An Introduction to Differentiable Manifolds and Riemannian Geometry

William M. Boothby

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William M. Boothby

DEPARTMENT OF MATHEMATICS
WASHINGTON UNIVERSITY
ST. LOUIS, MISSOURI

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Preface

Apart from its own intrinsic interest, a knowledge of differentiable manifolds has become useful—even mandatory—in an ever-increasing number of areas of mathematics and of its applications. This is not too surprising, since differentiable manifolds are the underlying, if unacknowledged, objects of study in much of advanced calculus and analysis. Indeed, such topics as line and surface integrals, divergence and curl of vector fields, and Stokes's and Green's theorems find their most natural setting in manifold theory. But however natural the leap from calculus on domains of Euclidean space to calculus on manifolds may be to those who have made it, it is not at all easy for most students. It usually involves many weeks of concentrated work with very general concepts (whose importance is not clear until later) during which the relation to the already familiar ideas in calculus and linear algebra become lost—not irretrievably, but for all too long. Simple but nontrivial examples that illustrate the necessity for the high level of abstraction are not easy to present at this stage, and a realization of the power and utility of the methods must often be postponed for a dismayingly long time.

This book was planned and written as a text for a two-semester course designed, it is hoped, to overcome, or at least to minimize, some of these difficulties. It has, in fact, been used successfully several times in preliminary form as class notes for a two-semester course intended to lead the student from a reasonable mastery of advanced (multivariable) calculus and a rudimentary knowledge of general topology and linear algebra to a solid fundamental knowledge of differentiable manifolds, including some facility in working with the basic tools of manifold theory: tensors, differential forms, Lie and covariant derivatives, multiple integrals, and so on. Although in overall content this book necessarily overlaps the several available excellent books on manifold theory, there are differences in presentation and emphasis which, it is hoped, will make it particularly suitable as an introductory text.

To begin with, it is more elementary and less encyclopedic than most books on this subject. Special care has been taken to review, and then to develop, the connections with advanced calculus. In particular, all of Chapter II is devoted to functions and mappings on open subsets of Euclidean space, including a careful exposition and proof of the inverse function theorem. Efforts are made throughout to introduce new ideas gradually and with as much attention to intuition as possible. This has led to a longer but more readable presentation of inherently difficult material. When manifolds are first defined, an effort is made to have as many nontrivial examples as possible; for this reason, Lie groups, especially matrix groups, and certain quotient manifolds are introduced early and used throughout as examples. A fairly large number of problems (almost 400) is included to develop intuition and computational skills.

Further, it may be said that there has been a conscious effort to avoid or at least to economize generality insofar as that is possible. Concepts are often introduced in a rather ad hoc way with only the generality needed and, if possible, only when they are actually needed for some specific purpose. This is particularly noticeable in the treatment of tensors—which is far from general—and in the brief introduction to vector bundles (more specifically to the tangent bundle). Thus it is not claimed that this is a comprehensive book; the student will emerge with gaps in his knowledge of various subjects treated (for example, Lie groups or Riemannian geometry). On the other hand it is hoped that he will acquire strong motivation, computational skills, and a feeling for the subject that will make it easy for him to proceed to more advanced work in any of a number of areas using manifold theory: differential topology, Lie groups, symmetric and homogeneous spaces, harmonic analysis, dynamical systems, Morse theory, Riemann surfaces, and so on.

Finally, it should be said that the author has tried to include at every stage results that illustrate the power of these ideas. Chapter VI is especially noteworthy in this respect in that it includes complete proofs of Brouwer's fixed point theorem and of the nonexistence of nowhere-vanishing continuous vector fields on even-dimensional spheres. In a similar vein, the existence of a bi-invariant measure on compact Lie groups is demonstrated and applied to prove the complete reducibility of their linear representations. Then, in a later chapter, compact groups are used as simple examples of symmetric spaces, and their corresponding geometry is used to prove that every element lies on a one-parameter subgroup. In the last two chapters, which deal with Riemannian geometry of abstract n-dimensional manifolds, the relation to the more easily visualized geometry of curves and surfaces in Euclidean space is carefully spelled out and is used to develop the general ideas for which such applications as the Hopf–Rinow theorem are given. Thus, by a selection of accessible but important applications, some truly nontrivial, unexpected (to the student) results are obtained from the abstract machinery so patiently constructed.

Briefly, the organization of the book is as follows. Chapter I is a very intuitive introduction and fixes some of the conventions and notation that are used. Chapter II is largely advanced calculus and may very well be omitted or skimmed by better prepared readers. In Chapter III, the basic concept of differentiable manifold is introduced along with mappings of manifolds and their properties; a fairly extensive discussion of examples is included. Chapter IV is particularly concerned with vectors and vector fields and with a careful exposition of the existence theorem for solutions of systems of ordinary differential equations and the related one-parameter group action. In Chapter V covariant tensors and differential forms are treated in some detail and then used to develop a theory of integration on manifolds in Chapter VI. Numerous applications are given. It would be possible to use Chapters II–VI as the basis of a one-semester course for students who wish to learn the fundamentals of differentiable manifolds without any Riemannian geometry. On the other hand, for students who already have some experience with manifolds, Chapters VII and VIII could serve as a brief introduction to Riemannian geometry. In these last two chapters, beginning from curves and surfaces in Euclidean space, the concept of Riemannian connection and covariant differentiation is carefully developed and used to give a fairly extensive discussion of geodesics—including the Hopf–Rinow theorem—and a shorter treatment of curvature. The natural (bi-invariant) geometry on compact Lie groups and Riemannian manifolds of constant curvature are both discussed in some detail as examples of the general theory. This discussion is based on a fairly complete treatment of covering spaces, discontinuous group action, and the fundamental group given earlier in the book.

This book, as do many of the books in this subject, owes much to the influence of S. S. Chern. For many years his University of Chicago notes——still an important reference (Chern [1])—were virtually the only systematic account of most of the topics in this text. Even more importantly his courses, lectures, published works, and above all his personal encouragement have had an impressive influence on a whole generation of differential geometers, among whom this author had the good fortune to be included. Another source of inspiration to the author was the work of John Milnor. The manner in which he has made exciting fundamental research in differential topology and geometry available to specialist and nonspecialist alike through many careful expository works (written in a style that this author particularly admires) certainly deserves gratitude. No better material for further or supplemental reading to this text could be suggested than Milnor's two books [1] and [2].

For part of the time during which this book was being written, the

author benefitted from a visiting professorship at the University of Strasbourg, France, and he is particularly grateful for the opportunity to work there, in an atmosphere so conducive to advancing in the task he had undertaken.

The author would also like to acknowledge with gratitude the help given to him by his son, Thomas Boothby, by students and colleagues at Washington University, especially Humberto Alagia and Eduardo Cattani, and by Mrs. Virginia Hundley for her careful work preparing the manuscript.

IV. Vector Fields on a Manifold

1. The Tangent Space at a Point of a Manifold 106
2. Vector Fields 115
3. One-Parameter and Local One-Parameter Groups Acting on a Manifold 122
4. The Existence Theorem for Ordinary Differential Equations 130
5. Some Examples of One-Parameter Groups Acting on a Manifold 138
6. One-Parameter Subgroups of Lie Groups 145
7. The Lie Algebra of Vector Fields on a Manifold 149
8. Frobenius' Theorem 156
9. Homogeneous Spaces 164
Notes 171
Appendix Partial Proof of Theorem 4.1 172

V. Tensors and Tensor Fields on Manifolds

1. Tangent Covectors 175
Covectors on Manifolds 176
Covector Fields and Mappings 178
2. Bilinear Forms. The Riemannian Metric 181
3. Riemannian Manifolds as Metric Spaces 185
4. Partitions of Unity 191
Some Applications of the Partition of Unity 193
5. Tensor Fields 197
Tensors on a Vector Space 197
Tensor Fields 199
Mappings and Covariant Tensors 200
The Symmetrizing and Alternating Transformations 201
6. Multiplication of Tensors 204
Multiplication of Tensors on a Vector Space 205
Multiplication of Tensor Fields 206
Exterior Multiplication of Alternating Tensors 207
The Exterior Algebra on Manifolds 211
7. Orientation of Manifolds and the Volume Element 213
8. Exterior Differentiation 217
An Application to Frobenius' Theorem 221
Notes 225

VI. Integration on Manifolds

1. Integration in R^n. Domains of Integration 227
Basic Properties of the Riemann Integral 228
2. A Generalization to Manifolds 233
Integration on Riemannian Manifolds 237
3. Integration on Lie Groups 241
4. Manifolds with Boundary 248
5. Stokes's Theorem for Manifolds with Boundary 256
6. Homotopy of Mappings. The Fundamental Group 263
Homotopy of Paths and Loops. The Fundamental Group 265
7. Some Applications of Differential Forms. The de Rham Groups 271
The Homotopy Operator 274

IV. Vector Fields on a Manifold

1. The Tangent Space at a Point of a Manifold 106
2. Vector Fields 115
3. One-Parameter and Local One-Parameter Groups Acting on a Manifold 122
4. The Existence Theorem for Ordinary Differential Equations 130
5. Some Examples of One-Parameter Groups Acting on a Manifold 138
6. One-Parameter Subgroups of Lie Groups 145
7. The Lie Algebra of Vector Fields on a Manifold 149
8. Frobenius' Theorem 156
9. Homogeneous Spaces 164

Notes 171

Appendix Partial Proof of Theorem 4.1 172

V. Tensors and Tensor Fields on Manifolds

1. Tangent Covectors 175
   - Covectors on Manifolds 176
   - Covector Fields and Mappings 178
2. Bilinear Forms. The Riemannian Metric 181
3. Riemannian Manifolds as Metric Spaces 185
4. Partitions of Unity 191
   - Some Applications of the Partition of Unity 193
5. Tensor Fields 197
   - Tensors on a Vector Space 197
   - Tensor Fields 199
   - Mappings and Covariant Tensors 200
   - The Symmetrizing and Alternating Transformations 201
6. Multiplication of Tensors 204
   - Multiplication of Tensors on a Vector Space 205
   - Multiplication of Tensor Fields 206
   - Exterior Multiplication of Alternating Tensors 207
   - The Exterior Algebra on Manifolds 211
7. Orientation of Manifolds and the Volume Element 213
8. Exterior Differentiation 217
   - An Application to Frobenius' Theorem 221

Notes 225

VI. Integration on Manifolds

1. Integration in R^n. Domains of Integration 227
   - Basic Properties of the Riemann Integral 228
2. A Generalization to Manifolds 233
   - Integration on Riemannian Manifolds 237
3. Integration on Lie Groups 241
4. Manifolds with Boundary 248
5. Stokes's Theorem for Manifolds with Boundary 256
6. Homotopy of Mappings. The Fundamental Group 263
   - Homotopy of Paths and Loops. The Fundamental Group 265
7. Some Applications of Differential Forms. The de Rham Groups 271
   - The Homotopy Operator 274

8. Some Further Applications of de Rham Groups 278
The de Rham Groups of Lie Groups 282
9. Covering Spaces and the Fundamental Group 286
Notes 292

VII. Differentiation on Riemannian Manifolds

1. Differentiation of Vector Fields along Curves in R^n 294
The Geometry of Space Curves 297
Curvature of Plane Curves 301
2. Differentiation of Vector Fields on Submanifolds of R^n 303
Formulas for Covariant Derivatives 308
$\nabla_{X_p} Y$ and Differentiation of Vector Fields 310
3. Differentiation on Riemannian Manifolds 313
Constant Vector Fields and Parallel Displacement 319
4. Addenda to the Theory of Differentiation on a Manifold 321
The Curvature Tensor 321
The Riemannian Connection and Exterior Differential Forms 324
5. Geodesic Curves on Riemannian Manifolds 326
6. The Tangent Bundle and Exponential Mapping. Normal Coordinates 331
7. Some Further Properties of Geodesics 338
8. Symmetric Riemannian Manifolds 347
9. Some Examples 353
Notes 360

VIII. Curvature

1. The Geometry of Surfaces in E^3 362
The Principal Curvatures at a Point of a Surface 366
2. The Gaussian and Mean Curvatures of a Surface 370
The Theorema Egregium of Gauss 373
3. Basic Properties of the Riemann Curvature Tensor 378
4. The Curvature Forms and the Equations of Structure 385
5. Differentiation of Covariant Tensor Fields 391
6. Manifolds of Constant Curvature 399
Spaces of Positive Curvature 402
Spaces of Zero Curvature 404
Spaces of Constant Negative Curvature 405
Notes 410

REFERENCES 413

INDEX 417

I INTRODUCTION TO MANIFOLDS

In this chapter, we establish some preliminary notations and give an intuitive, geometric discussion of a number of examples of manifolds—the primary objects of study throughout the book. Most of these examples are surfaces in Euclidean space; for these—together with curves on the plane and in space—were the original objects of study in classical differential geometry and are the source of much of the current theory.

The first two sections deal primarily with notational matters and the relation between Euclidean space, its model $\boldsymbol{R}^n$, and real vector spaces. In Section 3 a precise definition of topological manifolds is given, and in the remaining sections this concept is illustrated.

1 Preliminary Comments on $\boldsymbol{R}^n$

Let $\boldsymbol{R}$ denote the real numbers and $\boldsymbol{R}^n$ their n-fold Cartesian product

$$\overbrace{\boldsymbol{R} \times \cdots \times \boldsymbol{R}}^{n},$$

the set of all ordered n-tuples $(x^1, \ldots, x^n)$ of real numbers. Individual n-tuples may be denoted at times by a single letter. Thus $x = (x^1, \ldots, x^n)$, $a = (a^1, \ldots, a^n)$, and so on. We agree once and for all to use on $\boldsymbol{R}^n$ its topology as a metric space with the metric defined by

$$d(x, y) = \left(\sum_{i=1}^{n} (x^i - y^i)^2 \right)^{1/2}.$$

The neighborhoods are then open balls $B_\varepsilon^n(x)$, or $B_\varepsilon(x)$ or, equivalently, open cubes $C_\varepsilon^n(x)$, or $C_\varepsilon(x)$ defined for any $\varepsilon > 0$ as

$$B_\varepsilon(x) = \{y \in \boldsymbol{R}^n \mid d(x, y) < \varepsilon\},$$

and

$$C_\varepsilon(x) = \{y \in \boldsymbol{R}^n \mid |x^i - y^i| < \varepsilon, i = 1, \ldots, n\},$$

a cube of side 2ε and center x. Note that $\boldsymbol{R}^1 = \boldsymbol{R}$ and we define $\boldsymbol{R}^0$ to be a single point.

We shall invariably consider $\boldsymbol{R}^n$ with the topology defined by the metric. This space $\boldsymbol{R}^n$ is used in several senses, however, and we must usually decide from the context which one is intended. Sometimes $\boldsymbol{R}^n$ means merely $\boldsymbol{R}^n$ as *topological* space, sometimes $\boldsymbol{R}^n$ denotes an n-dimensional vector space, and sometimes it is identified with Euclidean space. We will comment on this last identification in Section 2 and examine here the other meanings of $\boldsymbol{R}^n$.

We assume that the definition and basic theorems of vector spaces over $\boldsymbol{R}$ are known to the reader. Among these is the theorem which states that any two vector spaces over $\boldsymbol{R}$ which have the same dimension n are isomorphic. It is important to note that this isomorphism depends on *choices* of *bases* in the two spaces; there is in general no *natural* or *canonical* isomorphism independent of these choices. However, there does exist one important example of an n-dimensional vector space over $\boldsymbol{R}$ which has a distinguished or canonical basis—a basis which is somehow given by the nature of the space itself. We refer to the vector space of n-tuples of real numbers with componentwise addition and scalar multiplication. This is, as a set at least, just $\boldsymbol{R}^n$; should we wish on occasion to avoid confusion, then we will denote it by V^n (and use boldface for its elements ($\mathbf{x}$ instead of x, and so forth). For this space the n-tuples $\mathbf{e}_1 = (1, 0, \ldots, 0), \ldots, \mathbf{e}_n = (0, 0, \ldots, 0, 1)$ are a basis, known as the *natural* or *canonical* basis. We may at times suppose that the n-tuples are written as rows, that is, $1 \times n$ matrices, and at other times as columns, that is, $n \times 1$ matrices. This only becomes important should we wish to use matrix notation to simplify things a bit, for example, to describe linear mappings, equations, and so on.

Thus $\boldsymbol{R}^n$ *may* denote a vector space of dimension n over $\boldsymbol{R}$. We sometimes mean even more by $\boldsymbol{R}^n$. An abstract n-dimensional vector space over $\boldsymbol{R}$ is called *Euclidean* if it has defined on it a positive definite inner product. In general there is no natural way to choose such an inner product, but in the case of $\boldsymbol{R}^n$ or V^n, again we have the natural inner product

$$(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} x^i y^i.$$

It is characterized by the fact that relative to this inner product the natural basis is orthonormal, $(\mathbf{e}_i, \mathbf{e}_j) = \delta_{ij}$.

Thus at times R^n is a Euclidean vector space, but one which has a built-in orthonormal basis and inner product. An abstract vector space, even if Euclidean, does not have any such preferred basis. The metric in R^n discussed at the beginning can be defined using the inner product on R^n. We define $\|\mathbf{x}\|$, the *norm* of the vector x, by $\|\mathbf{x}\| = ((\mathbf{x}, \mathbf{x}))^{1/2}$. Then we have

$$d(x, y) = \|\mathbf{x} - \mathbf{y}\|.$$

This notation is frequently useful even when we are dealing with R^n as a metric space and not using its vector space structure. Note, in particular, that $\|\mathbf{x}\| = d(x, 0)$, the distance from the point x to the origin. In this equality $\mathbf{x}$ is a vector on the left-hand side, and x is the corresponding point on the right-hand side; an illustration of the way various interpretations of R^n can be mixed together.

Exercises

1. Show that if A is an $m \times n$ matrix, then the mapping from V^n to V^m (with elements written as $n \times 1$ and $m \times 1$ matrices), which is defined by $\mathbf{y} = A\mathbf{x}$, is continuous. Identify the images of the canonical basis of V^n as linear combinations of the canonical basis of V^m.
2. Find conditions for the mapping of Exercise 1 to be onto; to be one-to-one.
3. Show that if W is an n-dimensional Euclidean vector space, then there exists an isometry, that is, an isomorphism preserving the inner product, of W onto R^n interpreted as Euclidean vector space.
4. Show that C^n, the space of n-tuples of complex numbers, may be placed in one-to-one correspondence with R^{2n}. Can this correspondence be a vector space isomorphism?
5. Exhibit an isomorphism between the vector space of $m \times n$ matrices over R and the vector space R^{mn}. Show that the map $X \to AX$, where A is a fixed $m \times m$ matrix and X is an arbitrary $m \times n$ matrix (over R), is continuous in the topology derived from R^{mn}.
6. Show that $\|\mathbf{x}\|$ has the following properties:

   (a) $\|\mathbf{x} \pm \mathbf{y}\| \le \|\mathbf{x}\| + \|\mathbf{y}\|$;
   (b) $\|\mathbf{x}\| - \|\mathbf{y}\| \le \|\mathbf{x} - \mathbf{y}\|$;
   (c) $\|\alpha\mathbf{x}\| = |\alpha| \, \|\mathbf{x}\|, \alpha \in R$; and
   (d) explain how (a) is related to the triangle inequality of $d(x, y)$.

7. Show that an isometry of a Euclidean vector space onto itself has an orthogonal matrix relative to any orthonormal basis.
8. Prove that every Euclidean vector space V has an orthonormal basis. Construct your proof in such a way that if W is a given subspace of V, $\dim W = r$, then the first r vectors of the basis of V are a basis of W.

2 R^n and Euclidean Space

Another role which R^n plays is that of a model for n-dimensional *Euclidean* space E^n, in the sense of Euclidean geometry. In fact many texts simply refer to R^n with the metric $d(x, y)$ as Euclidean space. This identification is misleading in the same sense that it would be misleading to identify *all* n-dimensional vector spaces with R^n; moreover unless clearly understood, it is an identification that can hamper clarification of the concept of manifold and the role of coordinates. Certainly Euclid and the geometers before the seventeenth century did not think of the Euclidean plane or three-dimensional space—which we denote by E^2 and E^3—as pairs or triples of real numbers. In fact they were defined axiomatically beginning with undefined objects such as points and lines together with a list of their properties—the axioms—from which the theorems of geometry were then deduced. This is the path which we all follow in learning the basic ideas of Euclidean plane and solid geometry, about which most of us know quite a bit before studying analytic or coordinate geometry at all. The identification of R^n and E^n came about after the invention of analytic geometry by Fermat and Descartes and was eagerly seized upon since it is very tricky and difficult to give a suitable definition of Euclidean space, of any dimension, in the spirit of Euclid, that is, by giving axioms for (abstract) Euclidean space as one does for abstract vector spaces. This difficulty was certainly recognized for a very long time, and has interested many great mathematicians. It led in part to the discovery of non-Euclidean geometries and thus to manifolds. A careful axiomatic definition of Euclidean space is given by Hilbert [1]. Since our use of Euclidean geometry is mainly to aid our intuition, we shall be content with assuming that the reader "knows" this geometry from high school.

Consider the Euclidean plane E^2 as studied in high school geometry; definitions are made, theorems proved, and so on, *without* coordinates. One later introduces coordinates using the notions of length and perpendicularity in choosing two mutually perpendicular "number axes" which are used to define a one-to-one mapping of E^2 onto R^2 by $p \to (x(p), y(p))$, the coordinates of $p \in E^2$. This mapping is (by design) an isometry, preserving distances of points of E^2 and their images in R^2. Finally one obtains further correspondences of essential geometric elements, for example, lines of E^2 with subsets of R^2 consisting of the solutions of linear equations. Thus we carry each geometric object to a corresponding one in R^2. It is the existence of such coordinate mappings which make the identification of E^2 and R^2 possible. But caution! An *arbitrary choice* of coordinates is involved, there is no *natural, geometrically determined* way to identify the two spaces. Thus, at best, we can say that R^2 may be identified with E^2 *plus a coordinate system*. Even then we need to define in R^2 the notions of line, angle of lines, and other attributes of the Euclidean plane before thinking of it as Euclidean space. Thus, with qualifications, we may identify E^2 and R^2 or E^n and R^n, especially remembering that they carry a choice of rectangular coordinates.

We conclude with a brief indication of why we might not always wish to make the identification, that is, to use the analytic geometry approach to the study of a geometry. Whenever E^n and R^n are identified it involves the choice of a coordinate system, as we have seen. It then becomes difficult at times to distinguish underlying geometric properties from those which depend on the choice of coordinates. An example: Having identified E^2 and R^2 and lines with the graphs of linear equations, for instance,

$$L = \{(x, y) \mid y = mx + b\},$$

we define the *slope m* and the *y-intercept b*. Neither has geometric meaning; they depend on the choice of coordinates. However, given two such lines of slope m_1, m_2, the expression $(m_2 - m_1)/(1 + m_1 m_2)$ does have geometric meaning. This can be demonstrated by directly checking independence of the choice of coordinates—a tedious process—or determining that its value is the tangent of the angle between the lines, a concept which is independent of coordinates! It should be clear that it can be difficult to do geometry, even in the simplest case of Euclidean geometry, working with coordinates alone, that is, with the model R^n. We need to develop both the coordinate method and the coordinate-free method of approach. Thus we shall often seek ways of looking at manifolds and their geometry which do not involve coordinates, but will use coordinates as a useful computational device (and more) when necessary.

However, being aware now of what is involved, we shall usually refer to R^n as Euclidean space and make the identification. This is especially true when we are interested only in questions involving topology—as in the next section—or differentiability.

Exercises

1. Using standard equations for change of Cartesian coordinates, verify that $(m_2 - m_1)/(1 + m_1 m_2)$ is independent of the choice of coordinates.
2. Similarly, show that $((x_2 - x_1)^2 + (y_2 - y_1)^2)^{1/2}$ is a function of points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ which does not depend on the choice of coordinates.
3. How do we describe the subset of R^n which corresponds to a segment $\overline{pq}$ in E^n? to a line? to a 2-plane not through the origin?

If we wish to prove the theorems of Euclidean geometry by analytical geometry methods, we need to define the notion of congruence. We say that two figures are *congruent* if there is a *rigid motion* of the space, that is, an isometry or distance-preserving transformation, which carries one figure to the other.

4. Identifying E^2 with $\boldsymbol{R}^2$, describe analytically the rigid motions of $\boldsymbol{R}^2$. Show that they form a group.
5. Using Exercise 4 prove that two triangles are congruent if and only if corresponding sides are of equal length.

3 Topological Manifolds

Of all the spaces which one studies in topology the Euclidean spaces and their subspaces are the most important. As we have just seen, the metric spaces $\boldsymbol{R}^n$ serve as a *topological model* for Euclidean space E^n, for finite-dimensional vector spaces over $\boldsymbol{R}$ or C, and for other basic mathematical systems which we shall encounter later. It is natural enough that we are led to study those spaces which are *locally* like $\boldsymbol{R}^n$, more precisely those spaces for which each point p has a neighborhood U which is homeomorphic to an open subset U' of $\boldsymbol{R}^n$, n fixed. We say that a space with this property is *locally Euclidean of dimension n*, and in order to stay as close as possible to Euclidean spaces, we will consider spaces called *manifolds*, defined as follows.

(3.1) Definition *A manifold M of dimension n*, or *n-manifold*, is a topological space with the following properties:

(i) M is Hausdorff,
(ii) M is locally Euclidean of dimension n, and
(iii) M has a countable basis of open sets.

As a matter of notation dim M is used for the *dimension* of M; when $\dim M = 0$, then M is a countable space with the discrete topology. It follows from the homeomorphism of U and U' that *locally Euclidean* is equivalent to the requirement that each point p have a neighborhood U homeomorphic to an n-ball in $\boldsymbol{R}^n$. Thus a manifold of dimension 1 is locally homeomorphic to an open interval, a manifold of dimension 2 is locally homeomorphic to an open disk, and so on. Our first examples will remove any lingering suspicion that an n-manifold is necessarily globally equivalent, that is, homeomorphic, to E^n.

(3.2) Example Let M be an open subset of $\boldsymbol{R}^n$ with the subspace topology; then M is an n-manifold.

Indeed properties (i) and (iii) of Definition 3.1 are hereditary, holding for any subspace of a space which possesses them; and we see that (ii) holds with $U = U' = M$ and with the homeomorphism of U to U' being the identity map. A bit of imagination, assisted perhaps by Fig. I.1, will show that even

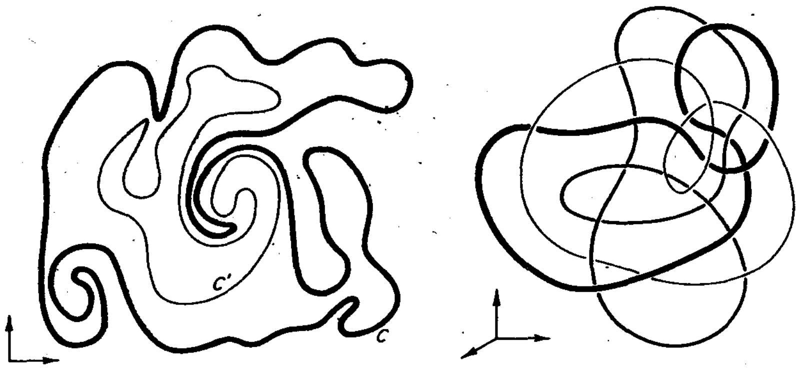

Figure I.1

(a) The manifold is the open set M of $\boldsymbol{R}^2$ between the curves C and C'. (b) The manifold is the open subset of $\boldsymbol{R}^3$ obtained by removing the knots.

when $n = 2$ or 3 these examples can be rather complicated and certainly not equivalent to Euclidean space in general, although they may be in special cases: a trivial such case is $M = E^n$.

(3.3) Example The simplest examples of manifolds not homeomorphic to open subsets of Euclidean space are the circle S^1 and the 2-sphere S^2, which may be defined to be all points of E^2, or of E^3, respectively, which are at unit distance from a fixed point 0.

These are to be taken with the subspace topology so that (i) and (iii) are immediate. To see that they are locally Euclidean we introduce coordinate axes with 0 as origin in the corresponding ambient Euclidean space. Thus in the case of S^2 we identify $\boldsymbol{R}^3$ and E^3, and S^2 becomes the unit sphere centered at the origin. At each point p of S^2 we have a tangent plane and a unit normal vector N_p. There will be a coordinate axis which is *not* perpendicular to N_p and some neighborhood U of p on S^2 will then project in a continuous and one-to-one fashion onto an open set U' of the coordinate plane perpendicular to that axis. In Fig. I.2a, N_p is not perpendicular to the x_2-axis so for $q \in U$, the projection is given quite explicitly by $\varphi(q) = (x^1(q), 0, x^3(q))$, where $(x^1(q), x^2(q), x^3(q))$ are the coordinates of q in E^3. Similar considerations show that S^1 is locally Euclidean. Note that S^2 and R^2 cannot be homeomorphic since one is compact while the other is not.

(3.4) Example Our final example is that of the surface of revolution obtained by revolving a circle around an axis which does not intersect it. The figure we obtain is the *torus* or "inner tube" (denoted T^2) as shown in Fig. I.2b. This figure can be studied analytically; it is easy to write down an