The Finite Element Method

in Structural and Continuum Mechanics

Zienkiewicz

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Numerical solution of problems in structural and continuum mechanics

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Preface

The engineer of the present day is faced with solving structural and other problems of growing complexity. Classical mathematics, despite its ever-increasing sophistication, is capable of solving only severely idealized situations while placing at the same time a heavy burden on his skilled time, which could more usefully be employed in the process of design. Fortunately, the rapid development of digital computers, with a progressively greater capacity and a decreasing cost of performing arithmetical operations, has come to his rescue, allowing the use of relatively simple numerical formulations and revolutionizing his approach to the process of analysis.

It is now often no more expensive to perform a more accurate analysis instead of producing approximate calculations of doubtful validity. With the aid of such methods as the one described in this book, solution of previously intractable problems has become possible. Expensive experimental models now often used in the design of important structures are rapidly becoming displaced by more economic computation.

With this progress in the field of *analysis*, automatic optimization of component *design* is rapidly becoming a reality. On the other hand, such new devices as a computer 'sketch pad', by which the designer can interact with the machine, are being developed. Both will allow the future engineer to make the best use of his creative and scientific talents.

Before numerical, computer-based solutions of real problems dealing with complex continua can be solved, it is necessary to limit their infinite degrees of freedom to a finite, if large, number of unknowns. Such a process of discretization was first successfully performed by the now well-known method of *finite differences.*

Now an alternative approach, that of the *finite elements*, appears to offer considerable advantages and its relatively simple logic makes it ideally suited for the computer. Many papers illustrating the application of this process have been published, but it is felt that a fairly comprehensive, simple presentation is called for to make the procedures more widely understood. This is being attempted in the present volume.

The finite element method was developed originally as a concept of structural analysis, and the major part of the applications which will be illustrated belong to this field. However, the wider basis of the method will be stressed with applicability to such diverse problems as that of heat conduction, fluid flow, etc.

Although the book is primarily intended for the engineering profession it is hoped that it may be of interest to mathematicians, who may some day develop a *calculus* of *finite elements* in parallel with that of finite difference calculus. Because of this emphasis, mathematical demands made of the reader will not be exacting. An elementary knowledge of differential calculus coupled with some rudiments of matrix algebra are the basic requirements. For uninitiated readers a brief summary of the principles of matrix algebra is included in the Appendix.

The first chapter of the book has relatively little to do with 'finite elements'. It summarizes the basic principles of stiffness analysis of structures in a simple way so that reference to other structural textbooks is superfluous. It is a characteristic of the finite element process, whether used in a structural context or to describe other phenomena, that the standard procedures of structural assembly can always be followed.

Chapter 2 describes the essentials of the finite element formulation of elastic problems based on assumed displacement patterns. A careful study of this chapter lays the foundations of the method which, in Chapters 3 to 9, is applied to a variety of elasticity problems. It is important to note here two things. First, that the method is a general one based on an approximate solution of an extremum problem. Second, that, contrary to the well-known Ritz process, quantities with obvious physical meaning are chosen as the variable parameters.

The first fact permits an immediate extension to non-structural problems—some of which are dealt with in Chapter 10. The second allows the engineer to maintain at all times a direct physical 'contact' with the real problem being examined.

Obviously, the finite element method, because of its tremendous utility, is in rapid process of evolution. No book of this type can, therefore, hope to be complete. Although Chapter 14 is intended to throw light on some possible future developments, it is nevertheless hoped that as a text this work will remain of some permanent value, outlining the basic principles as well as some immediate applications.

Since simplicity of presentation has been the guiding motif in writing this text, it should also appeal to the beginner as well as the more experienced practitioner of the art, whose interest may be in the discussion of such topics as the use of numerical versus closed form integration and reference to other technical details. For the beginner, some indication of the preparation of a typical computer program is given in Chapter 15.

Here some knowledge of Fortran computer language will be useful, but clearly, since this is a rapidly developing field, the reader will need to keep abreast of new programming techniques.

As the engineer will be a primary user of the text, practical examples have been included whenever possible. The great majority of these refer to civil engineering problems with which the authors have been associated. Clearly, applications in all other branches of engineering can equally be envisaged—the major use of the methodology being in the field of 'aero-space' engineering.

Acknowledgements

First, I should like to express my appreciation of Professor R. W. Clough, whose infectious enthusiasm in many discussions has drawn my attention to this powerful method and inspired me to direct my work there.

While due reference has been made to the work of Professor Clough and other researchers, I should also like to acknowledge the work of many others who have contributed to the general development and who cannot all be mentioned here.

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Contents

1. Structural Stiffness Analysis

1.1 Introduction

Conventional engineering structures can be visualized as an assemblage of structural elements interconnected at a discrete number of nodal points. If the force-displacement relationships for the individual elements are known it is possible, by using various well-known techniques of structural analysis,[1, 2] to derive the properties and study the behaviour of the assembled structure.

In an elastic continuum the true number of interconnection points is infinite, and here lies the biggest difficulty of its numerical solution. The concept of finite elements, as originally introduced by Turner *et al.*,[3] attempts to overcome this difficulty by assuming the real continuum to be divided into elements interconnected only at a finite number of nodal points at which some fictitious forces, representative of the distributed stresses actually acting on the element boundaries, were supposed to be introduced. If such an idealization is permissible the problem reduces to that of a conventional structural type well amenable to numerical treatment.

At first glance the procedure, though intuitively appealing to a structural engineer, does not seem entirely convincing—in particular, leaving open the question of the load-displacement characteristics of the element. The problems of a consistent way of determination of these characteristics will be discussed in detail in Chapter 2, when a firm foundation for the method will be established. At this stage, however, it is important to recapitulate a general method of structural analysis which will be used throughout this book once the properties of the elements have been established.

The finite element method will be shown to apply to many problems of non-structural type. The essential properties of an element will even then be of the form encountered in structural analysis. Again, the general procedures of assembly and solution will follow a pattern for which the structural analogy provides a convenient basis.

1.2 The Structural Element

Let Fig. 1.1 represent a two-dimensional structure assembled from individual components and interconnected at the nodes numbered 1 to *n*. The joints at the nodes, in this case, are pinned so that moments cannot be transmitted.

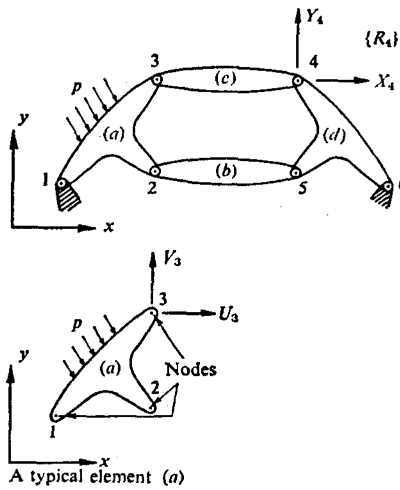

Fig. 1.1 A typical structure built up from interconnected elements

As a starting point it shall be assumed that by separate calculation, or for that matter from the results of an experiment, the characteristics of each element are precisely known. Thus, if a typical element labelled (*a*) and associated with nodes 1,2,3 is examined, the forces acting at the nodes are uniquely defined by the displacements of these nodes, the distributed loading acting on the element (*p*), and its initial strain. The last may be due to temperature, shrinkage, or simply an initial 'lack of fit'. The forces and the corresponding displacements are defined by appropriate components (U, V and u, v) in a common co-ordinate system.

Listing the forces acting on all the nodes (three in the case illustrated) of the element (*a*) as a matrix we have

$$\{F\}^a = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{Bmatrix} U_1 \\ V_1 \\ U_2 \\ V_2 \\ U_3 \\ V_3 \end{Bmatrix} \tag{1.1}$$

and for the corresponding nodal displacements as

$$\{\delta\}^a = \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{Bmatrix} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}. \tag{1.2}$$

Assuming elastic behaviour of the element, the characteristic relationship will always be of the form

$$\{F\}^a = [k]^a\{\delta\}^a + \{F\}^a{}_p + \{F\}^a{}_{\epsilon 0} \tag{1.3}$$

in which $\{F\}^a{}_p$, represents the nodal forces required to balance any distributed loads acting on the element, and $\{F\}^a{}_{\epsilon 0}$ the nodal forces required to balance any initial strains such as may be caused by temperature change if the nodes are not subject to any displacement. The first of the terms represents the forces induced by displacement of the nodes.

Similarly, the preliminary analysis or experiment will permit a unique definition of stresses or internal reactions at any specified point or points of the element in terms of the nodal displacements. Defining such stresses by a matrix $\{\sigma\}^a$ a relationship of the form

$$\{\sigma\}^a = [S]^a\{\delta\}^a + \{\sigma\}^a{}_p + \{\sigma\}^a{}_{\epsilon 0} \tag{1.4}$$

is obtained in which the last two terms are simply the stresses due to the distributed element loads or initial stresses respectively when no nodal displacement occurs.

The matrix $[k]^a$ is known as the element stiffness matrix and the matrix $[S]^a$ as the element stress matrix.

Relationships Eqs (1.3) and (1.4) have been illustrated on an example of an element with three nodes and with the interconnection points capable of transmitting only two components of force. Clearly, the same arguments and definitions will apply generally. An element (*b*) of the hypothetical structure will possess only two points of interconnection, others may have quite a large number of such points. Similarly, if the joins were considered as rigid, three components of generalized force and of generalized displacement would have to be considered, the last corresponding to a moment and a rotation respectively. For a rigidly jointed, three-dimensional structure the number of individual nodal components would be six. Quite generally therefore—

$$\{F\}^a = \begin{Bmatrix} F_i \\ \vdots \\ F_m \end{Bmatrix} \text{ and } \{\delta\}^a = \begin{Bmatrix} \delta_i \\ \vdots \\ \delta_m \end{Bmatrix} \tag{1.5}$$

with each F_i and δ_i possessing the same number of components or degrees of freedom.

The stiffness matrices of the element will clearly always be square and of the form

$$[k]^e = \begin{bmatrix} k_{ii} & k_{ij} & k_{im} \\ \vdots & \vdots & \vdots \\ k_{mi} & k_{mj} & k_{mm} \end{bmatrix} \tag{1.6}$$

in which k_{ii}, etc., are submatrices which are again square and of the size $l \times l$, where l is the number of force components to be considered at the nodes.

As an example, the reader can consider a pin-ended bar of a uniform section A and modulus E in a two-dimensional problem shown in Fig. 1.2. The bar is subject to a uniform lateral load p and a uniform thermal expansion strain

$$\epsilon_0 = \alpha T$$

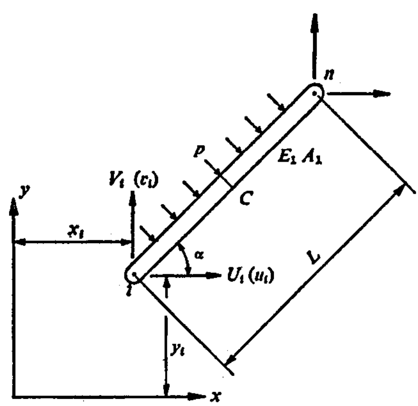

Fig. 1.2 A pin-headed bar

If the ends of the bar are defined by the co-ordinates x_i, y_i and x_n, y_n its length can be calculated as

$$L = \sqrt{\{(x_n - x_i)^2 + (y_n - y_i)^2\}}$$

and its inclination from the horizontal as

$$\alpha = \tan^{-1} \frac{y_n - y_i}{x_n - x_i}.$$

Only two components of force and displacement have to be considered at the nodes.

The nodal forces due to the lateral load are clearly

$$\{F\}_p^a = \begin{Bmatrix} F_i \\ F_n \end{Bmatrix}_p = \begin{Bmatrix} U_i \\ V_i \\ U_n \\ V_n \end{Bmatrix}_p = \begin{Bmatrix} -\sin\alpha \\ \cos\alpha \\ -\sin\alpha \\ \cos\alpha \end{Bmatrix} \cdot \frac{pL}{2}$$

and represent the appropriate components of simple beam reactions, $pL/2$. Similarly, to restrain the thermal expansion ϵ_0 an axial force $(E\alpha TA)$ is needed, which gives the components

$$\{F\}_{\epsilon 0}^a = \begin{Bmatrix} F_i \\ F_n \end{Bmatrix}_{\epsilon 0} = \begin{Bmatrix} U_i \\ V_i \\ U_n \\ V_n \end{Bmatrix}_{\epsilon 0} = \begin{Bmatrix} \cos\alpha \\ \sin\alpha \\ -\cos\alpha \\ -\sin\alpha \end{Bmatrix} (E\alpha TA).$$

Finally, the element displacements

$$\{\delta\}^a = \begin{Bmatrix} \delta_i \\ \delta_n \end{Bmatrix} = \begin{Bmatrix} u_i \\ v_i \\ u_n \\ v_n \end{Bmatrix}.$$

will cause an elongation $(u_n - u_i)\cos\alpha + (v_n - v_i)\sin\alpha$. This when multiplied by EA/L, gives the axial force whose components can again be found by substitution of this force in place of $E\alpha TA$ in the previous equation. Rearranging these in the standard form gives

$$\{F\}_\delta^a = \begin{Bmatrix} F_i \\ F_n \end{Bmatrix}_\delta = \begin{Bmatrix} U_i \\ V_i \\ U_n \\ V_n \end{Bmatrix}_\delta =$$

$$= \frac{EA}{L} \left[\begin{array}{cc|cc} \cos^2\alpha & \sin\alpha\cos\alpha & -\cos^2\alpha & -\sin\alpha\cos\alpha \\ \sin\alpha\cos\alpha & \sin^2\alpha & -\sin\alpha\cos\alpha & -\sin^2\alpha \\ \hline -\cos^2\alpha & -\sin\alpha\cos\alpha & \cos^2\alpha & \sin\alpha\cos\alpha \\ -\sin\alpha\cos\alpha & -\sin^2\alpha & \sin\alpha\cos\alpha & \sin^2\alpha \end{array} \right] \begin{Bmatrix} u_i \\ v_i \\ u_n \\ v_n \end{Bmatrix}$$

$$= [k]^a\{\delta\}^a.$$

The components of the general Eq. (1.3) have thus been established for the elementary case discussed. It is again quite simple to find the stresses at any section of the element in the form of relation Eq. (1.4). For instance, if attention is focused on the mid-section C of the beam the extreme fibre stresses determined from the axial tension to the element and the bending moment can be shown to be

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \end{Bmatrix}_C = \frac{E}{L} \begin{bmatrix} -\cos\alpha, & -\sin\alpha, & \cos\alpha, & \sin\alpha \\ -\cos\alpha, & -\sin\alpha, & \cos\alpha, & \sin\alpha \end{bmatrix} \{\delta\}^a$$

$$+ \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} \frac{pL^2}{8} \frac{d}{I} - \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} E\alpha T$$

in which d is the half depth of the section, I its second moment of area. All the terms of Eq. (1.4) can now be easily recognized.

For more complex elements more sophisticated procedures of analysis are required but the results are of the same form. The engineer will readily recognize that the so-called 'slope-deflection' relations used in analysis of rigid frames are only a special case of the general relations.

It may perhaps be remarked, in passing, that the complete stiffness matrix obtained for the simple element in tension turned out to be symmetric (as indeed was the case with all the submatrices). This is by no means fortuitous but follows from the principle of energy conservation and from its corollary—the well-known Maxwell-Betti reciprocal theorem.

The element properties were assumed to follow a simple linear relationship. In principle, similar relationships could be established for non-linear materials, but discussion of such problems will be held over at this stage.

1.3 Assembly and Analysis of a Structure

Consider again the hypothetical structure of Fig. 1.1. To obtain a complete solution the two conditions of

(*a*) displacement compatibility, and

(*b*) equilibrium

have to be satisfied throughout.

Any systems of nodal displacements $\{\delta\}$

$$\{\delta\} = \begin{Bmatrix} \delta_1 \\ \vdots \\ \delta_n \end{Bmatrix} \tag{1.7}$$

listed now for the whole structure in which all the elements participate, automatically satisfies the first condition.

As the conditions of overall equilibrium have already been satisfied *within* an element all that is necessary is to establish equilibrium conditions at the nodes of the structure. The resulting equations will contain the displacements as unknowns, and once these have been solved the structural problem is determined. The internal forces in elements, or the stresses, can easily be found by using the characteristics established *a priori* for each element by Eq. (1.4).

Consider the structure to be loaded by external forces $\{R\}$

$$\{R\} = \begin{Bmatrix} R_1 \\ \vdots \\ R_n \end{Bmatrix} \tag{1.8}$$

applied at the nodes in addition to the distributed loads applied to the individual elements. Again, any one of the forces R_i must have the same number of components as that of the element reactions considered. In the example in question

$$\{R_i\} = \begin{Bmatrix} X_i \\ Y_i \end{Bmatrix} \tag{1.9}$$

as the joints were assumed pinned, but at this stage a generality with an arbitrary number of components will be assumed.

If now the equilibrium conditions of a typical node, *i*, are to be established, each component of R_i has, in turn, to be equated to the sum of the component forces contributed by the elements meeting at the node. Thus, considering *all* the force components we have:

$$\{R_i\} = \Sigma\{F_i\} \tag{1.10}$$

the summation being taken over all the elements. Introducing the characteristics of the element given by Eq. (1.3) and taking note only of the appropriate forces F_i, by using the submatrices of Eq. (1.6), the above equations become

$$\{R_i\} = \sum_{m=1}^{m=n} \Sigma[k_{im}]^a\{\delta_m\} + \Sigma\{F_i\}^a{}_p + \Sigma\{F_i\}^a{}_{\epsilon 0}. \tag{1.11}$$

The inside summation is now taken over all the elements of the structure. If a particular element does not in fact include the node in question, it will contain no submatrices with an *i* suffix and, therefore, its contribution will simply be zero. This fact is of considerable convenience when computation schemes, either manual or for a digital machine, are being organized, because immediately on establishment of the characteristics of a particular element these can be summed in an appropriate location. Once all elements have been considered the overall system of equations is established.

This system of equations can be written simply as

$$[K]\{\delta\} = \{R\} - \{F\}_p - \{F\}_{\epsilon 0} \tag{1.12}$$

in which the submatrices are

$$\begin{aligned} [K_{im}] &= \Sigma[k_{im}]^a \\ \{F_i\}_p &= \Sigma\{F_i\}^a{}_p \\ \{F_i\}_{\epsilon 0} &= \Sigma\{F_i\}^a{}_{\epsilon 0} \end{aligned} \tag{1.13}$$

with summations including all elements.

If different types of structural elements are used and are to be coupled it must be remembered that the rules of matrix summation permit this to be done only if these are of identical size. The individual submatrices to be added have therefore to be built up of the same number of individual components of force or displacement. Thus, for example, if a member capable of transmitting moments to a node is to be coupled at that node