Contemporary Precalculus

through applications

functions, data analysis and matrices

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PREFACE

Contemporary Precalculus through Applications provides students with an applications-oriented, investigative mathematics curriculum in which they use technology to solve problems and to enhance their understanding of mathematics. The topics presented lay a foundation to support future course work in mathematics including calculus, finite mathematics, discrete mathematics, and statistics. The topics also provide an introduction to the mathematics used in engineering, the physical and life sciences, business, finance, and computer science. Whenever possible, new material is presented in the context of real-world applications. Students are active learners who generate ideas for both the development of problem statements and for the solution of problems. Students learn to use a variety of techniques to solve problems that are investigated from a number of perspectives as they proceed through the course. Problems are presented in the context of real-world applications, and, consequently, the interpretation of solutions is given strong emphasis.

The goals of *Contemporary Precalculus through Applications* parallel those of the National Council of Teachers of Mathematics *Curriculum and Evaluation Standards*, a document developed simultaneously with this textbook. A primary goal of the authors is to foster the development of mathematical power in students. Other student needs addressed include: exposure to real-world applications of mathematics in a wide variety of disciplines so that students can learn to value mathematics; preparation for future course work in mathematics; development of the self-confidence necessary to undertake further study; and opportunities to use modern technology to enhance understanding and to solve problems.

The fabric of *Contemporary Precalculus through Applications* is woven of six spiraling themes treated with increasing depth and breadth at each exposure as follows.

Mathematical Modeling

The use of mathematics to model a wide variety of phenomena is central to the course. The modeling approach is used in analyzing sets of data; introducing and applying the various elementary functions; and applying matrices to various problem-solving situations. Additionally, the modeling approach provides motivation for student study. Further, problem situations are often presented in such a way that the student must supply the mathematical framework required in the solution process.

Computers and Calculators as Tools

Technology has lessened the need for extensive computation using pencil and paper techniques. Now students can focus on mathematical concepts and structure while calculators or computers carry out the computations. In this text, calculators are used in evaluating expressions, in applying numerical algorithms, and as an important aid in making conjectures.

The graphing calculator and the computer are used for calculations related to functions and matrices and also for quick and accurate graphing. Empirical models are central to the course, and the graphing calculator and the computer are used to develop such models from actual data (which students sometimes gather themselves) through re-expression and curve fitting. Throughout, the design of the material is based on the assumption that appropriate technology is available to students. This might include a single microcomputer available for demonstrations by the teacher, graphing calculators for individual students, or a computer lab for class and individual use.

Applications of Functions

The overall goal of the study of elementary functions is to illustrate how functions serve as bridges between mathematics and the situations they model. The study of specific functions is motivated by the need for tools to build empirical models and to approximate trends in data. The importance of compositions and inverses of functions is heightened by the frequency of their use in building mathematical models. The focus on understanding the behavior of functions leads to an emphasis on graphing, using hand-drawn sketches and a computer or graphing calculator. The geometry of the functions is often used to enhance understanding of the algebra.

Data Analysis

The principal goal of data analysis is to deduce information from data. A conceptual treatment of resistant and least squares techniques of curve fitting is given here. Students are reminded that observation and measurement in the real world result in values for variables, rather than formulas for functions. Techniques of data analysis allow students to uncover what, if any, functional relationship exists between the relevant variables. Re-expression of data by means of elementary functions is helpful in extracting information and gaining insight from data. Through discussions of residuals and causation versus association, students learn that predictions based on techniques of data analysis have limits to their accuracy and reliability.

Discrete Phenomena

The traditional precalculus-calculus course is based primarily on the mathematics of continuous functions. In contrast, however, discrete techniques are required for the mathematical analysis of many phenomena. Discrete mathematics topics included in this text include finance, population growth, economic production, and Markov chains.

Numerical Algorithms

The pervasiveness of fast and inexpensive calculating tools requires that students be able to use numerical methods to solve problems for which no exact method exists or for which the exact method is beyond the ability of the student. Examples of such situations include determining extreme values of a function, identifying the zeros of a function, and finding the points of intersection of polynomial and transcendental functions. Such problems, which illustrate applications of elementary mathematics, can be solved with the assistance of a computer and appropriate software or a graphing calculator.

Contemporary Precalculus through Applications is designed to encourage students to approach mathematics in new and innovative ways. A conscious effort is made to combine several of the major themes in examples and problems, to ask familiar questions in new contexts, and to apply new concepts to familiar questions. Each theme that spirals through the course has an effect on the instructional approach, but none more than the use of the computer and graphing calculator as tools. Having a computer in front of the classroom or regularly using a graphing calculator during discussions enables students to ask and answer "What if ...?" questions, to make conjectures, to check their guesses and analyses, and to work with real data. All these activities are invaluable contributions to the learning process.

The models that are developed in the course come from many diverse areas. Naturally, models from the sciences are included, but, in addition, models from banking and finance, anthropology, economics, sociology, sports, and environmental issues also appear. The applications vary in complexity and depth and are often revisited as new techniques are learned. For example, characteristics of the simple exponential function are investigated and applied. Later, data analysis is used to model the phenomenon of population growth with the exponential function. Finally, the Leslie matrix is used to explore age-specific population growth, with the computer assisting in the consideration of long run expectation.

The course is not designed for a head-down march through the syllabus. The authors expect and have worked to create opportunities for discussions in class, additional questions, and reflection. For some problems there is no single "best" answer. The course has been constructed with the philosophy that the quality of learning is more important than completing a syllabus.

Acknowledgments

The curriculum and ideas of this textbook were developed with funding from the Carnegie Corporation of New York and the National Science Foundation. The endorsement and support of the Carnegie Corporation of New York through Alden Dunham encouraged us to develop a different precalculus course and to ask hard questions about what concepts are essential to the mathematical development of students. The National Science Foundation support gave us the opportunity to make revisions based on teaching experience and interaction with teachers across the United States. Any opinions, finding, and conclusions or recommendations expressed through this project are those of the authors and do not necessarily reflect the views of the Carnegie Corporation of New York or the National Science Foundation.

Three chapters of this book were published previously by the National Council of Teachers of Mathematics. NCTM has given permission for those chapters to be included in this textbook.

The years spent working on this project have confirmed the excitement of learning — by teachers and by students. A remarkable number of people have contributed. The students of the North Carolina School of Science and Mathematics from 1984 to 1991 played a significant role in the development of this book. A number of colleagues at NCSSM gave us support and help throughout the project; however, very special thanks go to Billie Bean, Nicole Holbrook, John Kolena, Mary Malinauskus, John Parker, Bonnie Ramey, Donita Robinson, and Dotte Williams. Their contributions included frequent feedback and ideas based on their classroom teaching experiences, proofreading and providing answers. Doug Shackelford, a 1985 graduate of NCSSM, created the software that enabled us to implement these ideas and continued to support that software until the NCTM publications were complete. A grant from IBM Corporation provided the equipment our students used and funded the beginning of the software development.

A number of teachers from across the United States became valued friends and colleagues through our work in this project. Landy Godbold from the Westminster Schools in Atlanta, Georgia, is an extraordinary and forward-looking teacher and was willing to gamble on this new curriculum in its earliest forms. We benefited from his creative ideas and findings throughout the project. David Bannard of The Collegiate Schools in Richmond, Virginia, Tom Seidenberg of Phillips Exeter Academy in Exeter, New Hampshire, Rick Jennings of Eisenhower High School in Yakima, Washington, Stan Mathes of Niskayuna High School in Schenectady, New York, Jon Choate of The Groton School in Groton, Massachusetts, Wally Green of Jordan High School in Durham, North Carolina and Jim Ebert of Durham Academy in Durham, North Carolina, were all very helpful in suggesting improvements and giving us constructive feedback. Teachers who took part in NCSSM summer math workshops, the Secondary School Mathematics and Computer Conferences at Phillips Exeter Academy, the Woodrow Wilson Institutes, and especially the NSF teachers were all essential to the development of the book. We learned a lot together.

Dr. Mary Kiely, of Stanford University and formerly with the Carnegie Corporation of New York, was a wonderfully supportive grant officer. Her enthusiasm and confidence in the project gave us the courage to pursue new ideas in secondary mathematics.

Zedra Williams and Cindy Colimon worked diligently in the production of the text and daily supported us in our work. Corlise Ferrell's technical expertise in the preparation of the manuscript was invaluable.

A very special thanks to the publisher, Barbara Janson, whose patience and expertise have brought this textbook to fruition. We particularly appreciate the freedom she gave us to produce a text that we believe supports the current needs in precalculus mathematics.

Our families — Rita, Doc, Jo-Ann, Judd, Chris, Emily, Scott, Bill, Kate, Ted, Lillian, and Nancy — deserve our greatest thanks for giving us the time and the support first to dream about and then to complete this project.

We are sad that Lawrence Gould, our colleague and co-author, did not see the completion of this textbook. Throughout the development of the course and in our daily work, Lawrence offered us the wisdom that experience brings. He passed away in May of 1990.

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CHAPTER 1

Data Analysis One

1

Some Thoughts about Models and Mathematics

When children think about models, they are generally considering some kind of a toy, perhaps an airplane or a dinosaur. When scientists and mathematicians think about models, they are generally considering a model as a tool, even though they may be thinking about the same airplane or dinosaur. Scientists and mathematicians use models to help them study and understand the physical world. People in all walks of life use models to help them solve problems; problems in this course will involve models used by bankers, city planners, anthropologists, geologists, economists, automobile makers, and many, many others.

So just what is a model? Models are representations of phenomena. In order to be useful, a model must share important characteristics with the phenomenon it represents, and it must also be simpler than what it represents. A model usually differs significantly from what it represents, but these differences are offset by the advantage that comes from simplifying the phenomenon. A good example is a road map, which models the streets and highways in a particular area. Clearly, a map has a lot in common with the actual streets and highways — it shows how roads are oriented and where they intersect. A road map simplifies the situation; it ignores stoplights, steep ascents, and back alleys and instead focuses on major thoroughfares. Such a map is very useful for traveling from one city to another, but is not much good for finding the quickest route to the shopping mall or the best street for skateboarding. Road maps, and most other models, are useful precisely because they ignore some extraneous information and thereby allow you to see other information more clearly.

Another fairly common model is an EKG, which models the electrical activity of the heart. The EKG is an excellent model when used to determine the heart rate or to find which regions of the heart may be damaged after a heart attack. It is a very poor model for determining the volume of blood flowing through the heart or the condition of the valves. Different models emphasize different aspects of a phenomenon; the choice of what model to use depends on what aspect is under investigation.

The ability to predict is the ultimate test for a model. A good model allows us to make accurate predictions about what will occur under certain conditions. If what actually occurs is very different from our prediction, then the model is of little use.

Scientists and mathematicians often need to update or revise models as more is learned about the phenomenon under study. Sometimes a model needs to be completely discarded and replaced with a new one. The pre-Columbian model of the flat world was discarded, and Ptolemy's geocentric model of the universe was revised by Copernicus and then overthrown by Galileo.

Even though Isaac Newton's models for the actions of a gravitational field have been replaced by Einstein's relativistic model, we still use Newtonian physics under everyday conditions because it is easier and because it gives reasonably accurate results. The aspects of Einsteinian mechanics that are ignored are largely irrelevant in most everyday applications, so the Newtonian model is still a good one.

As we move through the course, we will encounter phenomena that we want to know more about. Our task will be to find a mathematical expression or formula or picture that mimics the phenomenon we are interested in. This model must accurately represent the aspects of the phenomenon that we care about, but it may be very different from the phenomenon in other ways. To be able to find a model to represent a problem, we need to have a large toolkit of mathematical information and techniques at our disposal. The fundamental concepts learned in Algebra 1, Geometry, and Algebra 2 are all a part of our toolkit. We will also use the calculator and computer as tools to construct and analyze models for the phenomena we study. Probably the most important tools necessary for model making are an inquisitive mind and a determined spirit.

Often we will not stop after we have developed one model but will form two or three to get a better view of the subject. For example, suppose a rock is thrown into the air. How can its path be modeled? It would be informative if we had a picture of the flight. We can use a graph as a model for this phenomenon (see Figure 1).

We could also describe the flight by an equation. If the height above the ground is called h and the horizontal distance away from the thrower is called d, then we can represent the flight by the formula

$$\text{Model 2:} \quad h = -\frac{2}{5}d^2 + \frac{4}{5}d + 1.$$

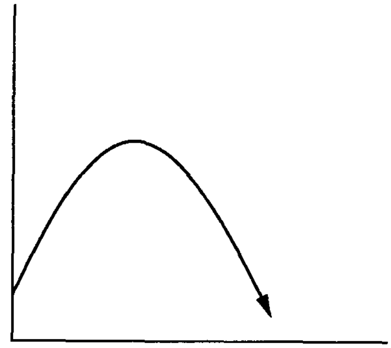

Figure 1 Model 1

Notice that these models do not give complete information about the problem. We cannot tell from the models what type of rock was thrown, who threw it, or why. These aspects of the problem are not relevant—all we really care about is distance and time. Both models give this information. The strength of any model is in what it emphasizes and what it ignores.

2

Analysis of Sets of Ordered Pairs

We begin our course with graphical models of data. These models provide information so that we can answer questions like the following:

- are high school graduation rates related to state spending on education?
- what will be the winning speed for the New York Marathon in the year 2000?
- is there a relationship between the amount of time a student spends studying for a test and the grade on the test?

Though the questions posed above are different in many respects, each requires the collection,

Table 2 Public School Spending per Pupil and High School Graduation Rates

	Spending	Graduation Rate		Spending	Graduation Rate
Alaska	$8,842	67.1%	California	$3,751	65.8%
New York	$6,299	62.7%	Iowa	$3,740	86.5%
Wyoming	$6,229	74.3%	Maine	$3,650	78.6%
New Jersey	$6,120	77.3%	Texas	$3,584	63.2%
Connecticut	$5,532	80.4%	New Mexico	$3,537	71.9%
Dist. of Columbia	$5,349	54.8%	North Carolina	$3,473	70.3%
Massachusetts	$4,856	76.3%	Nebraska	$3,437	86.9%
Delaware	$4,776	69.9%	New Hampshire	$3,386	75.2%
Pennsylvania	$4,752	77.2%	Indiana	$3,379	76.4%
Wisconsin	$4,701	84.0%	Missouri	$3,345	76.1%
Maryland	$4,659	77.7%	Louisiana	$3,237	54.7%
Rhode Island	$4,574	67.6%	North Dakota	$3,209	86.1%
Vermont	$4,459	83.4%	South Dakota	$3,190	85.1%
Hawaii	$4,372	73.8%	Georgia	$3,167	62.6%
Minnesota	$4,241	90.6%	Kentucky	$3,107	68.2%
Oregon	$4,236	72.7%	South Carolina	$3,005	62.4%
Kansas	$4,137	81.4%	West Virginia	$2,959	72.8%
Colorado	$4,129	72.2%	Tennessee	$2,842	64.1%
Montana	$4,070	82.9%	Arkansas	$2,795	75.7%
Florida	$4,056	61.2%	Arizona	$2,784	64.5%
Illinois	$3,980	74.0%	Oklahoma	$2,701	71.1%
Michigan	$3,954	71.9%	Alabama	$2,610	63.0%
Virginia	$3,809	73.7%	Idaho	$2,555	76.7%
Washington	$3,808	74.9%	Mississippi	$2,534	61.8%
Ohio	$3,769	76.1%	Utah	$2,455	75.9%
Nevada	$3,768	63.9%			

organization, and interpretation of data. To answer each question, we need to analyze the relationship between several variables. We will limit our attention to paired measurements, that is, the *analysis of two variables*. Sometimes one variable actually depends on the other; for example, we expect that blood pressure in adults of the same height in some way depends on weight and that crop yield depends on amount of rainfall. Other times there is a relationship between the variables, but it is not one of cause and effect or dependence. For example, we can show that there is a relationship between points scored and personal fouls committed by college basketball players, but we would certainly not consider one of these variables to be dependent on the other. Moreover, sometimes there is no relationship at all between the two variables; for example, we do not expect there to be a relationship between the distance a student lives from school and his or her height.

To determine whether there is a relationship between two variables, we must analyze data consisting of ordered pairs. Sometimes these data are gathered from a well-designed, carefully controlled scientific experiment. Other times we want to analyze data that exist in the world around us. The

September 7, 1987, *U.S. News and World Report* claims that "spending heavily on teachers doesn't always yield a bumper crop of graduates." The comment is followed by the data provided in Table 2. Study this list to determine whether you agree with the statement made by the magazine.

How do we get information out of the list of numbers in Table 2? Did you actually read all of it, or did you skip to this paragraph? Presented as just a table of numbers, the data are difficult to interpret. In analyzing data, we search for information hidden in the numbers. It should be obvious that we need to have some way of organizing and simplifying the data so that we can see its essential characteristics without getting lost in a jumble of numbers. Then we can decide for ourselves whether or not graduation rate is related to educational spending.

3
Scatter Plots

One step in analyzing the relationship between two variables is to make a *scatter plot.* A scatter plot is simply a graph in a rectangular coordinate system of all ordered pairs of data. Scatter plots display data so that we can see the general relationship between two variables. A computer-generated scatter plot of the fifty-one data points listed in Table 2 is shown in Figure 3. Each observation in the data set is represented by one point in the plane. State spending is plotted on the horizontal axis, and graduation rate is plotted on the vertical axis. For example, the ordered pair representing Montana is ($4070, 82.9%). When making a scatter plot, it really does not matter which variable is plotted on which axis. If we suspect that one variable depends on the other, however, we usually plot the dependent variable on the vertical axis and the independent variable on the horizontal axis.

Study the scatter plot in Figure 3. Do you agree with the *U.S. News and World Report* claim? Does there appear to be any relationship between state spending and graduation rate? Are

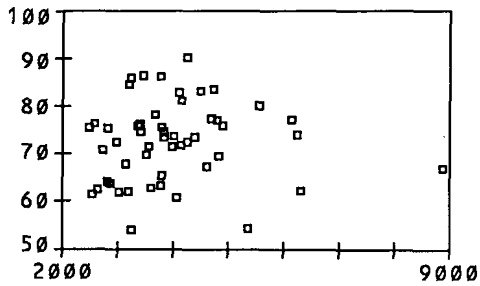

Figure 3 Graduation Rate versus Public School Spending

you surprised by the point that lies far to the right of the others? Which display do you find easier to interpret, the table or the graph?

A scatter plot is an effective tool for analyzing data. Special characteristics of data that may be unnoticed in a table are more obvious from a graph. If there is some relationship between the variables, a pattern or trend is usually apparent in the scatter plot.

Example 1

Basketball Statistics

The data in Table 4, provided by the Sports Information Offices at Duke University and the University of North Carolina, Chapel Hill, summarize the total number of rebounds, assists, personal fouls committed, and points scored by individual basketball players during the 1984–85 basketball season.

Do you think there is a relationship between the number of personal fouls that a basketball player commits in a season of play and the number of points he scores? The relationship (or lack thereof) should be more obvious if we plot the data. On the computer-generated scatter plot shown in Figure 5, page 8, personal fouls are plotted on the horizontal axis and points scored are on the vertical axis.

Looking at the scatter plot should convince us that there is a relationship between these variables. The points tend to slope upward to the right, so we

Table 4 Basketball Statistics

Player	Rebounds	Assists	Total Fouls Committed	Total Points Scored
Duke University				
Alarie	158	49	78	492
Amaker	69	184	55	253
Anderson	20	0	12	15
Bilas	186	14	96	312
Bryan	8	2	6	12
Dawkins	141	154	64	582
Henderson	98	51	69	317
King	50	22	34	51
Meagher	130	46	85	241
Nessley	18	3	15	16
Strickland	36	15	37	108
Williams	29	2	15	47
University of North Carolina, Chapel Hill				
Brust	4	3	2	2
Daugherty	349	77	112	623
Daye	3	2	1	6
Hale	119	168	102	349
Hunter	36	32	34	120
Martin	199	44	103	347
Morris	5	0	0	8
Peterson	65	57	31	198
Popson	87	19	66	211
Roper	1	0	2	0
K. Smith	92	235	54	444
R. Smith	19	7	18	65
Wolf	158	58	79	274

observe that the players who commit the most personal fouls also score the most points. When both variables increase together, we say there is a *positive association* between them. What is the general shape of the plot? There does not appear to be any obvious curvature; rather, the points seem to be increasing steadily. How strong is the relationship between these variables? To answer this question, you might want to consider how well you would be able to predict the points scored by a player committing 25 personal fouls. The scatter plot shows quite a bit of spread in the data. Though the variables are related, we would have to consider the relationship loose, or weak, in the sense that knowing a value of one variable does not give us confidence in predicting a value for the other variable. Are there any clusters of points on the graph? If so, where are they? Can you offer a reasonable explanation for the clusters? Are there any points that appear to stand out from the rest, that is, points that do not seem consistent with the other observations? If so, such points deserve special attention