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With 56 Illustrations

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PREFACE

One of the most useful and best known tools employed in the study of partial differential equations is the maximum principle. This principle is a generalization of the elementary fact of calculus that any function $f(x)$ which satisfies the inequality $f'' > 0$ on an interval $[a, b]$ achieves its maximum value at one of the endpoints of the interval. We say that solutions of the inequality $f'' > 0$ satisfy a *maximum principle.* More generally, functions which satisfy a differential inequality in a domain D and, because of it, achieve their maxima on the boundary of D are said to possess a maximum principle.

The study of partial differential equations frequently begins with a classification of equations into various types. The equations most frequently studied are those of elliptic, parabolic, and hyperbolic types. Because equations of these three types arise naturally in many physical problems, mathematicians interested in partial differential equations have tended to concentrate their efforts on those developments which are of both mathematical and physical interest. A reader who learns differential equations by studying physically oriented problems not only parallels the historical development of the subject, but also acquires a clear understanding of the reasons some equations are studied in great detail while others are virtually ignored. Since many problems associated with equations of elliptic, parabolic, and hyperbolic types exhibit maximum principles, we feel that a study of the methods and techniques connected with these principles forms an excellent introduction or supplement to the study of partial differential equations.

There is usually a natural physical interpretation of the maximum principle in those problems in differential equations that arise in physics. In such situations the maximum principle helps us apply physical intuition to mathematical models. Consequently, anyone learning about the maximum principle becomes acquainted with the classically important partial differential equations and, at the same time, discovers the reasons for their importance.

The proofs required to establish the maximum principle are extremely elementary. By concentrating on those applications which can be derived from the maximum principle by elementary methods, we have been able to write this book at a level suitable for the undergraduate science student. Anyone who has completed a course in advanced calculus is qualified to read the entire book. In fact, any student who, in addition to elementary calculus, knows line integration, Green's theorem, and some simple facts on continuity and differentiation should find almost all of the book within his grasp.

The maximum principle enables us to obtain information about solutions of differential equations without any explicit knowledge of the solutions themselves. In particular, the maximum principle is a useful tool in the approximation of solutions, a subject of great interest to many scientists. This book should prove useful, not only to professional mathematicians and students primarily interested in mathematics, but also to those physicists, chemists, engineers, and economists interested in the numerical approximation of solutions of ordinary and partial differential equations and in the determination of bounds for the errors in such approximations.

The maximum principles for partial differential equations can be specialized to functions of one variable, and we have devoted the first chapter to a treatment of this one-dimensional case. The statement of the results and the proofs of the theorems are so simple that the reader should find this introduction to the subject strikingly easy. Of course, the one-dimensional maximum principle is related to second-order ordinary differential equations rather than to partial differential equations. In Chapter 1 we show that portions of the classical Sturm-Liouville theory are a direct consequence of the maximum principle. This chapter is included primarily because it provides an attractive and simple introduction to the various forms of the maximum principle which occur later. It also provides new ways of looking at some topics in the theory of ordinary differential equations.

In Chapter 2 we establish the maximum principle for elliptic operators, state several generalizations, and give a number of applications. Although the maximum principle for Laplace's and some other equations has been known for about a hundred years, it was relatively recently that Hopf established strong maximum principles for general second-order elliptic operators. Many of the important applications which we present make use of these results.

The maximum principle for parabolic operators takes a form quite different from that for elliptic operators. In Chapter 3 we present Nirenberg's strong maximum principle for parabolic operators. We then show, as in the elliptic case, that the principle may be used to yield results on approximation and uniqueness. We conclude the chapter with a section on the maximum principle for a special class of parabolic systems.

The fourth and last chapter treats maximum principles for hyperbolic operators. The forms that these principles take reflect the structure of properly posed problems for hyperbolic equations. Both the statements of the theorems and the methods of proof for hyperbolic operators are quite different from those for elliptic and parabolic operators. In particular, the role of characteristic curves and surfaces becomes evident in the hyperbolic case.

The maximum principle occurs in so many places and in such varied forms that we have found it impossible to discuss some topics which we had originally hoped to treat. For example, the maximum principle for finite difference operators is omitted entirely. We do not mention the maximum principle for the modulus of an analytic function, a subject replete with important and interesting applications. Certain elliptic equations of order higher than the second are known to exhibit a maximum principle. (See, for example, Miranda [1] and Agmon [1].) We decided not to include this topic because advanced techniques of partial differential equations are needed.

Most of the notations and symbols we employ are fairly standard. A **domain** D in Euclidean space is an open connected set. The boundary of D is usually designated ∂D. The symbols $\cup$ and $\cap$ are used for the union and intersection of sets. Boldface letters denote vectors, and the customary notations u_{x_i} and $\partial u/\partial x_i$ are employed for partial derivatives.

We frequently use the letter L followed by brackets to denote a **linear operator** acting on functions. That is, L assigns to each function u of a certain class, a function $L[u]$ of another class. We say that L is **linear** if, whenever $L[u_1]$ and $L[u_2]$ are defined, the quantities $L[\alpha u_1 + \beta u_2]$ and $\alpha L[u_1] + \beta L[u_2]$ are also defined for all constants α and β, and the equation $L[\alpha u_1 + \beta u_2] = \alpha L[u_1] + \beta L[u_2]$ holds.

For those readers who may wish to explore the subject further, we have included at the end of each chapter a bibliographical discussion which contains historical references and a guide to other presentations and further results and applications relevant to the chapter. Since we have a continuing interest in the subject, we would enjoy hearing about results—new or old—which are related to the subject of this book.

We wish to thank the Air Force Office of Scientific Research and the National Science Foundation for their support of investigations leading to a number of results published here for the first time.

M. H. P.
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CONTENTS

CHAPTER 1

THE ONE-DIMENSIONAL MAXIMUM PRINCIPLE

SECTION 1. THE MAXIMUM PRINCIPLE

A function $u(x)$ that is continuous on the closed interval* $[a, b]$ takes on its maximum at a point on this interval. If $u(x)$ has a continuous second derivative, and if u has a relative maximum at some point c between a and b, then we know from elementary calculus that

$$u'(c) = 0 \text{ and } u''(c) \leq 0. \tag{1}$$

Suppose that in an open interval (a, b), u is known to satisfy a differential inequality of the form

$$L[u] \equiv u'' + g(x)u' > 0, \tag{2}$$

where $g(x)$ is any bounded function. Then it is clear that relations (1) cannot be satisfied at any point c in (a, b). Consequently, whenever (2) holds, the maximum of u in the interval cannot be attained anywhere except at the endpoints a or b. We have here the simplest case of a *maximum principle.*

An essential feature of the above argument is the requirement that the inequality (2) be strict; that is, we assume that $u'' + g(x)u'$ is never zero. In the study of differential equations and in many applications, such a requirement is overly restrictive, and it is important that we remove it if possible. We note, however, that for the nonstrict inequality

$$u'' + g(x)u' \geq 0,$$

the solution $u =$ constant is admitted. For such a constant solution the maximum is attained at every point. We shall prove that this exception is the only one possible.

*The symbol $[a, b]$ denotes the closed interval $a \leq x \leq b$; the symbol (a, b) denotes the open interval $a < x < b$.

THEOREM 1. (One-dimensional maximum principle). Suppose $u = u(x)$ satisfies the differential inequality

$$L[u] \equiv u'' + g(x)u' \geq 0 \text{ for } a < x < b, \tag{3}$$

with $g(x)$ a bounded function. If $u(x) \leq M$ in (a, b) and if the maximum M of u is attained at an interior point c of (a, b), then $u \equiv M$.

Proof. We suppose that $u(c) = M$ and that there is a point d in (a, b) such that $u(d) < M$. We shall show this leads to a contradiction. For convenience let $d > c$. We define the function

$$z(x) = e^{\alpha(x-c)} - 1$$

with α a positive constant to be determined. Note that $z(x) < 0$ for $a < x < c$, that $z(x) > 0$ for $c < x < b$, and that $z(c) = 0$. (See Fig. 1.)

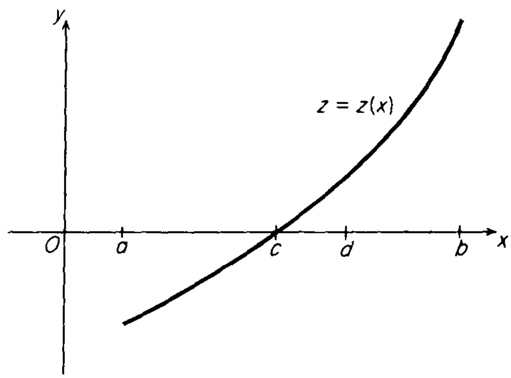

FIGURE 1

A simple computation yields

$$L[z] \equiv z'' + g(x)z' = \alpha[\alpha + g(x)]e^{\alpha(x-c)}.$$

We choose α so large that $L[z] > 0$ for $a < x < d$. That is, we select α so that it satisfies the inequality

$$\alpha > -g(x);$$

we can always do this since $g(x)$ is bounded. We now define

$$w(x) = u(x) + \epsilon z(x),$$

where ϵ is a positive constant chosen so that it satisfies the inequality

$$\epsilon < \frac{M - u(d)}{z(d)}.$$

The assumption $u(d) < M$ and the fact that $z(d) > 0$ make it possible to find such an ϵ. Then, since z is negative for $a < x < c$, we have

$$w(x) < M \text{ for } a < x < c;$$

by the definition of ϵ,

$$w(d) = u(d) + \epsilon z(d),$$
$$< u(d) + M - u(d),$$

so that

$$w(d) < M.$$

At the point c,

$$w(c) = u(c) + \epsilon z(c) = M.$$

Hence w has a maximum greater than or equal to M which is attained at an interior point of the interval (a, d). But

$$L[w] = L[u] + \epsilon L[z] > 0,$$

so that by our previous result concerning the strict inequality (2), w cannot attain its maximum in (a, d). We thereby reach a contradiction.

If $d < c$, we use the auxiliary function

$$z = e^{-\alpha(x-c)} - 1$$

with $\alpha > g(x)$ to reach the same conclusion.

The key to the above proof is the construction of the function $z(x)$ with the properties: (i) $L[z] > 0$; (ii) $z(x) < 0$ for $x < c$; (iii) $z(x) > 0$ for $x > c$; (iv) $z(c) = 0$. [If d is less than c, inequalities (ii) and (iii) are reversed.] The function z is by no means unique. For example, the function

$$z(x) = (x - a)^\alpha - (c - a)^\alpha$$

with α sufficiently large has the same four properties.

By applying Theorem 1 to $(-u)$ we have the *minimum principle* which asserts that a nonconstant function satisfying the differential inequality $L[u] \leq 0$ cannot attain its minimum at an interior point.

The boundedness condition for g in the statement of Theorem 1 may be relaxed. If g is bounded on every interval $[a', b']$ completely interior to (a, b), then the conclusion of Theorem 1 still holds. We simply apply the argument on any subinterval $[a', b']$ containing the points c and d in its interior. Note that it is possible for g to be bounded on every closed subinterval of (a, b) and yet unbounded as x tends to a or b. For example, $g(x) = 1/(1 - x^2)$ is bounded on every closed subinterval of $(-1, 1)$. This may seem to be a minor point, but it turns out that many of the differential equations of mathematical physics have coefficients g which become unbounded at the endpoints of the interval of definition.

The method employed to prove Theorem 1 enables us to obtain additional information about functions which satisfy an inequality such as (3). We might imagine that a solution u of (3) could have the appearance of the function shown in Fig. 2a. That is, the maximum of u on $[a, b]$ is

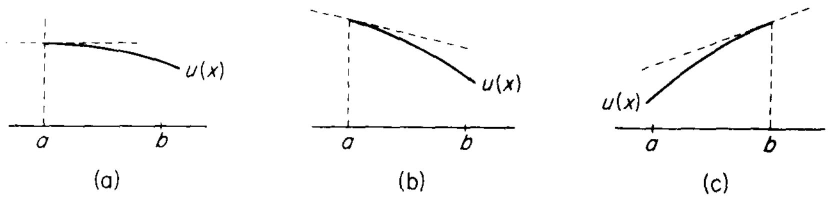

FIGURE 2

attained at a and $u'(a) = 0$. In fact, this situation never can occur. If the maximum occurs at the left endpoint, the slope at that point must be negative (Fig. 2b); if the maximum occurs at the right endpoint, the slope at that point must be positive (Fig. 2c). The next theorem establishes the precise result.

THEOREM 2. Suppose u is a nonconstant function which satisfies the inequality $u'' + g(x)u' \geq 0$ in (a, b) and has one-sided derivatives at a and b, and suppose g is bounded on every closed subinterval of (a, b). If the maximum of u occurs at $x = a$ and g is bounded below at $x = a$, then $u'(a) < 0$. If the maximum occurs at $x = b$ and g is bounded above at $x = b$, then $u'(b) > 0$.

Proof. Suppose that $u(a) = M$, that $u(x) \leq M$ for $a \leq x \leq b$, and that for some point d in (a, b) we have $u(d) < M$. Once again we define an auxiliary function

$$z(x) = e^{\alpha(x-a)} - 1 \text{ with } \alpha > 0.$$

We select $\alpha > -g(x)$ for $a \leq x \leq d$ so that $L[z] > 0$. Next, we form the function

$$w(x) = u(x) + \epsilon z(x)$$

with ϵ chosen so that

$$0 < \epsilon < \frac{M - u(d)}{z(d)}.$$

Because $L[w] > 0$, the maximum of w in the interval $[a, d]$ must occur at one of the ends. We have

$$w(a) = M > w(d),$$

so that the maximum occurs at a. Therefore, the one-sided derivative of w at a cannot be positive:

$$w'(a) = u'(a) + \epsilon z'(a) \leq 0.$$

However,

$$z'(a) = \alpha > 0,$$

and therefore

$$u'(a) < 0,$$

which is the desired result.

If the maximum occurs at $x = b$, the argument is similar.

Remarks. (i) If a function u which satisfies (3) has a relative maximum at an interior point c, there is an interval (a_1,b_1) containing c in its interior on which $u(x) \leq u(c)$. Then Theorem 1 shows that $u(x) = u(c)$ on this interval. By applying Theorem 2 to all intervals having c as an endpoint, we see that the value $u(c)$ at the relative maximum is actually the minimum value of u on the interval (a,b).

(ii) If a function u which satisfies (3) has relative minima at two points c_1 and c_2 of the interval (a,b), it must have a relative maximum at some point between c_1 and c_2. It then follows from Remark (i) that $u(c\ ) = u(c_2)$ and that $u(x)$ is constant on the interval (c_1,c_2).

(iii) A function satisfying (3) can have no horizontal point of inflection. (u has a **horizontal point of inflection** at $x = c$ if $u'(c) = 0$ while u is strictly increasing or strictly decreasing in some interval containing c.) If there were such a point, we could select a subinterval with this point as an endpoint (either a right or left endpoint, whichever is appropriate) on which u attains its maximum at c. Then Theorem 2 would be contradicted.

(iv) A result analogous to Theorem 2 holds for solutions of $L[u] \leq 0$, yielding an associated minimum principle. We obtain this principle by applying Theorem 2 to the function $(-u)$.

(v) It is possible to prove Theorem 2 before Theorem 1. Then the following argument yields Theorem 1 immediately. If u has a maximum at an interior point c, then $u'(c) = 0$. Applying Theorem 2 to the intervals (a, c) and (c, b), we conclude that u is constant.

(vi) The boundedness of g is required for the conclusion of Theorems 1 and 2. The equation

$$u'' + g(x)u' = 0$$

with

$$g(x) = \begin{cases} -3/x & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

has the solution

$$u = 1 - x^4.$$

Theorem 1 is clearly violated on the interval $-1 \leq x \leq 1$, as u has a maximum at $x = 0$. Theorem 2 is violated on $[0, 1]$ as $u'(0) = 0$. The results of Theorems 1 and 2 are not applicable because g is not bounded from below in $(0, 1)$.

We now take up the more general differential inequality

$$(L + h)[u] \equiv u'' + g(x)u' + h(x)u \geq 0. \tag{4}$$

The simplest examples show that at best we can only hope for a modified form of the maximum principle; the equation

$$u'' + u = 0$$

has the solution $u = \sin x$ which attains its maximum at $x = \pi/2$. Even the condition $h(x) \leq 0$ is not sufficient to yield an unrestricted maximum principle. We observe that the equation

$$u'' - u = 0$$

has the solution

$$u = -e^x - e^{-x},$$

which attains its maximum value (-2) at $x = 0$. We shall show that a nonconstant solution of (4) with $h \leq 0$ cannot attain a nonnegative maximum at an interior point.

It is easy to see that if the *strict* inequality

$$(L + h)[u] > 0, \text{ with } h \leq 0,$$

holds in an open interval (a, b), then u cannot have a nonnegative maximum in the interior of (a, b). In fact, at any such maximum, we have $u' = 0, u'' \leq 0, hu \leq 0$, contradicting the above strict inequality. This fact enables us to extend Theorems 1 and 2 without altering the argument in any way other than by choosing α so large that $(L + h)[z] > 0$.

The constant α in the function $e^{\alpha(x-c)} - 1$ (or the function $e^{-\alpha(x-c)} - 1$, if d is to the left of c) must only satisfy

$$\alpha^2 + \alpha g(x) + h(x)[1 - e^{-\alpha(x-c)}] > 0$$

$$(\text{or } \alpha^2 - \alpha g(x) + h(x)[1 - e^{\alpha(x-c)}] > 0).$$

Since $h(x) \leq 0$, it is sufficient in either case to select α so that

$$\alpha^2 - \alpha|g(x)| + h(x) > 0.$$

This can certainly be done if $g(x)$ and $h(x)$ are bounded. Again we can show that it suffices for them to be bounded on every closed subinterval of (a, b). In this way we arrive at the next two theorems, which are extensions of Theorems 1 and 2.

THEOREM 3. If $u(x)$ satisfies the differential inequality

$$(L + h)[u] \equiv u'' + g(x)u' + h(x)u \geq 0 \tag{4}$$

in an interval (a, b) with $h(x) \leq 0$, if g and h are bounded on every closed subinterval, and if u assumes a nonnegative maximum value M at an interior point c, then $u(x) \equiv M$.

Note that if h is not identically zero, then the only nonnegative constant M satisfying (4) is $M = 0$.

THEOREM 4. Suppose that u is a nonconstant solution of the differential inequality (4) having one-sided derivatives at a and b, that $h(x) \leq 0$, and that g and h are bounded on every closed subinterval of (a, b). If u has a nonnegative maximum at a and if the function $g(x) + (x - a)h(x)$ is bounded from below at $x = a$, then $u'(a) < 0$. If u has a nonnegative maximum at b and if $g(x) - (b - x)h(x)$ is bounded from above at $x = b$, then $u'(b) > 0$.

In extending the proof of Theorem 2 to Theorem 4, we need only observe that

$$\begin{aligned}(L + h)[e^{\alpha(x-a)} - 1] &= e^{\alpha(x-a)}[\alpha^2 + \alpha g + h(1 - e^{-\alpha(x-a)})] \\ &\geq e^{\alpha(x-a)}[\alpha^2 + \alpha g + \alpha(x - a)h].\end{aligned}$$

COROLLARY. If u satisfies (4) in (a, b) with $h(x) \leq 0$, if u is continuous on $[a, b]$, and if $u(a) \leq 0$, $u(b) \leq 0$, then $u(x) < 0$ in (a, b) unless $u \equiv 0$.

EXERCISES

1. Prove Theorem 1 by employing the function $z(x) = (x - a)^\alpha - (c - a)^\alpha$ instead of $z(x) = e^{\alpha(x-c)} - 1$.
2. The function $u = \cos x$ satisfies $u'' + g(x)u' = 0$ with $g(x) = -\cot x$, and yet u has a maximum at $x = 0$. Explain. Find a function u which has a horizontal point of inflection at $x = 0$ and which satisfies a differential inequality of the form $u'' + g(x)u' \geq 0$.
3. Show that if $u'' + e^u = -x$ for $0 < x < 1$, then u cannot attain a minimum in $(0, 1)$.
4. Show that a solution of $u'' - 2\cos(u') = 1$ cannot attain a local maximum.
5. Consider the problem

$$u'' + e^x u' = -1 \text{ for } 0 < x < 1,$$
$$u(0) = u(1) = 0.$$

Verify that the solution has no minimum in $(0, 1)$. Also show that $u'(0) > 0$, $u'(1) < 0$.

6. Consider the inequality

$$u'' + (\alpha/x)u' + (\beta/x^2)u \geq 0, \qquad 0 < x < 1,$$

with α and β constant. For what values of α and β are Theorems 3 and 4 applicable? Verify by considering solutions of the form $u = x^n$. What is the result if the interval is $-1 < x < 1$?

SECTION 2. THE GENERALIZED MAXIMUM PRINCIPLE

We investigate the differential inequality

$$(L + h)[u] \equiv u'' + g(x)u' + h(x)u \geq 0, \qquad a < x < b, \tag{1}$$

without the requirement that $h(x)$ be nonpositive. Suppose we can find a function w which has a continuous second derivative on $[a, b]$ and which satisfies the inequalities

$$w > 0 \text{ on } [a, b], \tag{2}$$

$$(L + h)[w] \leq 0 \text{ in } (a, b). \tag{3}$$

We define the new dependent variable

$$v = \frac{u}{w}.$$

A simple computation yields

$$(L + h)[u] = (L + h)[vw] = wv'' + (2w' + gw)v' + (L + h)[w]v \geq 0.$$

Dividing by the positive quantity w, we see that v satisfies the differential inequality

$$v'' + \left(2\frac{w'}{w} + g\right)v' + \frac{1}{w}(L + h)[w]v \geq 0. \tag{4}$$

Inequality (4), when taken in conjunction with (2) and (3), shows that $v = u/w$ satisfies Theorems 3 and 4.

The argument above depends on the existence of a function w which satisfies (2) and (3). We shall now show that if $h(x)$ is bounded, if $g(x)$ is bounded from below, and if the interval $[a, b]$ is sufficiently short, then there is a function w which fulfills inequalities (2) and (3). In fact, such a function is given by

$$w = 1 - \beta(x - a)^2, \tag{5}$$

if the constant β is determined suitably. To see this, we compute

$$(L + h)[w] = -2\beta[1 + (x - a)g(x) + \tfrac{1}{2}(x - a)^2h(x)] + h(x). \tag{6}$$

Since, by assumption, g and h are bounded from below, there are constants G and H such that $g \geq G$ and $h \geq H$. We suppose a and b are so close together that

$$1 + (x - a)G + \tfrac{1}{2}(x - a)^2H > 0 \text{ for } a \leq x \leq b.$$

Since $h(x)$ is also bounded from above, we can select β so that

$$\beta \geq \frac{1}{2}\left[\frac{h(x)}{1 + (x - a)G + \frac{1}{2}(x - a)^2H}\right].$$

Then, because of (6), we have $(L + h)[w] \leq 0$ in (a, b). If the length $(b - a)$ is also so small that

$$\beta(b - a)^2 < 1,$$

then (5) shows that $w > 0$ on $[a, b]$. In this way, the function w with the desired properties may always be constructed.

The preceding discussion leads to the following **generalized maximum principle.**

THEOREM 5. Suppose the operator $L + h$ is given by (1) with $h(x)$ bounded and with $g(x)$ bounded from below. For any sufficiently short interval $[a, b]$, a function w can be found which satisfies (2) and (3). Then if u is any function satisfying (1) in (a, b), the function u/w satisfies the maximum principles as given in Theorems 3 and 4.

Remark. Theorem 5 shows that a function u which satisfies (1) cannot oscillate too rapidly, for if $u > 0$ between two of its zeros $x = a$ and $x = b$, then u/w must have a positive maximum between them. Hence, Theorem 5 is violated unless the distance $b - a$ between these zeros is so large that this theorem doesn't hold. We thus find that u can have at most two zeros (between which u is negative) in any interval (a, b) where Theorem 5 holds.

If u is a solution of the **equation** $u'' + g(x)u' + h(x)u = 0$, we can apply the same reasoning to both u and $-u$ to find that u can have at most one zero in any interval (a, b) where Theorem 5 holds.

Let $r(x)$ be a solution of the differential equation

$$r'' + g(x)r' + h(x)r = 0, \tag{7}$$

with g and h bounded functions. Suppose that r is not identically zero, and that

$$r(a) = 0.$$

In the light of the remark following Theorem 5, we know that r cannot vanish for some distance to the right of a. If r has any zeros to the right of a, we denote the first one by a^* and call it the **conjugate point** of a. Thus r is of one sign in the interval (a, a^*), and for convenience we assume that

$$r(x) > 0 \text{ for } a < x < a^*.$$

If $w > 0$ on $[a, a^*]$, the function r/w vanishes at a and at a^* and is positive in (a, a^*). Hence it has a maximum in (a, a^*). Therefore by Theorem 5, w cannot satisfy (3). On the other hand, if b is any point in (a, a^*), a function w can be found so that r/w satisfies the maximum principle of Theorem 5. To see this, we observe first that $r(x)$ is bounded from below by a positive number on any subinterval $[c, b]$ contained in (a, a^*). Consequently, for sufficiently small $\epsilon > 0$, the function

$$w(x) = r(x) + \epsilon[2 - e^{\alpha(x-a)}]$$