Introduction to Holomorphic Functions of Several Variables

Volume III

Homological Theory

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Preface

These three volumes together comprise a revised version of the book *Analytic Functions of Several Complex Variables* that Hugo Rossi and I wrote some 20 years ago. The revisions are fairly extensive; indeed, they essentially amount to a complete rewriting. The attempt to include some more recent material, to incorporate a number of corrections and improvements, and to expand and simplify the treatment to make for easier reading have led quite naturally to rather drastic changes. I very much regret that this revision could not have been a joint effort as before; it suffers much in balance, scope, and clarity as a result, and I have no one else to blame for the inevitable errors it contains and the gaps it does not.

These volumes are intended as an extensive introduction to the Oka–Cartan theory of holomorphic functions of several variables and holomorphic varieties, just as was the earlier book, and they cover very much the same range of topics. Great advances have been made in function theory in the past 20 years. To survey all that has been done to a comparable extent would require a few more volumes; that is something I hope to return to later. In the meantime, though, a considerable number of books that treat many of these topics very well indeed are listed in the Bibliography, and some more detailed suggestions for further reading are given in the Outline of the present work that follows.

The Outline is intended as a survey and guide to reading these volumes. It is not likely that the reader will want or need to start at the beginning and read consecutively through to the end. The three volumes cover three somewhat different aspects of the subject and can to a certain extent be read independently. Alternatively, the first sections of each volume can be read as a shorter introduction to the subject.

This revision grew out of courses of lectures on various topics in complex analysis that I have given at Princeton over the years since the first book was written. I should like to express here my sincere gratitude to the many students and colleagues who attended these lectures for the many very helpful comments, suggestions, and corrections they have given me. I should like in particular to thank Jay Belanger for his careful reading of much of the final manuscript. Finally,

I should like to thank all those who have typed various parts, and revisions of parts, and parts of revisions, and revisions of revisions of the manuscript, and especially Maureen Kirkham, who has typed the greater part of the whole project.

Robert C. Gunning

Series Outline

The prerequisites for reading these books are basically a solid undergraduate training in analysis (especially in the theory of holomorphic functions of a single variable), topology, and algebra. The aim of the books is to introduce the reader to a wide range of topics in the theory of holomorphic functions of several variables, with fairly complete proofs, as preparation either for using this material as part of a general mathematical background or for continuing a more detailed and extensive study of this fascinating and active field of research. The Bibliography at the end lists only other books and general surveys in this and closely related areas; its length is a good indication of just how active a field this really is. An excellent historical discussion of the development of the field can be found in [6].

Volume I. Function Theory

The first volume deals with holomorphic functions defined in open subsets of the space $\mathbb{C}^n$ of n complex variables, focusing on their properties as ordinary complex-valued functions. **A.** There are various equivalent characterizations of holomorphic functions, the basic and most primitive of which is as those functions that can be represented locally as convergent power series. It is clear from this that a holomorphic function of several variables is holomorphic in each variable separately, when all the others are held fixed. So a great many of the familiar properties of holomorphic functions of a single variable, such as the Cauchy integral formula, can be applied quite easily and directly to obtain corresponding properties of holomorphic functions of several variables. **B.** On the other hand, though quite true, it is surprisingly nontrivial to prove that any function that is holomorphic in each variable separately is actually a holomorphic function of several variables. This result really rests on a closer analysis of the precise domains of convergence of power series of several variables, in the course of which some of the peculiar properties of functions of more than a single variable first become manifest. **C.** The most straightforward properties of differentiable functions of several variables, such as the implicit and inverse function theorems, extend very easily to holomorphic functions; complex manifolds and submanifolds can then be introduced in immediate analogy to the corresponding differentiable notions. These three sections cover the rudiments of the theory of holomorphic functions, the ABC's of the subject.

D. There is a natural analogue for holomorphic functions of several variables of Riemann's removable singularities theorem, but much more is true: sufficiently small subsets, such as an isolated point in $\mathbb{C}^n$ when $n > 1$, are automatically removable singularities for holomorphic functions in general, with no hypotheses of boundedness. Several more examples of removable singularities are given; these can be skipped if desired, but they do provide some useful intuitive feel for a property that is peculiarly characteristic of holomorphic functions of several variables. **E.** The differential operator that appears in the analogue for functions of several variables of the Cauchy–Riemann characterization of holomorphic functions can be extended as a linear mapping $\bar{\partial}$ from complex differential forms of bidegree (p, q) to those of bidegree $(p, q + 1)$ and has the property that $\bar{\partial}\bar{\partial} = 0$. Thus if $\phi = \bar{\partial}\psi$, then $\bar{\partial}\phi = 0$, and that raises the question whether, conversely, if $\bar{\partial}\phi = 0$, then there is a differential form ψ such that $\bar{\partial}\psi = \phi$. That is true locally and also for differential forms in such simple domains as polydiscs. What may seem to be a digression here is actually not one; this result has a very nice application to another removable singularities theorem, and this is in turn the model for a deep and powerful general use of the $\bar{\partial}$ operator in complex analysis, to be discussed in section O. **F.** Another application of properties of this differential operator yields an interesting and useful result about the approximation of holomorphic functions in special domains in $\mathbb{C}^n$ by polynomials, in extension of the Runge approximation theorem for functions of a single variable. For several variables, though, there can be such approximation theorems only for somewhat special domains, as indicated by an example.

G. The discussion of removable singularities indicates that there are some pairs of open subsets $D \subseteq E \subseteq \mathbb{C}^n$ such that every function holomorphic in D extends to a holomorphic function in E. Particularly natural and useful are those subsets $D \subseteq \mathbb{C}^n$ for which there is actually no nontrivial such extension—that is, for which necessarily $E = D$. Such sets are called domains of holomorphy and have a number of special properties that will be considered subsequently. Some alternative characterizations and various examples of domains of holomorphy are provided here. **H.** If D is not a domain of holomorphy, it is but natural to ask for the maximal set E to which all holomorphic functions in D extend. A complication is that the extended functions may be multiple-valued; but all of them can be viewed as single-valued holomorphic functions on a complex manifold spread out as a locally unbranched covering space over an open subset of $\mathbb{C}^n$. In the category of such manifolds, called Riemann domains, there is a unique maximal one to which all holomorphic functions in D extend, the envelope of holomorphy of D; it can be described intrinsically in terms of the ring of holomorphic functions on D. **I.** An envelope of holomorphy is itself maximal, the analogue of a domain of holomorphy among Riemann domains, virtually by definition. It is possible to extend many of the alternative characterizations of domains of holomorphy to hold on Riemann domains as well, as is done here with elementary but in part quite nontrivial arguments. A more complete and in many ways preferable treatment of this topic is given in section P, after some more machinery is developed, so this section can be omitted altogether by those who are willing to proceed further; it is really inserted just to conclude the discussion for those wishing to go no further in this direction.

The preceding portion is by itself a reasonable introduction to function theory in several variables. It is quite possible for those so inclined to omit the rest of this volume, and proceed to either Volume II or Volume III. On the other hand, those interested in functions as such may wish to plow on through the rest of Volume I, which concentrates on a particularly useful class of nonholomorphic functions.

J. Plurisubharmonic functions have played a considerable role in function theory of several variables. They are natural and useful extensions to several variables of the possibly familiar subharmonic functions of a single complex variable. A review of the relevant properties of subharmonic functions is included here for those who may not be that well acquainted with them. **K.** The basic properties of plurisubharmonic functions are discussed in some detail. **L.** Various special or generalized classes of plurisubharmonic functions are useful at some points and so are discussed. It is quite possible, and perhaps at first reading advisable, to skip this section until the need arises in references to it later.

M. There is a special class of open subsets of $\mathbf{C}^n$ described in terms of the plurisubharmonic functions in them, the pseudoconvex subsets. These are in some ways analogous to domains of holomorphy. Indeed, it is demonstrated in section P that they are precisely domains of holomorphy, although that is quite a nontrivial result. These sets have a number of special properties and alternative characterizations, all rather simple to demonstrate because of the flexibility and abundance of plurisubharmonic functions. If it is demonstrated that pseudoconvex subsets are really just domains of holomorphy, these special properties and alternative characterizations hold immediately for the latter subsets; that will yield some very useful and powerful results about domains of holomorphy. **N.** Similar arguments hold for Riemann domains, although with some further complications. The discussion here can be omitted altogether by those readers interested in only subsets of $\mathbf{C}^n$ rather than general Riemann domains. **O.** The main technical tool needed to establish the equivalence of pseudoconvexity and holomorphic convexity is the solvability of the $\bar{\partial}$ equation in pseudoconvex domains, which is established here. **P.** The equivalence and some of the immediate consequences are demonstrated quite easily.

Q. The preceding results rest eventually on the close relationship between plurisubharmonic and holomorphic functions; some other facets of this relationship are discussed here. **R.** In conclusion, and as an introduction to another vast area of great interest, the special case of domains with smooth boundaries is briefly discussed.

A very quick and convenient survey of various other and more detailed results about pseudoconvex sets can be found in [19]. Powerful tools in studying domains with smooth boundaries are versions of the Cauchy integral formula that involve integrating over the whole boundary; such formulas and their consequences are nicely treated in [48]. Closely related to this is the finer analysis of the $\bar{\partial}$ operator in smoothly bounded domains, in particular in connection with boundary value problems; this is treated in [17] and [25], where more references to the literature can also be found. There are analogues of other classical boundary value problems in function theory of a single variable treated, for instance, in [50] and [51]. There is an extensive theory of holomorphic mappings and differential-geometric methods, as discussed, for instance, in [35].

Volume II. Local Theory

A holomorphic or meromorphic function of a single variable has the appealingly simple form z^n in suitable local coordinates, for some integer n. The zero set of a holomorphic function of one variable or the singular set of a meromorphic function of one variable is an isolated point. The situation is much more complicated and hence much more interesting for functions of several variables. It leads to an extensive collection of results in a quite different direction from that of Volume I, indeed more akin to algebraic geometry than to classical function theory.

A. A basic tool for the local study of holomorphic functions of several variables is a convenient canonical form for those functions, the Weierstrass polynomials. From that it is easy to derive some general properties of the ring of local holomorphic functions at a point in $\mathbb{C}^n$. **B.** A holomorphic subvariety of an open subset in $\mathbb{C}^n$ is a subset that can be described locally as the set of common zeros of finitely many holomorphic functions. Such subvarieties are described locally in terms of ideals in the ring of local holomorphic functions, with interesting simple relations between the algebraic and geometric structures. It is convenient to consider some natural equivalence classes of holomorphic subvarieties as abstract holomorphic varieties. **C.** The simplest local form of a holomorphic mapping between holomorphic varieties is a finite branched holomorphic covering; any mapping is of this form in the case of a single variable. The general topological and algebraic properties of such mappings are considered first. **D.** It is demonstrated that every holomorphic variety can be represented locally as a finite branched holomorphic covering of $\mathbb{C}^n$, when it is locally irreducible. This provides a very convenient local description of holomorphic varieties. **E.** This description is used to establish some basic local properties of holomorphic varieties, such as the analytic form of Hilbert's zero theorem.

F. There are some very subtle but important properties of holomorphic functions and varieties that are really semilocal rather than local, in the sense that they involve relations between the local rings of holomorphic functions at all points of an open neighborhood of a fixed point. These are perhaps the most complicated results of the local theory, but they play a major role in its development. These results are reformulated as coherence theorems in Volume III; they can really be used without worrying about the proofs, although that approach to mathematics has both dangers and drawbacks and should not be encouraged.

G. Perhaps the basic invariant attached to a holomorphic variety is its dimension. This can be characterized either geometrically or algebraically. There are a number of relations between the dimension of a subvariety and the number of holomorphic functions needed to describe the subvariety. A useful digression here is the description of general divisors and divisors of functions.

The results obtained so far in this volume provide an introduction to the local theory of holomorphic functions and varieties, quite sufficient for many purposes. It is possible for those so interested to skip the remainder of this volume and pass to the global theory in Volume III. On the other hand, though, a great deal more can be said about the local theory.

H. It is natural to consider holomorphic functions on holomorphic varieties, but it is not altogether trivial to show that a limit of such functions is also holomorphic. That closure property and an extension to the closure of modules of various sorts are useful and important, if somewhat technical, results.

I. Another useful local invariant is not the dimension of a variety itself but rather the least value m such that the variety can be realized as a subvariety of $\mathbb{C}^m$; this is known as the tangential or imbedding dimension of the variety. This invariant is strictly larger than the ordinary dimension at singular points and is indeed a measure of the singularity. It can be described in terms of the natural notion of the tangent space to a variety, even at singular points, and is useful in examining differential notions such as the inverse mapping theorem at singular points. **J.** Global sections of the tangent bundle, or holomorphic vector fields, can be introduced on varieties with singularities, and they play a role much like the one they play on manifolds.

K. With the additional machinery that has been developed here, it is possible to complete the discussion of holomorphic extensions of holomorphic functions from section D of Volume I. There are analogous results for the holomorphic extension of holomorphic varieties.

L. Finite holomorphic mappings were discussed in section C. Local properties of general holomorphic mappings are examined in some detail here. The simplest classes of holomorphic mappings are those for which the dimension of the inverse image of a point is independent of the point; they have relatively simple standard representations, extending those examined earlier for finite holomorphic mappings. **M.** For completeness, and in preparation for the further discussion of local properties of holomorphic mappings, a review of some standard properties of complex projective spaces is given. This digression can be skipped by readers already familiar with such spaces. **N.** The discussion of holomorphic mappings continues with an analysis of proper holomorphic mappings, those for which the inverse image of a compact set is compact. The images of these mappings, and of those of a natural generalization of this class of mappings, are always holomorphic varieties themselves, a useful and nontrivial result. Among these mappings is the special class of monoidal or quadratic transforms, which play a major role in the detailed analysis of singularities of holomorphic varieties.

O. Meromorphic functions were earlier considered locally; here they are examined globally, first in open subsets of $\mathbb{C}^n$. **P.** The extension of this discussion to holomorphic functions on varieties is more complicated. **Q.** There is a special class of varieties of great importance here; those for which the bounded meromorphic functions are necessarily holomorphic. They are called normal varieties. R. To any holomorphic variety there is naturally associated a unique one-sheeted branched covering that is itself a normal variety. This normalization is a convenient and useful partial step in the classification of singularities of holomorphic varieties.

Further discussion of general properties of holomorphic varieties with singularities and other references to the literature can be found in [1], [16], [23], [28], [43], and [60]. The topological properties of singularities of one-dimensional varieties are a classical topic in algebraic geometry, while the higher-dimensional study was pioneered in [41]. The case of two-dimensional singularities is the next most extensively studied class, as surveyed, for instance, in [37]. For many purposes it is important to study holomorphic varieties with more general rings of holomorphic functions than those considered here, rings with nilpotent elements, reflecting a description of subvarieties by particular families of defining equations; these topics are surveyed nicely in [23].

Volume III. Homological Theory

Sheaves have proved to be very useful tools in organizing and simplifying various arguments and calculations in function theory, as well as in algebraic geometry, and should be part of the equipment of those working in this area. **A.** The definitions and basic properties of sheaves are developed in a fairly general form, assuming no previous acquaintance; those readers already familiar with sheaves can skip or skim through this section. **B.** Sheaves provide a particularly convenient mechanism for handling the semilocal properties of holomorphic functions and varieties discussed in the preceding volume. The notion of coherence plays a prominent role here.

C. Sheaves provide an equally convenient mechanism for handling some of the basic global problems of complex analysis through cohomology theory. Again no previous acquaintance with this theory is presupposed. The algebraic structure underlying cohomology theory is developed first, including some general properties of cohomology theories and the technique of spectral sequences. **D.** The cohomology groups of a space with coefficients in a sheaf are examined in some detail, together with techniques for calculating these groups by using convenient auxiliary sheaves. **E.** There is another technique for calculating these groups, a useful technique in some analytic applications; it involves sections of the sheaf over the sets forming various open covers of the space. **F.** The behavior of sheaves and sheaf cohomology groups under mappings of the base spaces is examined only as necessary for applications here.

G. The discussion turns next to the use of sheaf cohomology in complex analysis. A basic technical tool is developed first, a result about matrices of holomorphic functions, a variant of the possibly familiar Riemann–Hilbert problem. **H.** This auxiliary result is used to show that an open polydisc has trivial cohomology groups in all positive dimensions, when the coefficient sheaf is any holomorphic sheaf satisfying the basic semilocal coherence condition. **I.** The vanishing of all of these cohomology groups has a wide range of useful and interesting consequences. In general, a holomorphic variety such as a polydisc for which these cohomology groups vanish is called a Stein variety. Some of the fundamental properties of Stein varieties, including approximation results reminiscent of the Runge theorem, are established first. **J.** The special properties of the Frechet algebra of holomorphic functions on a Stein variety are examined. **K.** The global properties of meromorphic functions on a Stein variety, and their relationship to holomorphic functions, are discussed. **L.** Alternative characterizations of Stein varieties are established to show that there actually do exist a wide range of interesting such varieties. For instance, an open subset of $\mathbb{C}^n$ is a Stein variety precisely when it is a domain of holomorphy; any subvariety of a Stein variety is itself a Stein variety, and certain increasing unions of Stein varieties are Stein varieties.

The results established up to this point in Volume III provide a good introduction to the use of homological methods in complex analysis, as well as to the properties of Stein varieties, so some readers may be content to stop here.

There are, however, further interesting properties and alternative characterizations of Stein varieties. **M.** For instance, any holomorphic variety that admits a finite proper holomorphic mapping into a Stein variety is itself a Stein variety. That generalizes the earlier observation that any subvariety of a Stein variety is a Stein variety, and can be used to simplify some of the other characterizations of Stein varieties discussed in the preceding section. **N.** For another instance, a holomorphic variety is Stein if and only if its normalization is Stein; more generally, if $F: V \to W$ is a finite branched holomorphic covering, then V is Stein if and only if W is Stein. **O.** There are in addition various cohomological characterizations of Stein varieties; some of these have applications to the problem of the number of holomorphic functions needed to describe a holomorphic subvariety.

P. One of the basic properties of Stein varieties, as made manifest in the preceding discussion, is that they have a plenitude of global holomorphic functions. Indeed, it is noted here that a general n-tuple of holomorphic functions on an n-dimensional Stein variety V determines a finite holomorphic mapping from V into $\mathbb{C}^n$. **Q.** Moreover, a dense set of $(n + 1)$-tuples of holomorphic functions on V determines a proper holomorphic mapping from V to $\mathbb{C}^{n+1}$. **R.** These results can be combined to yield yet more characterizations of Stein varieties, perhaps the most appealing and satisfactory of all the characterizations. For instance, a holomorphic variety V is Stein precisely when there is a holomorphic homeomorphism from V to a holomorphic subvariety of some space $\mathbb{C}^n$.

A very important topic in the continuation of the discussion of global methods is another solution of the Levi problem using sheaf-theoretic techniques, due to Grauert; this was discussed in the first version of the present book. It is possible to extend the notion of plurisubharmonic functions to general holomorphic varieties as well and approach the Levi problem this way. These topics have not yet appeared very accessibly in textbooks, but a useful survey of the literature can be found in [6]. Other topics are treated in [4] and [24], among other places. There are extension theorems for holomorphic sheaves, analogues of the extension theorems for holomorphic functions and varieties, discussed in [54].

Series Contents

VOLUME I
Function Theory

VOLUME II
Local Theory

VOLUME III

Homological Theory