Advanced Calculus for Users

ALAIN ROBERT

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About the Author

A. Robert was born in 1941 and studied in Neuchâtel where he acquired his Ph.D. He visited Paris (1967/68), Princeton University (1968/70), Princeton IAS (1970/71), Rio de Janeiro PUC (1977), Berkeley (1983/84) and Queen's University (Kingston, Ont.) on several occasions. He is thus familiar with the university education system of several countries.

His research activities range from algebra to analysis and he is the author of several books and monographs:

- Elliptic Curves, Lecture Note, Springer-Verlag (2nd ed. 1986);
- Representation Theory, Cambridge University Press (1983);
- Nonstandard Analysis, John Wiley (1988) – (first appeared in French at PPR in 1985).

He is currently full professor at the University of Neuchâtel.

FOREWORD

Mathematics are made to be ***understood***, *enjoyed* and *applied*...

This text grew out of a course that I gave several years consecutively. It contains more material than can be covered in a two semesters course and, in particular, a selection of topics was made each year. It is *applications* oriented and —I hope— will be consulted by users (physicists, electrotechnicians, microtechnicians, engineers, ...). For this last purpose, I have added a few appendices and lists of formulae, even if they are not proved in the text.

I believe that excellent textbooks are now available for the deductive point of view of analysis. Let me only mention

- ▷ W.Rudin "Principles of mathematical analysis" and "Real and complex analysis" (on a higher level),
- ▷ H.Cartan "Cours de calcul différentiel",
- ▷ S.Lang "Analysis I"

(cf. Bibliography at the end of this volume).

My attempt is not to duplicate these, but rather delve into the wealth of applications (including historical ones).

Consequently, even if my formalization (or axiomatization) is not pushed to its maximum, I hope that users will be able to grasp the *meaning* of the mathematical concepts developed. Like proofs, applications can enlighten the comprehension of a mathematical result. An example will illustrate this point. Stokes' theorem can be understood through its proof (via partitions of unity on manifolds, simplices, showing the way to homology...). But in this optic, the Archimedes principle is lost and most modern treatises avoid the *surface element* $d\vec{\sigma}$ and its meaning (thus certainly losing an important part of its applications).

My attempt here has been to recover these classical applications and to present them in an updated fashion.

* * *

The central idea in this calculus course is that of

LINEARIZATION.

It occurs in the notion of derivative (tangent linear map) and differential linear forms (fields of linear forms). In these first two parts, *finite dimensional* vector spaces play the central role (although $\Omega^k(\mathbb{R}^n)$ is already *infinite dimensional*). The third part constitutes an introduction to *functional analysis* and thus, many *infinite dimensional function spaces* are introduced and examined. Convergence (and in particular uniform convergence) for sequences and series of functions is studied more or less systematically : the importance of these concepts in analysis cannot be overemphasized (definition of functions by means of series or parametric integrals, to mention only these). Finally, the fourth part on Fourier series studies the linear operators which associate to a periodic function f its Fourier sequence $(c_k(f))_{k\in\mathbb{Z}}$, and to a sequence $(c_k)_{k\in\mathbb{Z}}$ the series $\sum_{k\in\mathbb{Z}} c_k e^{ikt}$.

Since students were supposed to have already followed a first calculus course and a linear algebra course, I have taken for granted the Cauchy-Schwarz inequality in $\mathbb{R}^n$ (and this, from the first chapter onwards). However this inequality, in the more general context of scalar products, is proved later on (chapter 17). In a similar vein, I have already used the possibility of differentiating a parametric integral in the first part, although the Leibniz rule is only given and proved in chapter 16. I believe that these transgressions to a strict deductive order will not cause any difficulty. My purpose was to make a reasonably short book with many applications of the theory. This forced me to raise the level of sophistication within each part. For example, PART 3 on functional spaces starts quite elementarily with the notion of uniform convergence for continuous functions and ends with a few notions and results for the Lebesgue spaces L^p.

* * *

Let me now explain a few *C O N V E N T I O N S.*

For simplicity, I have adopted a single numbering for all sections. Thus, 8.5 refers to Chap.8, sec.5 (potentials : definitions). Important results are given a special name, and if a section contains several theorems, this name will help in finding which particular result is referred to (my experience shows that three —or four!— figure cross references are awkward and difficult to remember whereas names are more suggestive). Figures are numbered separately.

The symbol ∎ denotes the end of a proof, or the end of a statement whose proof has already been given or whose proof will not be given. The symbol □ indicates the end of a statement whose proof follows.

The term *canonical* is used for algorithmic constructions (independent of arbitrary choices). For example, *the canonical basis* of $\mathbb{R}^n$ is $e_1 = {}^t(1,0,\ldots,0)$, $e_2 = {}^t(0,1,0,\ldots,0)$, ... But there is no canonical basis in the two-dimensional vector space of solutions of the differential equation $y'' + y = 0$. Similarly, the term *intrinsic* refers to a construction or definition which is made independently from choices of bases.

In the statement of a theorem, the numeration i), ii),... is reserved for *equivalent* properties (this is repeated in each case) and thus *different* assertions would be numbered differently e.g. as a), b),... or 1), 2),...

* * *

Finally, it is a pleasure to acknowledge the help that I got during the writing of this book.

Let me thank especially D. Straubhaar and M. Lanz who helped me with first versions, D. Jeandupeux who carefully read the proofs of the final version and suggested a few improvements, and last but not least, my wife Ann who checked my English and corrected many misprints (but I take full responsibility for remaining mistakes...).

Neuchâtel, August 1988 A. Robert

PREREQUISITES AND NOTATIONS

Although it would be difficult to make an exhaustive list of prerequisites for this book (I have already mentioned that the level of sophistication increases somehow inside each part), it may be helpful to list some terminology used throughout the book.

Generalities

injective = one to one into (i.e. $f(x) = f(y) \Longrightarrow x = y$).

${}^{t}A$ denotes the transpose of a matrix, and in particular, if $x \in \mathbb{R}^n$ denotes a column vector, ${}^{t}x = (x_1, \dots, x_n)$ is a row vector with n components and $x = {}^{t}(x_1, \dots, x_n)$. In particular ${}^{tt}A = {}^{t}({}^{t}A) = A$.

$f|_X$: restriction of a map $f : E \longrightarrow F$ to a subset $X \subset E$.

id_E : identity function $E \longrightarrow E$.

If f is a map, $x \longmapsto f(x)$ is the correspondence for elements.

The basic numerical sets are denoted by

$\mathbb{N} = \{0,1,2,3,\dots\}$: natural numbers,

$\mathbb{Z} = \{\dots,-1,0,1,2,\dots\}$ ring of rational integers,

$\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$: fields of rational, resp. real and complex numbers.

$I \times J$ denotes a Cartesian product (e.g. a rectangle in $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$)

sinx, cosx, tan x, cot x = 1/tan x, Arctan x $\in\ ]-\pi/2, \pi/2[$ denote the usual trigonometric functions whereas Shx, Chx, Thx denote the hyperbolic functions (e.g. $Ch^2x - Sh^2x = 1$).

Linear algebra

$\mathbb{R}^n$: Euclidean space of n-tuples of real numbers (column vectors)

$\mathbb{C}^n$: vector space consisting of n-tuples of complex numbers

All vector spaces E to be considered have field of scalars $\mathbb{R}$ or $\mathbb{C}$ (and the context should always make it clear which!)

Form = scalar valued homogeneous function $E \longrightarrow \mathbb{R}$ (or $\mathbb{C}$) (e.g. linear form, quadratic form,...)

E' : dual of E, space of linear forms on E,
E" = (E')' : dual of E', bi-dual of E
$\varepsilon : E \longrightarrow E''$ Dirac evaluation map, $\varepsilon_a(\varphi) = \varphi(a)$

Kronecker symbol $\delta_{ij} = \delta^i_j = \delta^j_i$ (= 0 if $i \neq j$ and = 1 if $i = j$)

$\mathcal{L}\langle a : a \in A \rangle$ linear span of A (in a vector space E)

$\mathcal{L}(E,F) = \mathrm{Hom}(E,F)$: space of linear maps $E \longrightarrow F$,

$M_n(\mathbb{R})$, $M_n(\mathbb{C})$: ring of $n \times n$ matrices with real (resp. $\mathbb{C}$) entries

$G\ell(E) \subset \mathcal{L}(E) = \mathcal{L}(E,E)$: group of invertible linear transformations of E in itself

$G\ell_n(\mathbb{R}) \subset M_n(\mathbb{R})$: group of n by n invertible matrices with real entries (similarly for $G\ell_n(\mathbb{C}) \subset M_n(\mathbb{C})$ for complex entries)

Matrix representation of an operator in a basis : $A = (a^i_j)$

The j^{th} column of A is made up with the components of the j^{th} basis vector : $A(\mathbf{e}_j) = \sum_i a^i_j \mathbf{e}_i$,

$Tr(A) = \sum_i a^i_i$: trace of **A** ,

det(A) : determinant of **A** .

Some familiarity with orthonormal bases, eigenvectors and characteristic vectors (Jordan reduced form) is assumed.

Analysis

Derivative at 0 of a function $f : [0,1] \longrightarrow \mathbf{R}$ means

right derivative (similarly at 1 : *left* derivative)

$o(x^k)$ represents a function f such that $|f(x)/x^k| \longrightarrow 0$ for $x \longrightarrow a$ (and a is given explicitely in each context).

$O(x^k)$ represents a function f such that $|f(x)/x^k|$ remains bounded for $x \longrightarrow a$ (and a is given explicitely in each context).

$f(x) \sim g(x)$ means $f(x)/g(x) \longrightarrow 1$, (for $x \longrightarrow a$ as before)

Topology

We assume that the reader is familiar with the intuitive notions of open and of closed subsets of $\mathbf{R}^n$. Neighborhoods of points and limits of sequences are also used here without comment. A compact set in $\mathbf{R}^n$ is simply a closed and bounded subset : on a compact set, a continuous function always attains a maximum (hence is bounded).

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PART ONE

DIFFERENTIABILITY

CHAPTER 1
VECTOR MAPPINGS

1.1 CONVENTIONS AND NOTATIONS

In this chapter we shall study mappings from a subset $U \subset \mathbb{R}^n$ to some other $\mathbb{R}^m$. The vector spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ will be considered as real vector spaces whose elements are *column vectors*

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = {}^t(x_1, \dots, x_n) \in \mathbb{R}^n \quad (x_i \in \mathbb{R}).$$

(For typographical reasons, we shall often use the row notation preceded by the upper "t" meaning "transpose" instead of the column notation.) Special values of the exponent n (or m) will lead to interesting applications. For example, $n = 3$ leads to the usual physical space $\mathbb{R}^3$ whose elements will more conveniently be denoted by

$$\vec{r} = r = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = {}^t(x,y,z) \in \mathbb{R}^3.$$

Occasionally, we shall even identify a vector r to its extremity P once a fixed origin 0 has been chosen. Thus the components of $r = \overrightarrow{OP}$ are taken as coordinates of the point P.

We shall also have to use row vectors $a = (a_1, \dots, a_n)$ and denote by $\mathbb{R}_n$ their vector space (observe the position of the index n in this vector space). Thus this space $\mathbb{R}_n$ is also a real vector space of dimension n, but its elements have a different representation from those of $\mathbb{R}^n$. We shall identify row vectors $a \in \mathbb{R}_n$ to linear forms on $\mathbb{R}^n$:

$$a : x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \longmapsto (a_1,\dots,a_n)\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = a_1x_1 + \dots + a_nx_n.$$

In other words, we identify the vector space $\mathbb{R}_n$ of row vectors to the *dual* of the vector space $\mathbb{R}^n$ of column vectors.

The vector spaces $\mathbb{R}^n$ will also be endowed with their usual scalar product

$$x \cdot y = x_1y_1 + \dots + x_ny_n \quad (x \text{ and } y \in \mathbb{R}^n).$$

This scalar product gives the *Euclidean structure* of the space $\mathbb{R}^n$. Let us also recall that the length of a vector $x \in \mathbb{R}^n$ is given by its *norm*

$$\|x\| = \sqrt{(x \cdot x)} = (x_1^2 + \dots + x_n^2)^{1/2} \geq 0.$$