Probabilistic Methods of Signal and System Analysis

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Preface

This book presents an introduction to probability theory, random processes, and the analysis of systems with random inputs. It is written at a level that is suitable for junior and senior engineering students and presumes that the student is familiar with conventional methods of system analysis such as convolution and transform techniques. However, it may also serve graduate students as a concise review of material that they previously encountered in widely scattered sources.

Since this is an engineering text, the treatment is heuristic rather than rigorous, and the student will find many examples of the application of these concepts to engineering problems. However, it is not completely devoid of the mathematical subtleties, and considerable attention has been devoted to pointing out some of the difficulties that make a more advanced study of the subject essential if one is to master it. The authors believe that the educational process is best served by repeated exposure to difficult subject matter; this text is intended to be the first exposure to probability and random processes, but hopefully, will not be the last. Thus, the book is not comprehensive, but deals selectively with those topics that the authors have found most useful in the solution of engineering problems.

A brief discussion of some of the significant features of this book will help set the stage for a discussion of the various ways it can be used.

Elementary concepts of discrete probability are introduced in Chapter 1; first from the intuitive standpoint of the relative-frequency approach and then from the more rigorous standpoint of axiomatic probability. Simple examples illustrate all these concepts and are more meaningful to engineers than are the traditional examples of selecting red and white balls from urns.

The concept of a random variable is introduced in Chapter 2 along with the ideas of probability distribution and density functions, mean values, and conditional probability. A significant feature of this chapter is a rather extensive discussion of many different probability density functions and the physical situations in which they may occur. Chapter 3 extends the random variable concept to situations involving two or more random variables and introduces the concepts of statistical independence and correlation.

A general discussion of random processes and their classification is given in Chapter 4. The emphasis here is on selecting probability models that are useful in solving engineering problems. Accordingly, a great deal of attention is devoted to the physical significance of the various process classifications, with no attempt at mathematical rigor. A unique feature of this chapter, which is continued in subsequent chapters, is an introduction to the practical problem of estimating the mean of a random process from an observed sample function.

Properties and applications of autocorrelation and cross-correlation functions are discussed in Chapter 5. Many examples are presented in an attempt to develop some insight into the nature of correlation functions. The important problem of estimating autocorrelation functions is discussed in some detail.

Chapter 6 turns to a frequency-domain representation of random processes by introducing the concept of spectral density. Unlike most texts, which simply define spectral density as a Fourier transform of the correlation function, a more fundamental approach is adopted here in order to bring out the physical significance of the concept. This chapter is the most difficult one in the book, but the authors believe the material should be presented in this way. Instructors who wish to by-pass some of the more fundamental problems may omit Section 6-2 and bridge the gap by defining spectral density simply as the Fourier transform of the correlation function.

Chapter 7 utilizes the concepts of correlation functions and spectral density to analyze the response of linear systems to random inputs. In a sense, this chapter is a culmination of all that preceded it, and is particularly significant to engineers who must use these concepts. Hence, it contains a great many examples that are relevant to engineering problems and emphasizes the need for mathematical models that both are realistic and manageable.

Chapter 8 extends the concepts of systems analysis to consider systems that are optimum in some sense. Both the classical matched filter for known signals and the Wiener filter for random signals are considered from an elementary standpoint.

In a more general vein, each chapter contains references that the student can use to extend his knowledge. There is also a wide selection of problems at the end of each chapter, and, at the end of the book, a number of Appendixes that he will find useful in solving these problems, along with a set of selected answers,

As an additional aid to learning and using the concepts and methods discussed in this book, there are exercises at the end of many of the sections. The student should consider these exercises as part of the reading assignments and should make every effort to solve each one before going on to the next section. Answers are provided in order that he may know when his efforts have been successful. *It should be noted, however, that the answers to each exercise may not be listed in the same order as the question.* This is intended to provide an additional challenge to the student. The presence of these exercises should substantially reduce the number of additional problems that need to be assigned by the instructor.

The material in this text has been used in a one-semester, three-credit course offered in the first semester of the junior year. Not all sections of the text are used in this course but at least 90% of it is covered in reasonable detail. The sections usually omitted include 3-6, 4-6, 5-4, 5-9, 6-9, and 8-6; but other choices may be made at the discretion of the instructor. There are, of course, many other ways in which the text material could be utilized. For example, a one-semester course with a more relaxed pace could be given by omitting all of Chapter 8 in addition to the sections noted above. For those schools on the quarter-system, the material noted above could be covered in a four-credit hour course. If a three-credit hour course were desired, in addition to the omissions noted above, it is suggested that Sections 1-4, 1-5, 1-6, 1-8, 2-6, 2-7, 3-5, 6-2, 6-8, 6-10, 7-9, and all of Chapter 8 can be omitted if the instructor supplies a few explanatory words to bridge gaps. Obviously, there are also many other possibilities that are open to the experienced instructor.

It is a pleasure for the authors to acknowledge the very substantial aid and encouragement that they have received from their colleagues and students. A complete list is too lengthy to include here, but it is appropriate to mention the valuable suggestions and comments received from Professors J. Y. S. Luh and P. A. Wintz of Purdue University.

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CHAPTER

1

Introduction to Probability

1-1 Engineering Applications of Probability

Before embarking on a study of elementary probability theory, it is desirable to motivate such a study by considering why probability theory is useful in the solution of engineering problems. This will be done in two different ways. The first is to suggest a viewpoint, or philosophy, concerning probability that emphasizes its universal physical reality rather than treating it as another mathematical discipline which may be useful occasionally. The second is to note some of the many different types of situations that arise in normal engineering practice in which the use of probability concepts is indispensible.

A characteristic feature of probability theory is that it concerns itself with situations that involve uncertainty in some form. The popular conception of this relates probability to such activities as tossing dice, drawing cards, and spinning roulette wheels. Because the rules of probability are not widely known, and because such situations can become quite complex, the prevalent attitude is that probability theory is a mysterious and esoteric branch of mathematics that is accessible only to trained mathematicians and is of only limited value in the real world. Since probability theory does deal with uncertainty, another prevalent attitude is that a probabilistic treatment of physical problems is an inferior substitute for a more desirable exact analysis and is forced on the analyst by a lack of complete information. *Both of these attitudes are false.*

Regarding the alleged difficulty of probability theory, it is doubtful if there is any other branch of mathematics or analysis which is so completely based on such a small number of basic concepts that are so easily understood. Subsequent discussion will reveal that the major body of probability theory can be deduced from only three axioms that are almost self-evident. Once these axioms and their applications are understood, the remaining concepts follow in a logical manner.

The attitude that regards probability theory as a substitute for exact analysis stems from the current educational practice of presenting physical laws as deterministic, immutable, and strictly true under all circumstances. Thus, a law that describes the response of a dynamical system is supposed to predict that response exactly if the system excitation is known exactly. For example, Ohm's law

$$v(t) = Ri(t)$$

is assumed to be exactly true at every instant of time, and on a macroscopic basis this assumption may be well justified. On a microscopic basis, however, this assumption is patently false—a fact that is immediately obvious to anyone who has tried to connect a large resistor to the input of a high-gain amplifier and listened to the resulting noise.

In the light of modern physics and our emerging knowledge of the nature of matter, the viewpoint that natural laws are deterministic and exact is untenable. They are at best a representation of the average behavior of nature. In many important cases this average behavior is close enough to that actually observed so that the deviations are unimportant. In such cases the deterministic laws are extremely valuable because they make it possible to predict system behavior with a minimum of effort. In other equally important cases the random deviations may be significant—perhaps even more significant than the deterministic response. For these cases, analytic methods derived from the concepts of probability are essential.

From the above discussion it should be clear that the so-called exact solution is not exact at all, but in fact represents an idealized special case which actually never arises in nature. The probabilistic approach, on the other hand, far from being a poor substitute for exactness, is actually the method which most nearly represents physical reality. Furthermore, it includes the deterministic result as a special case.

It is now appropriate to discuss the types of situations in which probability concepts arise in engineering. The examples presented here emphasize situations that arise in systems studies; but they do serve to illustrate the essential point that engineering applications of probability tend to be the rule rather than the exception.

Random input signals. In order for a physical system to perform a useful task, it is usually necessary that some sort of forcing function (the input signal) be applied to it. Input signals that have simple mathematical representations are convenient for pedagogical purposes or for certain types of system analysis, but they seldom arise in actual applications. Instead, the input signal is more likely to involve a certain amount of uncertainty and unpredictability that justifies treating it as a *random* signal. There are many examples of this: speech and music signals that serve as inputs to communication systems; random digits applied to a computer; random command signals applied to an aircraft flight control system; random signals derived from measuring some characteristic of a manufactured product, and used as inputs to a process control system; steering wheel movements in an automobile power-steering system; the sequence in which the call and operating buttons of an elevator are pushed; the number of vehicles passing various checkpoints in a traffic control system; outside and inside temperature fluctuations as inputs to a building heating and airconditioning system; and many others.

Random disturbances. Many systems have unwanted disturbances applied to their input or output in addition to the desired signals. Such disturbances are almost always random in nature and call for the use of probabilistic methods even if the desired signal does not. A few specific cases serve to illustrate several different types of disturbances. If, for a first example, the output of a high-gain amplifier is connected to a loudspeaker, one frequently hears a variety of snaps, crackles, and pops. This random noise arises from thermal motion of the conduction electrons in the amplifier input circuit or from random variations in the number of electrons (or holes) passing through the tubes and transistors. It is obvious that one cannot hope to calculate the value of this noise at every instant of time since this value represents the combined effects of literally billions of individual moving charges. It is possible, however, to calculate the average power of this noise, its frequency spectrum, and even the probability of observing a noise value larger than some specified value. As a practical matter, these quantities are more important in determining the quality of the amplifier than is a knowledge of the instantaneous waveforms.

As a second example, consider a radio or television receiver. In addition to noise generated within the receiver by the mechanisms noted, there is random noise arriving at the antenna. This results from distant electrical storms, man-made disturbances, radiation from space, or thermal radiation from surrounding objects. Hence, even if perfect receivers and amplifiers were available, the received signal would be combined with random noise. Again, the calculation of such quantities as average power and frequency spectrum may be more significant than the determination of instantaneous value.

A different type of system is illustrated by a large radar antenna, which may be pointed in any direction by means of an automatic control system. The wind blowing on the antenna produces random forces that must be compensated for by the control system. Since the compensation is never perfect, there is always some random fluctuation in the antenna direction; it is important to be able to calculate the effective value of this fluctuation.

A still different situation is illustrated by an airplane flying in turbulent air, a ship sailing in stormy seas, or an army truck traveling over rough terrain. In all these cases random disturbing forces, acting on complex mechanical systems, interfere with the proper control or guidance of the system. It is important to be able to determine how the system responds to these random input signals.

Random system characteristics. The system itself may have characteristics that are unknown and that vary in a random fashion from time to time. Some typical examples are: aircraft in which the load (that is, the number of passengers or the weight of the cargo) varies from flight to flight; troposcatter communication systems in which the path attenuation varies radically from moment to moment; an electric power system in which the load (that is, the amount of energy being used) fluctuates randomly; and a telephone system in which the number of users changes from instant to instant.

System reliability. All systems are composed of many individual elements, and one or more of these elements may fail, thus causing the entire system, or part of the system, to fail. The times at which such failures will occur is unknown, but it is often possible to determine the probability of failure for the individual elements and from these to determine the "mean time to failure" for the system. Such reliability studies are deeply involved with probability and are extremely important in engineering design. As systems become more complex, more costly, and contain larger numbers of elements, the problems of reliability become more difficult and take on added significance.

Quality control. An important method of improving system reliability is to improve the quality of the individual elements, and this can often be done by an inspection process. As it may be too costly to inspect every element after every step during its manufacture, it is necessary to develop rules for inspecting elements selected at random. These rules are based on probabilistic concepts and serve the valuable purpose of maintaining the quality of the product with the least expense.

Information theory. A major objective of information theory is to provide a quantitative measure for the information content of messages such as printed pages, speech, pictures, graphical data, numerical data, or physical observations of temperature, distance, velocity, radiation intensity, and rainfall. This quantitative measure is necessary in order to be able to provide communication channels that are both adequate and efficient for conveying this information from one place to another. Since such messages and observations are almost invariably unknown in advance and random in nature, they can be described only in terms of probability. Hence, the appropriate information measure is a probabilistic one. Furthermore, the communication channels are subject to random disturbances (noise) that limit their ability to convey information, and again a probabilistic description is required.

It should be clear from the above partial listing that almost any engineering endeavor involves a degree of uncertainty or randomness that makes the use of probabilistic concepts an essential tool for the present-day engineer. In the case of system analysis, it is necessary to have some description of random signals and disturbances. There are two general methods of describing random signals mathematically. The first, and most basic, is a probabilistic description in which the random quantity is characterized by a probability model. This method is discussed later in this chapter.

The probabilistic description of random signals cannot be used directly in system analysis since it tells very little about how the random signal varies with time or what its frequency spectrum is. It does, however, lead to the statistical description of random signals, which is useful in system analysis. In this case the random signal is characterized by a statistical model, which consists of an appropriate set of average values such as the mean, variance, correlation function, spectral density, and others. These average values represent a less precise description of the random signal than that offered by the probability model, but they are more useful for system analysis because they can be computed by using straightforward and relatively simple methods. Some of the statistical averages will be discussed in subsequent chapters.

1-2 Definitions of Probability

One of the most serious stumbling blocks in the study of elementary probability is that of arriving at a satisfactory definition of the term "probability." There are, in fact, four or five different definitions for probability that have been proposed and used with varying degrees of success. They all suffer from deficiencies in concept or application. Ironically, the most successful "definition" leaves the term probability *undefined.*

Of the various approaches to probability, the two that appear to be most useful are the *relative-frequency* approach and the *axiomatic* approach. The relative-frequency approach is useful because it attempts to attach some physical significance to the concept of probability and thereby makes it possible to relate probabilistic concepts to the real world. Hence the application of probability to engineering problems is almost always accomplished by invoking the concepts of relative frequency, even when the engineer may not be conscious that he is doing so.

The limitation of the relative-frequency approach is the difficulty of using it to deduce the appropriate mathematical structure for situations that are too complicated to be analyzed readily by physical reasoning. This is not to imply that this approach cannot be used in such situations, for it can, but it does suggest that there may be a much easier way to deal with these cases. The easier way turns out to be the axiomatic approach.

The axiomatic approach treats the probability of an event as a number that satisfies certain postulates but is otherwise undefined. Whether or not this number relates to anything in the real world is of no concern in developing the mathematical structure that evolves from these postulates. Engineers may object to this approach as being too artificial and too removed from reality, but they should remember that the whole body of circuit theory was developed in essentially the same way. In the case of circuit theory the basic postulates are Kirchhoff's laws and the conservation of energy. The same mathematical structure emerges regardless of what physical quantities are identified with the abstract symbols—or even if *no* physical quantities are associated with them. It is the task of the engineer to relate this mathematical structure to the real world in a way that is admittedly not exact, but that leads to useful solutions to real problems.

From the above discussion it appears that the most useful approach to probability for engineers is a two-pronged one, in which the relative-frequency concept is employed in order to relate simple results to physical reality, and the axiomatic approach is employed to develop the appropriate mathematics for more complicated situations. It is this philosophy that will be presented here.

1-3 The Relative-Frequency Approach

As its name implies, the relative-frequency approach to probability is closely linked to the frequency of occurrence of some particular event. The term *event* is used for one of the most basic concepts of probability theory. An event is something that may or may not happen. For example, if a coin is tossed the result may be a head or a tail and each of these is an event.

In order to examine this concept more precisely, however, it is necessary to introduce the idea of an *experiment* and the *outcomes* of that