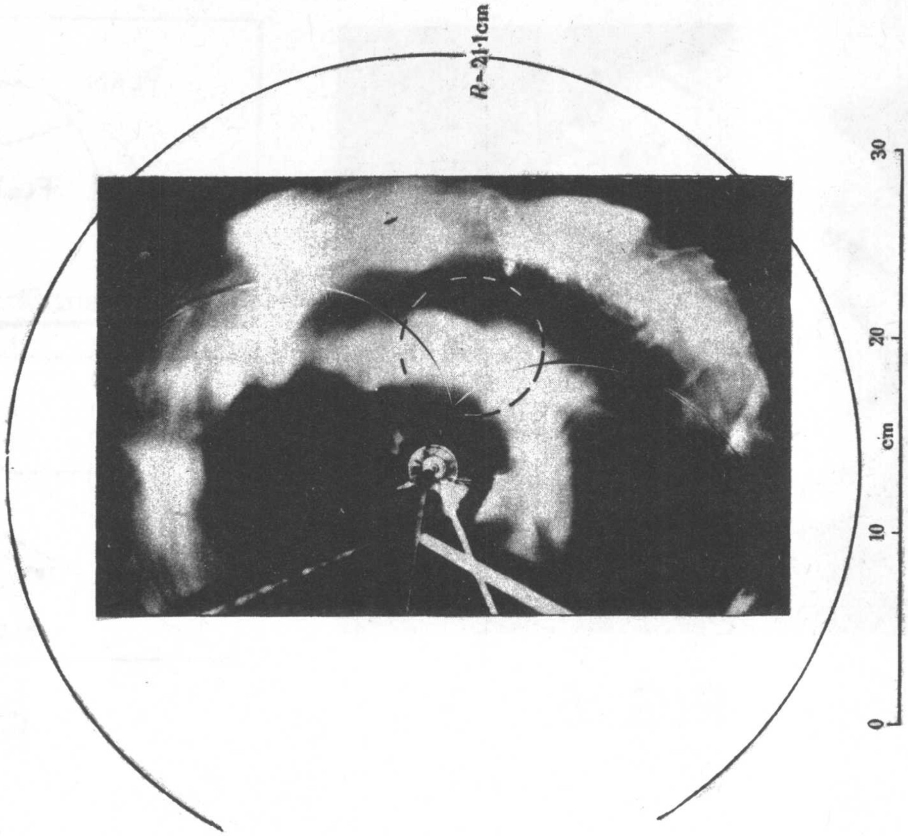

FIG. 10

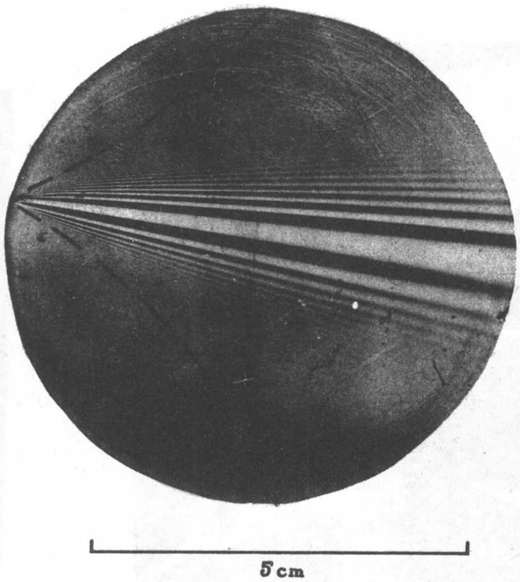

FIG. 11

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GENERAL REPORTS AND DISCUSSIONS

The General Reports and Discussions will be found on the following pages:-

I.A.H.R. Congress
LONDON, 1963

SOME CONSIDERATIONS ON THE ANALYSIS AND DESIGN OF HYDRAULIC MACHINERY FOR NON-NEWTONIAN FLUIDS

George Bugliarello
Carnegie Institute of Technology, U.S.A.

SYNOPSIS

Some of the aspects to be considered in the design and operation of hydraulic machines for non-Newtonian fluids are presented. After a brief discussion of the problems encountered in determining the proper rheological characteristics of the fluid and the characteristics of the flow in the piping system to which the machine is connected, the treatment is focused on the losses in turbomachines. The major emphasis is on the determination of disk friction characteristics for a simple power-law fluid, both under laminar and turbulent conditions, and on the way in which these characteristics differ from corresponding Newtonian cases. The problem of the internal losses, and the significance of the non-Newtonian characteristics of the fluid with regard to cavitation phenomena are also briefly discussed.

RÉSUMÉ

Le but de cette communication est de traiter aucuns des aspects qui doivent être pris en consideration dans le calcul et l'exercise des machines hydrauliques pour fluides non-newtoniens. Après une brève discussion des problemes que l'on rencontre dans la determination des caractéristiques rheologiques du fluide et des caractéristiques de l'écoulement dan les systeme de conduites ou la machine est inserée, l'on passe a' traiter plus en detail le probleme des pertes d'energie dans les turbomachines. L'on s'entretient particulièrement sur le calcul des caracteristiques du frottement de disque dans conditions de regime laminaire et turbulent pour fluides obeissant la simple loi de puissance, est sur le paragon entre cettes caractéristiques et les mêmes caractéristiques pour fluides newtoniens. Le probleme des pertes d'energie internes, et la signification des caractéristiques non-newtoniennes du fluide au regard des phenomenes de cavitation sont aussi discutés en bref.

1. INTRODUCTION

The handling of non-Newtonian fluids, i.e. fluids characterized by a non-linear relationship between the stress-and rate of strain-tensors when flowing under laminar conditions, is acquiring an increasing importance in hydraulic engineering (1). Although there is a rapidly growing body of literature on non-Newtonian fluid mechanics (e.g. (2)) and on the design of pipe lines for non-Newtonian fluids (e.g. (3),(4)), the design of hydraulic machinery for non-Newtonian fluids appears thus far to have received very little attention, except for a few pioneering studies of some specialized aspects. (e.g. (5), (6) and previous ref. (2)).

This paper will attempt to discuss-by necessity in a summary and often qualitative fashion and by making use of an extremely simplified rheological equation-some of the principal ways in which the non-linear viscosity of the fluids can affect the design and operation of a hydraulic machine. Although attention will be focused primarily on turbomachines, the concepts discussed should also be relevant for positive-displacement machines, which find employ for the handling of the more viscous non-Newtonian fluids. Among the main problems to be considered in the design of a machine for non-Newtonian fluids are: a) the characterization of the fluid; b) the determination of the energy losses in the conduits leading to and from the machine; c) the determination of the external losses in the machine proper; d) the determination of the internal losses in the machine; e) the influence of the rheological characteristics of the fluid on cavitation phenomena.

2. CHARACTERIZATION OF THE FLUID

The determination of the rheological characteristics of the fluid and their description by a suitable equation is the first step in the design. This problem, far yet from being solved in its most general form - the providing of a description of a given fluid valid for all ranges of state parameters and stress conditions - has been discussed by several authors (e.g. (7) (8)) and summarized in standard references (See ref. (2)). The characterization becomes, in general, further complicated when - as in many cases of practical interest in hydraulic engineering - the fluid is a mixture.

Fortunately, from the viewpoint of design, a rigorous characterization will not always be necessary. Empirical or semiempirical descriptions of validity limited to the range of conditions prevailing in the machine can provide useful approximations. Examples are the well known Bingham plastic equation and the power-law rheological equation

$$\tau = K(du/dy)^n \qquad (1)$$

where τ is the shear stress and u the velocity at a distance y from a surface, K is a rheological "consistency" index and n a rheological "flow behavior" index (which become, respectively, the dynamic viscosity μ and 1.0 for the Newtonian case. In using such equations it is important to test their validity over the whole range of shear stresses prevailing in the machine (9). It may thus be found that in eq. (1) different sets of constants K and n describe the flow at different shear stress levels. For mixtures, furthermore, the possible occurrence of phase separation effects in variable-shear flow situations must be considered, which may not be detected through constant-shear viscometric measurements and may lead to erroneous flow pattern predictions. Hence, the greatest caution must be exerted by the machine designer in using for the rheological constants either published values - limited as they are - which are not accompanied by a specific description of their range of validity and of the conditions under which they

---

(1) Bugliarello, G., "I Moti Non-Newtoniani Nell' Ingegneria Idraulica," (Non-Newtonian Flows in Hydraulic Engineering), Chapter in "Scritti in Onore del Prof. Francesco Marzolo," Istituto d' Idraulica dell' Universita, ed., Stediv, Padova, 1962.
(2) Metzner, A.B., "Flow of Non-Newtonian Fluids," Chapter in Handbook of Fluid Mechanics," V. L. Streeter, ed., McGraw-Hill, New York, 1961.
(3) Alves, G.E., Boucher, D.F. and Pigford, R.L., Chemical Eng. Progress, Vol. 48, No. 8, p. 385, 1952.
(4) Savins, J.C., J. Institute of Petroleum, Vol. 47, No. 454, p. 329, 1961.
(5) Schultz-Grunow, F., Chemie-Ingenieur Technik, Vol. 26, No. 1, p. 18, 1954.
(6) Rotem, Z. and Shinnar, R., Chemical Eng. Science, Vol. 15, p. 130, 1961.
(7) Truesdell, C., "Principles of Continuum Mechanics," Socony Mobil Oil Co., Dallas, Tex., 1960.

were obtained, or - a more recommendable design practice - in designing an ad hoc program of rheological tests.

3. ENERGY LOSSES IN THE CONDUITS TO AND FROM THE MACHINE

This problem, in a strict sense foreign to the design of the machine proper, but nevertheless very important in establishing the machine's specifications and operating rules, has been discussed recently by Savins (ref. (4)) and, previously, by others (e.g. ref. (3)). Here it will be emphasized that from the pipeline design viewpoint the main difficulties with respect to the Newtonian case are 1) the current impossibility to describe the friction factor in terms of a single function of the rheological characteristics akin to the Reynolds number for the whole laminar and turbulent range in smooth tubes, even for simple power-law fluids (10), and 2) the lack, at present, of comprehensive experimental friction factor data in the turbulent regime, particularly for rough pipes and, for all pipes, at high values of the Reynolds number.

Within the framework of these limitations, if reference is made with Savins (see ref. (4)) to the well known Metzner's generalized friction factor diagram for power-law fluids, the following trends are significant from the viewpoint of the machine design and operation: 1) In the laminar range - which has often greater practical significance than in the Newtonian case - the progressive reduction in sensitivity of the head losses to velocity or diameter changes when $\underline{n}$ decreases. 2) In the turbulent range, the progressively lower friction factor values as $\underline{n}$ decreases. (Thus it may occur that a non-Newtonian fluid requiring a much greater energy expenditure in the laminar range than a Newtonian one at the same flow rate may require a lower expenditure than the Newtonian in the turbulent range). 3) The influence of the index $\underline{n}$ on the shape of the velocity profiles and hence, in turn, on the kinetic energy correction factor α in $\alpha\, V^2/2g$. In the laminar range the profiles are blunter than a corresponding Newtonian, for $\underline{n} < 1$ (so that α varies from 2 for $\underline{n} = 1$ to 1 for $\underline{n} = 0$) and sharper for $\underline{n} > 1$ (so that α varies from 1 to 2.7 for $\underline{n} = \infty$). The same trends apply - but with considerably less pronounced variations - to the turbulent range. (These trends may become significant, e.g., in the determination of the head at the suction side of a pump). Qualitatively analogous - and easily derived - is the trend of the momentum coefficient.

4. EXTERNAL LOSSES

DISK FRICTION

Of the various external losses in the turbomachine, those associated with disk-friction type of phenomena are likely to be the most significant in the case of radial and mixed flow machines handling non-Newtonian fluids of high viscosity.

The case of a disk rotating in a Newtonian fluid has been the object of several studies (see, e.g., the review in (11)), and the implications in terms of the overall behavior of a centrifugal pump have been investigated in detail by Ippen (12). For non-Newtonian fluids Schultz-Grunow presented a procedure for the correlation of friction losses for a smooth disk rotating under laminar conditions in a Prandtl-type of fluid (see ref. (5)). Further studies on power requirements for mixers have been performed by Brown and Petsiavas for a Bingham plastic fluid and by Otto and Metzner for power-law fluids (as summarized in ref. (13)). The latter correlated the results in terms of a generalized Reynolds number. Thus far, however, a comprehensive analysis for the entire range of flow and boundary conditions of significance to the machine designer has been lacking, even for the simple case of a power-law fluid.

This section will present the results of such an analysis, for the case of a smooth disk,

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(9) Bugliarello, G., Discussion, J. Sanitary Eng. Div., ASCE, Vol. 87, No. SA3, p. 89, May 1961.
(10) Bugliarello, G. and Daily, J.W., TAPPI (Technical Assoc. of the Pulp and Paper Industry), Vol. 44, No. 12, p. 881, 1961.
(11) Schlichting, H., "Boundary Layer Theory," McGraw-Hill, New York, 1960.
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both free and encased, rotating under laminar and turbulent conditions in a power-law fluid, i.e. in a fluid governed by the one-dimensional rheological eq. (1). The method followed in obtaining the results has been an approximate one, based essentially on dimensional considerations (so that the numerical values of several coefficients is still left unspecified) and on the assumption of a simplified flow picture. The results are however significant in bringing out the essential features of the non-Newtonian power-law case. For lack of space only a tabular summary of the most important results can be presented here.

The free disk, both under laminar and turbulent conditions, and the enclosed disk have been considered. For the latter, following the classification proposed by Daily and Nece(14), 4 regimes (I,II,III, and IV) have been considered, denoting conditions of flow between the rotor and stator which are respectively laminar with merging boundary layers, laminar with separate boundary layers, turbulent with merging boundary layers and turbulent with separate boundary layers.

The problems involved in obtaining laminar boundary layer solutions for power-law fluids have been discussed recently by Acrivos et al (15) and by Schowalter (16). For two-dimensional flows and $n < 2$, solutions can be obtained, provided the generalized Reynolds number is sufficiently large; for all n the Newtonian behavior is approached if U_∞ is sufficiently small. The behavior for $n > 2$ is more complex (but has a more limited practical significance) - as more complex is also the three-dimensional case.

In the laminar case and within the framework of these limitations, expressions for the boundary layer thickness δ on the disk have been obtained here through direct application of eq. (1). In the turbulent case, δ has been obtained from an expression for the boundary shear stress τ_0 derived as follows:

The friction factor for flow in a pipe can be expressed (17), for not too large values of the generalized Reynold's number $\mathbb{R} = (D^n V^{2-n}/K' 8^{n-1})$ in the form

$$f = (a/\mathbb{R}'_n)^b \qquad (2)$$

where a and b are functions of n (18). In practice the limitation to moderate Reynolds numbers is not very confining because usually much higher flows correspond to the same value of the Reynolds number for a non-Newtonian fluid than for a Newtonian one (See ref. (4)). From eq. (2) the following expression can be obtained for the shear stresses on a flat plate:

$$\tau_o/\rho U_\infty^2 = A_1 (n) \left[K/\rho\, \delta^n U_\infty^{2-n} \right]^b \qquad (3)$$

where A_1 (n) is a numerical coefficient, function of n. For a rotating disk, setting $U_\infty = \omega r$, eq. (3) becomes

$$\tau_o/\rho\, \omega^2 r^2 = A_2 (n)\, \mathbb{R}_{n,\delta}^{-b} \qquad (4)$$

where A_2 (n) is another numerical coefficient function of n, and $R_{n,\delta}$ is a generalized Reynolds number given by

$$\mathbb{R}_{n,\delta} = \rho\, \omega^{2-n} r^{2-n} \delta^n / K \qquad (5)$$

which for the Newtonian case (n=1) reduces to $\rho \omega r \delta / \mu$.

Table I gives the results for the boundary layer thickness δ for the free disk, under laminar and turbulent conditions, and, for comparison, the results for the Newtonian case, i.e. for the special case n = 1.

(14) Daily, J.W. and Nece, R.E., Trans., ASME, (J. of Basic Eng.), March 1960, p. 217.
(15) Acrivos, A., Shah, M. J. and Petersen, E.E., J. Amer. Inst. Chem. Eng., Vol. 6, No. 2, p. 312, 1960.
(16) Schowalter, W.R., J. Amer. Inst. Chem. Eng., Vol. 6, No. 1, 1960.
(17) Dodge, D.W. and Metzner, A.B., J. Amer. Inst. Chem. Eng., Vol. 5, p. 189, 1959.
(18) Some representative values of b are 0.35 for n= 0.2, 0.29 for n=0.5 , 0.25 for n= 1.0 and 0.21 for n=2.0.