Wilhelm Klingenberg

A Course in Differential Geometry

Translated by David Hoffman

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Preface to the English Edition

This English edition could serve as a text for a first year graduate course on differential geometry, as did for a long time the Chicago Notes of Chern mentioned in the Preface to the German Edition. Suitable references for ordinary differential equations are Hurewicz, W. *Lectures on ordinary differential equations*. MIT Press, Cambridge, Mass., 1958, and for the topology of surfaces: Massey, *Algebraic Topology*, Springer-Verlag, New York, 1977.

Upon David Hoffman fell the difficult task of transforming the tightly constructed German text into one which would mesh well with the more relaxed format of the Graduate Texts in Mathematics series. There are some elaborations and several new figures have been added. I trust that the merits of the German edition have survived whereas at the same time the efforts of David helped to elucidate the general conception of the Course where we tried to put Geometry before Formalism without giving up mathematical rigour.

I wish to thank David for his work and his enthusiasm during the whole period of our collaboration. At the same time I would like to commend the editors of Springer-Verlag for their patience and good advice.

Bonn
June, 1977

Wilhelm Klingenberg

From the Preface to the German Edition

This book has its origins in a one-semester course in differential geometry which I have given many times at Göttingen, Mainz, and Bonn.

It is my intention that these lectures should offer an introduction to the classical differential geometry of curves and surfaces, suitable for students in their middle semester who have mastered the introductory courses. A course such as this would be an alternative to other middle semester courses such as complex function theory, abstract algebra, or algebraic topology.

For the most part, these lectures assume nothing more than a knowledge of basic analysis, real linear algebra, and euclidean geometry. It is only in the last chapters that a familiarity with the topology of compact surfaces would be useful. Nothing is used that cannot be found in Seifert and Threlfall's classic textbook of topology.

For a summary of the contents of these lectures, I refer the reader to the table of contents. Of course it was necessary to make a selection from the profusion of material that could be presented at this level. For me it was clear that the preferred topics were precisely those which contributed to an understanding of two-dimensional Riemannian geometry. Nonetheless, I think that my lectures provide a useful basis for the understanding of all the areas of differential geometry.

The structure of these lectures, including the organization of some of the proofs, has been greatly influenced by S. S. Chern's lecture notes entitled "Differential Geometry," published in Chicago in 1954. Chern, in turn, was influenced by W. Blaschke's "Vorlesungen über Differentialgeometrie." Chern had studied with Blaschke in Hamburg between 1934 and 1936, and, nearly twenty years later, it was Blaschke who gave me strong support in my career as a differential geometer.

So as I take the privilege of dedicating this book to Shiing-shen Chern, I would at the same time desire to honor the memory of W. Blaschke.

Bonn-Röttgen
January 1, 1972

Wilhelm Klingenberg

Contents

Calculus in Euclidean Space 0

We will start with a brief outline of the essential facts about $\mathbb{R}^n$ and the vector calculus.[1] The reader familiar with this subject may wish to begin with Chapter 1, using this chapter as the need arises.

0.1 Euclidean Space

As usual, $\mathbf{R}^n$ is the vector space of all real n-tuples $x = (x^1, \ldots, x^n)$. The scalar product of two elements x, y in $\mathbf{R}^n$ is given by the formula

$$x \cdot y := \sum_i x^i y^i.$$

We will write $x \cdot x = x^2$ and $\sqrt{x^2} = |x|$. The real number $|x|$ is called the *length* or the *norm* of x. The *Schwarz inequality*,

$$(x \cdot y)^2 \leq |x|^2 |y|^2, \qquad |x|^2 = x^2,$$

is satisfied by the scalar product and from it is derived the *triangle inequality*:

$$|x + y| \leq |x| + |y| \quad \text{for all } x, y \in \mathbf{R}^n.$$

The distinguished basis of $\mathbf{R}^n$ will be denoted by (e_i), $1 \leq i \leq n$. The vector e_i is the n-tuple with 1 in the ith place and 0 in all the other places.

We shall also use $\mathbf{R}^n$ to denote the n-dimensional *Euclidean space*. More precisely, $\mathbf{R}^n$ is the Euclidean space with origin $= (0, 0, \ldots, 0)$, and an orthonormal basis at this point, namely (e_i), $1 \leq i \leq n$.

[1] Some standard references for material in this chapter are: Dieudonné, J. *Foundations of Modern Analysis*. New York: Academic Press, 1960. Edwards, C. H. *Advanced Calculus of Several Variables*. New York: Academic Press, 1973. Spivak, M. *Calculus on Manifolds*. Reading, Mass.: W. Benjamin, 1966.

The *distance* between two points $x, y \in \mathbf{R}^n$ will be denoted by $d(x, y)$ and defined by $d(x, y) := |x - y|$. Clearly $d(x, y) \geq 0$, ($d(x, y) = 0$ if and only if $x = y$) and $d(x, y) = d(y, x)$. Also, the triangle inequality for the norm implies the triangle inequality for the distance function,

$$d(x, z) \leq d(x, y) + d(y, z), \qquad x, y, z \in \mathbf{R}^n.$$

These three conditions satisfied by d imply that $\mathbf{R}^n$, with d as distance function, is a metric space.

The transformations of Euclidean space which preserve the Euclidean structure, i.e., the metric preserving transformations of $\mathbf{R}^n$, are called *isometries*. One type of isometry is a *translation*: $T_{x_0}: \mathbf{R}^n \to \mathbf{R}^n$ defined by $x \mapsto x + x_0$, where x_0 is a fixed element of $\mathbf{R}^n$. Another type is an *orthogonal transformation*:

$$R: \mathbf{R}^n \to \mathbf{R}^n, \ R \text{ is linear and } R(x) \cdot R(y) = x \cdot y, \qquad x, y \in \mathbf{R}^n.$$

If an orthogonal motion is orientation preserving (i.e., the matrix whose columns are $Re_1, \ldots, Re_n$, $i = 1, \ldots, n$, has determinant $+1$), it is a *rotation*. An example of an orthogonal motion which is not a rotation is given by the reflection

$$\rho: \mathbf{R}^n \to \mathbf{R}^n \qquad x \mapsto -x$$

when n is odd.

Any isometry B of Euclidean space may be written

$$B: \mathbf{R}^n \to \mathbf{R}^n, \qquad x \mapsto Rx + x_0$$

where $x_0 \in \mathbf{R}^n$ and R is an orthogonal motion. In other words, every isometry of Euclidean space consists of an orthogonal motion R, followed by a translation T_{x_0}. We will call R the *orthogonal component* of B. If R is a rotation we will say that B is a *congruence*. If not, we will say that B is a *symmetry*.

0.2 The Topology of Euclidean Space

The distance function d allows us, in the usual way, to define the metric topology on $\mathbf{R}^n$. For $x \in \mathbf{R}^n$ and $\epsilon > 0$, the *ϵ-ball centered at x* is denoted $B_\epsilon(x)$ and is defined by

$$B_\epsilon(x) := \{y \in \mathbf{R}^n \mid d(x, y) < \epsilon\}.$$

A set $U \subset \mathbf{R}^n$ is called *open* if for every $x \in U$ there exists an $\epsilon = \epsilon(x) > 0$ such that $B_\epsilon(x) \subset U$. A set $V \subset \mathbf{R}^n$ is *closed* if $\mathbf{R}^n \setminus V$ is open. Given a set $W \subset \mathbf{R}^n$, $\mathring{W}$ denotes its *interior*, i.e., the set of all $x \in W$ for which there exists some $\epsilon > 0$ with $B_\epsilon(x) \subset W$.

A set $U \subset \mathbf{R}^n$ is said to be a neighborhood of $x_0 \in \mathbf{R}^n$ if $x_0 \in \mathring{U}$. A mapping $F: U \to \mathbf{R}^n$ is *continuous at* x_0 if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $F(U \cap B_\delta(x)) \subset B_\epsilon(Fx_0)$. F is said to be *continuous* if it is continuous at all $x \in U$.

Example. Linear functions are continuous

Let L be a linear function, i.e., $L(ax + by) = aL(x) + bL(y)$ for $a, b \in \mathbb{R}$, $x, y \in \mathbf{R}^n$. L may be written in terms of a matrix (a_i^j), $1 \le i \le n$, $1 \le j \le m$, where $(L(x))^j = \sum_i a_i^j x^i$. To show that L is continuous, we use the Schwarz inequality. Writing $|L|^2$ for $\sum_{i,j} (a_i^j)^2$,

$$|Lx|^2 = \sum_j \left(\sum_i a_i^j x^i\right)^2 \le \sum_j \left(\sum_i (a_i^j)^2\right) \cdot \sum_i (x^i)^2 = |L|^2 \cdot |x|^2.$$

Therefore $|Lx - Lx_0| \le |L| \cdot |x - x_0|$. From this, the continuity of L is easily seen. *Note*: It follows that isometries B: $\mathbf{R}^n \to \mathbf{R}^n$ are continuous: for $Bx - Bx_0 = R(x - x_0)$, R being the orthogonal component of B, and R is linear.

0.3 Differentiation in $\mathbf{R}^n$

Consider the set $L(\mathbf{R}^n, \mathbf{R}^m)$ of linear transformations from $\mathbf{R}^n$ to $\mathbf{R}^m$. This set has a natural real vector-space structure of dimension $n \cdot m$. Addition of two linear transformations L_1, L_2 is defined by adding in the range; $(L_1 + L_2)x := L_1x + L_2x$. Scalar multiplication by $\alpha \in \mathbf{R}$ is defined by $(\alpha L_1)x := \alpha(L_1x)$.

In terms of the matrices (a_i^j) which represent elements $L \in L(\mathbf{R}^n, \mathbf{R}^m)$, addition corresponds to the usual matrix addition and scalar multiplication to multiplication of matrices by scalars.

The bijection of $L(\mathbf{R}^n, \mathbf{R}^m)$ onto $\mathbf{R}^{n \cdot m}$, given by considering the matrix representation (a_i^j) of a linear map L and identifying (a_i^j) with the vector $(a_1^1, \ldots, a_1^m, a_2^1, \ldots, a_2^m, \ldots, a_n^1, \ldots, a_n^m)$, is norm-preserving. The norm $|L|$ agrees with the length (= norm) of its image vector in $\mathbf{R}^{n \cdot m}$.

Let $U \subset \mathbf{R}^n$ be an open set, and suppose F: $U \to \mathbf{R}^m$ is any continuous map. F is said to be *differentiable at* $x_0 \in U$ if there exists a linear mapping $L = L(F, x_0) \in L(\mathbf{R}^n, \mathbf{R}^m)$ such that

$$\lim_{x \to x_0} \frac{|Fx - Fx_0 - L(x - x_0)|}{|x - x_0|} = 0.$$

It will be convenient to denote by $o(x)$ an arbitrary function with

$$\lim_{x \to 0} \frac{o(x)}{|x|} = 0.$$

In terms of this notation, the equation above may be rewritten as

$$|Fx - Fx_0 - L(x - x_0)| = o(x - x_0).$$

If such an $L = L(F, x_0)$ exists, it is unique. Suppose L and L' are two such linear mappings with the required properties. Then, using the triangle inequality,

$$\begin{aligned}|(L - L')(x - x_0)| &= |(L - L')(x - x_0) + Fx - Fx + Fx_0 - Fx_0| \\ &\le |Fx - Fx_0 - L(x - x_0)| + |Fx - Fx_0 - L'(x - x_0)| \\ &= o(x - x_0) + o(x - x_0) = o(x - x_0).\end{aligned}$$

Thus $|(L - L')(x - x_0)|$ is $o(x - x_0)$. In particular, if $x - x_0 = re_i$, then

$$r\left(\sum_j (a_i^j - a_i'^j)^2\right)^{1/2} = o(r).$$

Therefore, $a_i^j = a_i'^j$ for all i, j.

The unique linear map $L = L(F, x_0)$ is called the *differential of F at x_0*, which will also be denoted by dF_{x_0}, or simply dF.

If A is an arbitrary (not necessarily open) set in $\mathbf{R}^n$, a mapping $F: A \to \mathbf{R}^m$ is said to be differentiable on A if there exists an open set $U \subset \mathbf{R}^n$ containing A and a mapping $G: U \to \mathbf{R}^n$ such that $G|_A = F$, and G is differentiable at each $x_0 \in U$.

Examples of differentiable mappings

1. $L: \mathbf{R}^n \to \mathbf{R}^m$, any linear map. $dL_x = L$, for all $x \in \mathbf{R}^n$.
2. $B: \mathbf{R}^n \to \mathbf{R}^m$, an isometry. $dB_x = R$, the orthogonal component of B.
3. All the elementary functions encountered in calculus of one variable are differentiable; polynomials, rational functions, trigonometric functions, the exponential and logarithm.
4. The maps $(x, y) \mapsto x \cdot y$ from $\mathbf{R}^n \times \mathbf{R}^n$ into $\mathbf{R}$ and $x \mapsto |x|^2$ from $\mathbf{R}^n$ into $\mathbf{R}$ are differentiable.
5. The familiar vector cross-product $(x, y) \mapsto x \times y \in \mathbf{R}^3$, considered as a map from $\mathbf{R}^3 \times \mathbf{R}^3$ into $\mathbf{R}^3$, is differentiable. In terms of a basis for $\mathbf{R}^3$, if $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$, then $x \times y = (x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1)$.

It is an easy exercise to prove that the composition of two differentiable mappings is differentiable.

A mapping $F: U \to \mathbf{R}^m$, U open in $\mathbf{R}^n$, is said to be continuously differentiable, or C^1, if F is differentiable at each $x \in U$ and the map $dF: U \to L(\mathbf{R}^n, \mathbf{R}^m)$, given by $x \to dF_x$, is continuous.

A mapping $F: U \to \mathbf{R}^m$, $U \subset R^n$ is said to be twice continuously differentiable, or C^2, if $dF: U \to L(\mathbf{R}^n, \mathbf{R}^m)$ is differentiable, and its derivative is continuous.

In an analogous manner, we may define k-times continuously differentiable mappings, or C^k mappings. If f is k-times differentiable for any $k = 1, 2, \ldots$, f is said to be C^∞ (read "C infinity"). Sometimes we will refer to C^∞ mappings as differentiable mappings when there is no possibility of confusion.

If $U \subset \mathbf{R}^m$, $V \subset \mathbf{R}^n$ are open sets and $F: U \to V$ is a bijective, differentiable function such that $F^{-1}: V \to U$ is also differentiable, then F is called a *diffeomorphism* (between U and V).

If $F: U \to \mathbb{R}^m$, $U \subset \mathbb{R}^n$ is differentiable, then the m coordinate functions $F^j(x^1, \ldots, x^n)$ have partial derivatives $\partial F^j/\partial x^i = F^j_{x^i}$ with respect to each of the n coordinates x^i. From our definition of $dF_{x_0}: \mathbb{R}^m \to \mathbb{R}^n$, it follows that the matrix of this linear map is given by the matrix of first derivatives of F at x_0, $(F^j_{x^i})_{x_0}$, the familiar Jacobian matrix.

The differential $d^2F = d(dF)$ of the differentiable function $dF: U \to L(\mathbb{R}^n, \mathbb{R}^m)$ at the point $x_0 \in U$ has the following matrix representation: dF is determined by the $n \cdot m$ real valued functions $\partial F^j/\partial x^i$. Therefore $d^2F_{x_0}$ is determined by the $(m \times n \cdot m)$-matrix $(\partial^2 F^j/\partial x^i \, \partial x^k)|_{x_0}$. The row-index in this notation is $\{^j_i\}$ and k is the column-index. (The pairs $\{^j_i\}$ are ordered lexicographically.)

0.4 Tangent Space

The concept of a tangent space will play a fundamental role in our study of differential geometry. For $x_0 \in \mathbf{R}^n$, the *tangent space of* $\mathbf{R}^n$ *at* x_0, written $\mathrm{T}_{x_0}\mathbf{R}^n$ or $\mathbf{R}^n_{x_0}$, is the n-dimensional vector-space whose elements consist of pairs $(x_0, x) \in \{x_0\} \times \mathbf{R}^n$. The vector-space structure is defined by means of the bijection

$$\mathrm{T}_{x_0}\mathbf{R}^n \to \mathbf{R}^n, \qquad (x_0, x) \mapsto x,$$

i.e., $(x_0, x) + (x_0, y) = (x_0, x + y)$ and $a(x_0, x) = (x_0, ax)$.

Let U be a subset of $\mathbf{R}^n$. *The tangent bundle of* U, denoted $\mathrm{T}U$, is the disjoint union of the tangent spaces $\mathrm{T}_{x_0}\mathbf{R}^n$, $x_0 \in U$, together with the canonical projection $\pi: \mathrm{T}U \to U$, given by $(x_0, x) \mapsto x_0$. $\mathrm{T}U$ is in 1-1 correspondence with $U \times \mathbf{R}^n$ via the bijection

$$(x_0, x) \in \mathrm{T}_{x_0}\mathbf{R}^n \subset \mathrm{T}U \mapsto (x_0, x) \in U \times \mathbf{R}^n.$$

In view of the generalizations we will make in subsequent chapters, the interpretation of $\mathrm{T}U$ as the disjoint union of the tangent spaces $\mathrm{T}_{x_0}\mathbb{R}^n$, $x_0 \in U$, is preferable to that of $\mathrm{T}U$ as $U \times \mathbf{R}^n$. On the other hand, the interpretation of $\mathrm{T}U$ as $U \times \mathbf{R}^n$ shows that $\mathrm{T}U$ may be considered as a subset of $\mathbf{R}^n \times \mathbf{R}^n = \mathbf{R}^{2n}$. If U is open, then $U \times \mathbf{R}^n$ is also open in $\mathbf{R}^{2n}$, so it is clear what it means for a function $G: \mathrm{T}U \to \mathbf{R}^k$ to be continuous or differentiable. We may now define the notion of the *differential* of a differentiable mapping $F: U \to \mathbf{R}^m$ in terms of the tangent bundle.

Let U be an open set in $\mathbf{R}^n$ and let $F: U \to \mathbf{R}^m$ be a differentiable function. For each $x_0 \in U$ we define the map $\mathrm{T}F_{x_0}: \mathrm{T}_{x_0}\mathbf{R}^n \to \mathrm{T}_{F(x_0)}\mathbf{R}^m$ by $(x_0, x) \mapsto (F(x_0), dF_{x_0}(x))$. The map $\mathrm{T}F: \mathrm{T}U \to \mathrm{T}\mathbf{R}^m$ is now defined by $\mathrm{T}F|\mathrm{T}_{x_0}\mathbb{R}^n := \mathrm{T}F_{x_0}$. $\mathrm{T}F$ is called the *differential* of F.

A word about notation: If we identify $\mathrm{T}x_0\mathbf{R}^n$ with $\mathbf{R}^n$ in the canonical way, and likewise $\mathrm{T}_{F(x_0)}\mathbf{R}^m$ with $\mathbf{R}^m$, then instead of $\mathrm{T}F_{x_0}: \mathrm{T}_{x_0}\mathbf{R}^n \to \mathrm{T}_{F(x_0)}\mathbb{R}^m$ we write $dF_{x_0}: \mathbf{R}^n \to \mathbf{R}^m$.

0.5 Local Behavior of Differentiable Functions (Injective and Surjective Functions)

We shall need to use the following basic theorem:

0.5.1 Theorem (Inverse function theorem). *Let U be an open neighborhood of $0 \in \mathbf{R}^n$. Suppose $F: U \to \mathbf{R}^n$ is a differentiable function with $F(0) = 0 \in \mathbf{R}^n$. If $dF_0: \mathbf{R}^n \to \mathbf{R}^n$ is bijective, then there is an open neighborhood $U' \subset U$ of 0 such that $F|_{U'}: U' \to FU'$ is a diffeomorphism.*

Such a function F is said to be a *local diffeomorphism* (or, more precisely, a local diffeomorphism at 0).

In order to state and prove an important consequence of the inverse function theorem, it is necessary to recall some facts about linear maps. A linear map $L: \mathbf{R}^n \to \mathbf{R}^m$ is injective, or 1-1, if and only if $\ker L := \{x \in \mathbf{R}^n \mid Lx = 0\} = \{0\}$. This is equivalent, in turn, to the requirement that $\mathbf{R}^m$ has a direct sum decomposition $\mathbf{R}^m = \mathbf{R}'^n \oplus \mathbf{R}''^{m-n}$ (into subspaces of dimension n and $m - n$, respectively) such that $L: \mathbf{R}^n \to \mathbf{R}'^n$ is a bijection.

Similarly, a linear map $L: \mathbf{R}^n \to \mathbf{R}^m$ is surjective, or onto, if and only if $n - m = \dim \ker L$. This condition is equivalent to the existence of a direct sum decomposition $\mathbf{R}^n = \mathbf{R}'^m \oplus \mathbf{R}''^{n-m}$ into subspaces of dimension m and $n - m$, respectively, such that $\mathbf{R}''^{n-m} = \ker L$ and $L|_{\mathbf{R}'^m}: \mathbf{R}'^m \to \mathbf{R}^m$ is a bijection.

The next theorem shows that, locally, differentiable functions behave in a manner analogous to linear maps, at least with respect to the injectivity and surjectivity properties described above.

0.5.2 Theorem (Local linearization of differentiable mappings). *Let U be an open neighborhood of $0 \in \mathbf{R}^n$. Suppose $F: U \to \mathbf{R}^m$ is a differentiable function with $F(0) = 0$.*

i) *If $\mathrm{T}F_0: \mathrm{T}_0\mathbf{R}^n \to \mathrm{T}_0\mathbf{R}^m$ is injective, then there exists a diffeomorphism g of a neighborhood W of $0 \in \mathbf{R}^m$ onto a neighborhood $g(W)$ of $0 \in \mathbf{R}^m$ such that $g \circ F$ is an injective linear map from some neighborhood of $0 \in \mathbf{R}^n$ into $\mathbf{R}^m$. In fact, $g \circ F(x_1, \ldots, x_n) = (x_1, \ldots, x_n, 0, \ldots, 0)$.*

ii) *If $\mathrm{T}F_0: \mathrm{T}_0\mathbf{R}^n \to \mathrm{T}_0\mathbf{R}^m$ is surjective, there exists a diffeomorphism h of a neighborhood V of $0 \in \mathbf{R}^n$ onto a neighborhood $h(V)$ of $0 \in \mathbf{R}^n$ such that $F \circ h$ is a surjective linear map from some neighborhood of $0 \in \mathbf{R}^n$ onto a neighborhood of $0 \in \mathbf{R}^m$. In fact, $F \circ h(x_1, \ldots, x_m, \ldots, x_n) = (x_1, \ldots, x_m)$.*

Remark. The converse of each of the above statements is clearly true.

Proof. i) Suppose $dF_0: \mathbf{R}^n \to \mathbf{R}^m$ is injective. Write $\mathbf{R}^m = \mathbf{R}'^n \oplus \mathbf{R}''^{m-n}$ with $dF_0(\mathbf{R}^n) = \mathbf{R}'^n$. Define $\tilde{g}: \mathbf{R}^m = \mathbf{R}'^n \oplus \mathbf{R}''^{m-n} \to \mathbf{R}^m = \mathbf{R}'^n \oplus \mathbf{R}''^{m-n}$ in a neighborhood of 0 by $v = (v', v'') \mapsto F(v') + (0, v'')$. Here the $\mathbf{R}'^n$ on the left-hand side is identified with $\mathbf{R}^n$. Clearly, $d\tilde{g}_0 = dF_0 + \mathrm{id} \mid \mathbf{R}''^{m-n}$.

Therefore $d\tilde{g}_0$ is bijective and we may use the inverse function theorem (0.5.1) to assert the existence of a local differentiable inverse $g = \tilde{g}^{-1}$.

Since $g \circ \tilde{g} = \text{id}$, $g \circ \tilde{g} \mid \mathbf{R}'^n = \text{id} \mid \mathbf{R}'^n$ locally, and thus $g \circ F(v') = (v', 0)$. This proves $g \circ F$ is a linear injective function from a neighborhood of 0 in $\mathbf{R}^n$ into $\mathbf{R}'^n \subset \mathbf{R}'^n \oplus \mathbf{R}''^{m-n} = \mathbf{R}^m$.

ii) Suppose $dF_0 : \mathbf{R}^n \to \mathbf{R}^m$ is surjective. Decomposing $\mathbf{R}^n = \mathbf{R}'^m \oplus \mathbf{R}''^{n-m}$ so that $dF_0 \mid \mathbf{R}'^m : \mathbf{R}'^m \to \mathbf{R}^m$ is a bijection, define $\tilde{h}: \mathbf{R}^n = \mathbf{R}'^m \oplus \mathbf{R}''^{n-m} \to \mathbf{R}^n = \mathbf{R}'^m \oplus \mathbf{R}''^{n-m}$ in a neighborhood of zero by $v = (v', v'') \mapsto (Fv, v'')$. Here we have identified $\mathbf{R}'^m$ on the right-hand side with $\mathbf{R}^m$.

Since $d\tilde{h}_0 = dF_0 \mid \mathbf{R}'^m + \text{id} \mid \mathbf{R}''^{n-m}$ is bijective, $\tilde{h}$ has a local inverse $h = \tilde{h}^{-1}$. Since $h \circ \tilde{h} = \text{id}$ locally, $h(F(v', v''), v'') = (v', v'')$ and therefore $F \circ h(F(v', v''), v'') = F(v', v'')$. This means that $F \circ h$ is given locally by the projection $\mathbf{R}^n = \mathbf{R}'^m \oplus \mathbf{R}''^{n-m} \to \mathbf{R}'^m$ onto the first m coordinates, which, of course, is linear and surjective. □

0.6 Exercise

Prove that any distance-preserving mapping $B: \mathbf{R}^n \to \mathbf{R}^n$ may be written in the form

$$Bx = Rx + x_0,$$

an orthogonal motion followed by a translation.

1 Curves

1.1 Definitions

1.1.1 Definitions. Let $I \subseteq \mathbf{R}$ be an interval. For our purposes, a (parametrized) *curve in* $\mathbf{R}^n$ is a C^∞ mapping $c: I \to \mathbf{R}^n$. c will be said to be *regular* if for all $t \in I$, $\dot{c}(t) \neq 0$.

Remarks. 1. If I is not an open interval, we need to make explicit what it means for c to be C^∞. There exists an open interval I^* containing I and a C^∞ mapping $c^*: I^* \to \mathbf{R}^n$ such that $c = c^*|I$.
2. The variable $t \in I$ is called the *parameter* of the curve.
3. The tangent space $\mathbf{R}_{t_0} = \mathrm{T}_{t_0}\mathbf{R}$ of $\mathbf{R}$ at $t_0 \in I$ has a distinguished basis $1 = (t_0, 1)$. As an alternate notation we will sometimes write d/dt for $(t_0, 1) = 1$.
4. If $c: I \to \mathbf{R}^n$ is a curve, the vector $dc_{t_0}(1) \in \mathrm{T}_{c(t_0)}\mathbf{R}^n$ is well defined. Since $|c(t) - c(t_0) - dc_{t_0}(1)(t - t_0)| = o(t - t_0)$, it follows immediately that $dc_{t_0}(1) = \lim_{t \to t_0}[c(t) - c(t_0)]/(t - t_0) = \dot{c}(t_0)$, the derivative of the $\mathbf{R}^n$-valued function $c(t)$ at $t_0 \in I$.

1.1.2 Definitions. i) A *vector field along* $c: I \to \mathbf{R}^n$ is a differentiable mapping $X: I \to \mathbf{R}^n$. The vector $X(t)$, that is the value of X at a given $t \in I$, will be thought of as lying in the copy of $\mathbf{R}^n$ identified with $\mathrm{T}_{c(t)}\mathbf{R}^n$ (see Figure 1.1).
ii) The *tangent vector field of* $c: I \to \mathbf{R}^n$ is the vector field along $c: I \to \mathbf{R}^n$ given by $t \mapsto \dot{c}(t)$.

1.1.3 Definition. Let $c: I \to \mathbf{R}^n$, $\tilde{c}: \tilde{I} \to \mathbf{R}^n$ be two curves. A diffeomorphism $\phi: \tilde{I} \to I$ such that $\tilde{c} = c \circ \phi$ is called a *parameter transformation* or a *change of variables* relating c to $\tilde{c}$. The map ϕ is called *orientation preserving* if $\phi' > 0$.

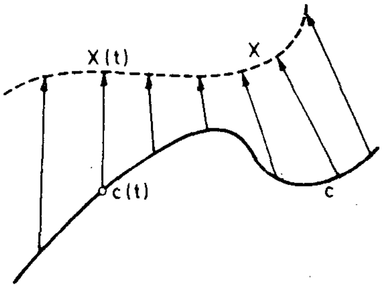

Figure 1.1

Remark. Relationship by a parameter transformation is clearly an equivalence relation on the set of all curves in $\mathbb{R}^n$. An equivalence class of curves is called an *unparameterized curve*.

1.1.4 Definitions. i) The curve $c(t)$, $t \in I$, is said to be *parameterized by arc length* if $|\dot{c}(t)| = 1$. We will sometimes refer to such a curve as a *unit-speed* curve.

ii) *The length of* c is given by the integral $L(c) := \int_I |\dot{c}(t)|\, dt$.

iii) The integral $E(c) := \frac{1}{2} \int_I \dot{c}(t)^2\, dt$ is called the *energy integral of* c or, simply, *the energy of* c.

1.1.5 Proposition. *Every regular curve* $c\colon I \to \mathbb{R}^n$ *can be parameterized by arc length. In other words, given a regular curve* $c\colon I \to \mathbb{R}^n$ *there is a change of variables* $\phi\colon J \to I$ *such that* $|(c \circ \phi)'(s)| = 1$.

Proof. The desired equation for ϕ is $|dc/ds| = |dc/dt| \cdot |d\phi/ds| = 1$. Define $s(t) = \int_{t_0}^{t} |\dot{c}(t')|\, dt'$, $t_0 \in I$, and let $s(t) = \phi^{-1}(t)$. Since c is regular, ϕ exists

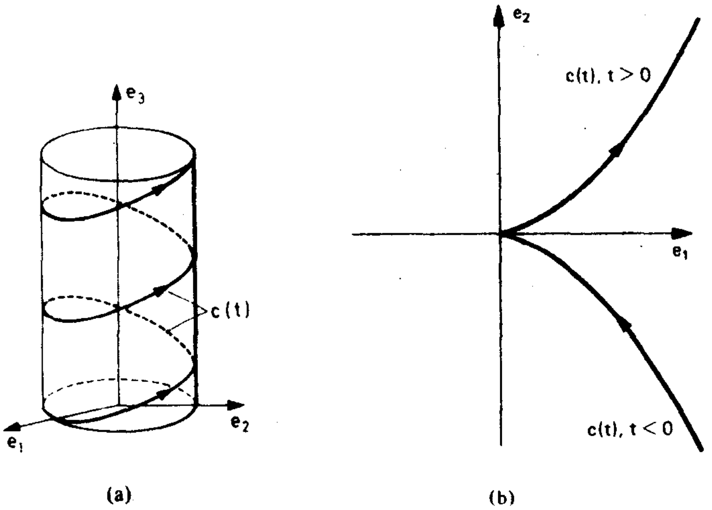

Figure 1.2 (a) Helix; (b) cusp

and satisfies the desired equation. Clearly, $c \circ \phi$ is parameterized by arc length. □

Examples

1. *Straight line.* For $v, v_0 \in \mathbf{R}^n$ let $c(t) = tv + v_0$, $t \in \mathbf{R}$. The curve $c(t)$ is regular if and only if $v \neq 0$ and, in this case, is a straight line.
2. *Circle and helix.* $c(t) = (a \cos t, a \sin t, bt)$, $a, b, t \in \mathbf{R}$, $a^2 + b^2 \neq 0$. When $b = 0$, $c(t)$ is a plane circle of radius a. When $a = 0$, $c(t)$ is a straight line. In general, $c(t)$ is a helix. In all cases, $c(t)$ is a regular curve.
3. *Parameterization of a cusp.* The curve $c(t) = (t^2, t^3)$, $t \in \mathbf{R}$, is regular when $t \neq 0$. The image of $c(t)$ is a cusp.
4. *Another parameterization of a straight line.* The curve $c(t) = (t^3, t^3)$, $t \in \mathbf{R}$, is regular when $t \neq 0$. The image of $c(t)$ is a straight line.

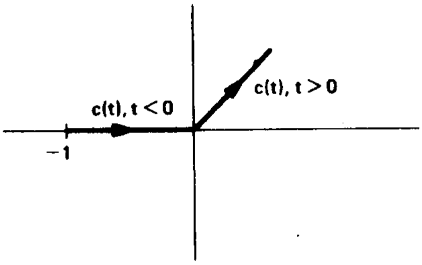

Figure 1.3 Image of c

1.2 The Frenet Frame

1.2.1 Definition. Let $c: I \to \mathbf{R}^n$ be a curve. i) A *moving n-frame* along c is a collection of n differentiable mappings

$$e_i: I \to \mathbf{R}^n, \qquad 1 \leq i \leq n,$$

such that for all $t \in I$, $e_i(t) \cdot e_j(t) = \delta_{ij}$, where $\delta_{ij} = \{{}^{1:\, i=j}_{0:\, i\neq j}\}$. Each $e_i(t)$ is a vector field along c, and $e_i(t)$ is considered as a vector in $T_{c(t)}\mathbf{R}^n$.

ii) A moving n-frame is called a *Frenet-n-frame*, or simply *Frenet frame*, if for all k, $1 \leq k \leq n$, the kth derivative $c^{(k)}(t)$ of $c(t)$ lies in the span of the vectors $e_1(t), \ldots, e_k(t)$.

Remark. Not every curve possesses a Frenet-n-frame. Consider

$$c: \mathbf{R} \to \mathbf{R}^2, \qquad c(t) = \begin{Bmatrix} (-e^{-1/t^2}, 0), & \text{if } t < 0 \\ (e^{-1/t^2}, e^{-1/t^2}), & \text{if } t > 0 \\ (0, 0), & \text{if } t = 0 \end{Bmatrix}.$$

Because the image of c has a crease at (0, 0) it is impossible to find a differentiable unit vector field $e_1(t)$ along c such that $\dot{c}(t) = |\dot{c}(t)| e_1(t)$.

1.2.2 Proposition (The existence and uniqueness of a distinguished Frenet-frame). *Let $c: I \in \mathbb{R}^n$ be a curve such that for all $t \in I$, the vectors $\dot{c}(t), c^{(2)}(t), \ldots, c^{(n-1)}(t)$ are linearly independent. Then there exists a unique Frenet-frame with the following properties*:

i) *For $1 \le k \le n-1$, $\dot{c}(t), \ldots, c^{(k)}(t)$ and $e_1(t), \ldots, e_k(t)$ have the same orientation.*

ii) *$e_1(t), \ldots, e_n(t)$ has the positive orientation.*

This frame is called the *distinguished Frenet-frame*.

Remark. Recall that two bases for a real vector space have the same orientation provided the linear transformation taking one basis into the other has positive determinant. A basis for $\mathbb{R}^n$ is *positively oriented* if it has the same orientation as the canonical basis of $\mathbb{R}^n$.

Proof. We will use the Gram-Schmidt orthogonalization process. The assumption that $\dot{c}(t), \ddot{c}(t), \ldots$ are linearly independent implies that $\dot{c}(t) \neq 0$ and so we may set $e_1(t) = \dot{c}(t)/|\dot{c}(t)|$. Suppose $e_1(t), \ldots, e_{j-1}(t)$, $j < n$, are defined. Let $\tilde{e}_j(t)$ be defined by

$$\tilde{e}_j(t) := -\sum_{k=1}^{j-1} (c^{(j)}(t) \cdot e_k(t)) e_k(t) + c^{(j)}(t)$$

and let $e_j(t) := \tilde{e}_j(t)/|\tilde{e}_j(t)|$.

Clearly, the $e_j(t)$, $j < n$, are well defined and satisfy the first assertion of the theorem. Furthermore, we may define $e_n(t)$ so that $e_1(t), \ldots, e_n(t)$ has positive orientation. The differentiability of $e_j(t)$, $j < n$, is clear from its definition. To see that $e_n(t)$ is differentiable, observe that each of the components $e_n^i(t)$, $1 \le i \le n$, of $e_n(t)$ may be expressed as the determinant of a minor of rank $(n-1)$ in the $n \times (n-1)$-matrix $(e_j^i(t))$, $1 \le i \le n$, $1 \le j \le n-1$. □

1.3 The Frenet Equations

1.3.1 Proposition. *Let $c(t)$, $t \in I$, be a curve in $\mathbb{R}^n$ together with a moving frame $(e_i(t))$, $1 \le i \le n$, $t \in I$. Then the following equations for the derivatives hold*:

$$\dot{c}(t) = \sum_i \alpha_i(t) e_i(t),$$

$$\dot{e}_i(t) = \sum_j \omega_{ij}(t) e_j(t),$$

where

$$(*) \qquad \omega_{ij}(t) := \dot{e}_i(t) \cdot e_j(t) = -\omega_{ji}(t).$$

If $(e_i(t))$ is the distinguished Frenet-frame defined in (1.2.2),

$$(**) \qquad \alpha_1(t) = |\dot{c}(t)|, \qquad \alpha_i(t) = 0 \quad \text{for } i > 1,$$

and $$\omega_{ij}(t) = 0 \quad \text{for } j > i+1.$$