Field Theory, the Renormalization Group and Critical Phenomena

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This book was set in IBM Press Roman

British Library Cataloging in Publication Data

Amit, Daniel J
Field theory, the renormalization group,
and critical phenomena.
(International series in pure and applied physics)
1. Critical point – Mathematical models
2. Phase transformations (Statistical physics)
– Mathematical models 3. Field theory (Physics)
I. Title
536'.401 QC307 77-30555

ISBN 0-07-001575-9

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1 2 3 4 5 W.J.M. 8 0 7 9 8

Printed and bound in Great Britain

PREFACE

The idea to write this book was conceived during a graduate course in field theory which I taught at the Hebrew University in Jerusalem 1974–75. The course was a collaboration with Dr. J. Katz. He presented the canonical theory of fields; my part was to show that the approach actually works as a practical tool.

By 1973 Wilson's pioneering work, showing that the renormalization group in field theory is a general technique for dealing with systems with an infinite number of degrees of freedom, had been fully integrated into the beautiful framework developed previously by Symanzik, Čallan, 't Hooft etc. This framework, beyond being aesthetically attractive, proved also, through its unifying power, to be a source of new results. There was then a general feeling that the field was mature and integrated enough to be summarized. This feeling gave rise to the article of Brézin, Le Guillou and Zinn-Justin in Volume VI of the series of Domb and Green.†

With a background in statistical physics the transition to the world of

†See list of General Sources, page xv.

concepts and experience of field theory has proven rather difficult. This explains, perhaps, why the number of workers in statistical physics to adopt the new faith has been increasing rather slowly. Despite the fact that the benefit of crossing the potential barrier is quite significant. Most work in critical phenomena using the techniques of field theory has been done by field theorists – usually oriented towards problems of elementary particles – who learnt the problems facing statistical physics in this domain. In fact, Wilson's original foray into this field had been motivated by the idea that the domain of critical phenomena, formulated as questions in field theory, can serve as a laboratory in which ideas in field theory proper can be tested.

The present book was written as a description of the process undergone by a statistical physicist in converting to the very elegant and efficient new faith, with the intention of demystifying the formalism for many other statistical physicists active in the field, or in becoming such. In brief, one aspect was to try another instance of Kadanoff and Baym,† who so successfully introduced in 1962 the techniques developed by Schwinger, Matsubara, Martin etc. to the many-body problem. This explains, as well as apologizes for, the fact that the number of physical phenomena which are described, and the amount of discussion of the intuitive content, of the results is very limited. These are amply discussed in Toulouse and Pfeuty, Ma, and in Vol. VI of DG. It also accounts for the fact that very little effort has been taken to make the list of references either complete or to give due credits of priority; these can be found in the relevant reviews. On the other hand an effort has been made to stay away from the attractions of rigor, and thus of lengthy proofs. Instead, I have tried to emphasize the other "phenomenology", which resides within the theoretical framework. This is the phenomenology of graphs, of power-counting, of the role of the number of space dimensions, of internal symmetry, of symmetry factors, of ultraviolet vs. infrared, etc. .

But it seems to me that there is much more at stake than a gentle introduction of the methods of field theory to statistical physicists. The impact of Wilson's work has already been called a revolution – and justifiably so. The ushering of a new and powerful insight has by now totally shifted the language, moved the frontiers and revitalized not only the field of critical phenomena – from magnetism, and spin-glasses to polymers to turbulence – but the field of fundamental particle physics as well. The great impetus experienced by the descriptions of nature in terms of gauge fields can hardly be imagined without the discovery of asymptotic freedom. This started a chain of events on whose trail one can discern possibilities such as unifications of theories of different interactions (a canonical dream in physics), quantization of gravity, confinement of quarks etc. – all of which seemed beyond the reach of our conceptual framework only a couple of years ago.

†L. P. Kadanoff and G. Baym, *Quantum Statistical Mechanics* (W. A. Benjamin, N.Y., 1962).

This ambitious new program, of attacking the most difficult problems in statistical physics as well as in particle physics using one single set of concepts, is far from completion. In the process the dialectical cross-fertilization of the two fields is still very much at work. To mention just a couple of examples one can cite the work of Migdal and Polyakov† on extending the mechanism which prevents phase-transitions in systems with a continuous, non-abelian global symmetry in two dimensions to systems with non-abelian local gauge symmetry in four dimensions, and from there to confinement. Likewise, the attempts to calculate the spectrum of the hadrons draws much from the experience in statistical physics once the field theory is placed on a lattice, as was proposed by Wilson,‡ and ideas of solid state physics become useful. Looking in the opposite direction one has learned from field theory how to obtain a systematic expansion of thermodynamic quantitie near two dimensions, as well as the recent extension of Lypatov's work by Brézin, Le Guillou and Zinn-Justin¶ which provides an understanding of the nature of the mysterious series in the number of dimensions in critical phenomena, and thus a tool for an improved exploitation of these series.

All the exciting subjects mentioned above are not treated in the book. They are introduced only to support my feeling that it may still be useful for field theorists to see how the ideas, developed in their domain, operate in the territory in which they enjoyed their most impressive successes. To them the level of field theoretic technology may seem elementary but, I hope, the context may prove stimulating. It is with this in mind that I tried to intermingle the vocabularies of the two fields wherever possible. Anyone familiar with standard texts in field theory can surely skip Chapters 3 to 5.

To conclude the comments on the contents and the intentions of the book I should add that I deliberately refrained from including the most general cases of critical phenomena which can be treated, nor did I try to describe the vast culture of applications to different systems. The treatment of time-dependent problems was also left out. This was done with the view that once the mechanics of the field theoretic formulation of the renormalization group are successfully communicated by a detailed study of a few simple models, the reader will find the literature of reviews and original articles accessible. Many exercises are included to provide the reader with a way to test that his sense of understanding is not merely superficial, or restricted to the model discussed in the text.

As far as my indebtednesses go I owe all I know in this field to Edouard Brézin and to Jean Zinn-Justin. In fact, it would not be unreasonable to consider this book a poor man's version of their review article in the series of Domb and Green. This gigantic debt should by no means implicate them in any of the faults

†A. A. Migdal, *Z.E.T.F.*, 69 1457 (1975); A. M. Polyakov, *Physics Letters*, 59B, 79 (1975).

‡K. G. Wilson, *Physical Review*, D10, 2445 (1974).

¶E. Brézin, J. C. Le Guillou and J. Zinn-Justin, "Perturbation Theory at Large Order," *Physical Review* D15, 1544 and 1558 (1977).

of the book, be they of substance or of presentation. The bulk of the manuscript was written when I spent an unforgettable year at Saclay. During this year I taught this course a second time as a graduate course in theoretical physics at the University of Paris at Orsay. It was during that year that the final form of the manuscript crystallized, and where I learned that most of the material can be covered in about fifty course hours. Teaching this course a second time was indispensible and for the opportunity I am grateful to Prof. Bernard Jancovici and to *the audience.* Beside the hospitality I enjoyed at Saclay, I had many enlightening discussions with Cyrano DeDominicis, with Luca Peliti and with Michael Mirkowich. The latter also suffered through many of the exercises and made many valuable comments on an early version of the manuscript. Back in Jerusalem, Yadin Goldschmidt and Hadassa Jacobson made many comments on the manuscript – reducing the number of errors and clarifying many points.

Finally, the enthusiastic assistance extended to me by the secretarial staff at Saclay, in particular that of Mme. Francine Lefevre and Mme. Madeleine Porneuf went far beyond their professional duties, and its value cannot be exaggerated.

CONTENTS

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1

PERTINENT CONCEPTS AND IDEAS IN THE THEORY OF CRITICAL PHENOMENA

1-1 DESCRIPTION OF CRITICAL PHENOMENA

There is a rich variety of systems which exhibit second orde: phase transitions. By a second order transition we mean one in which the system approaches, continuously, a state at which the scale of correlations becomes unbounded. In the language of field theory, one is approaching a zero mass theory. At such points the first derivatives of the free energy – like entropy, volume, magnetization, etc. – behave continuously. The name phase transition stems probably from the fact that often there is a change in symmetry occurring at the same point – as, for example, in the magnetic cases, or in changes of crystal structures, superconductivity, superfluidity, etc. But the classic second order transition – that of the critical point of the gas–liquid transition – involves no symmetry change at all.

One notices the large scale correlations by observing, for example, critical opalescence at the gas–liquid critical point. The dramatic increase in the scattering of light is a direct result of the fact that regions of the size of microns – the wavelength of visible light – are fluctuating coherently. The divergence of the susceptibility in ferromagnets reflects the long-range nature of correlations, since, as we shall see, the susceptibility is given by the integral of the correlation function (see Sec. 2-5).

Various physical quantities either vanish or diverge as one approaches the transition point. We have mentioned already the correlation length and magnetic susceptibility. Thus, if the correlation function of the fluctuating quantity at two different points is

$$\langle \delta s(x) \delta s(0) \rangle \underset{|x| \to \infty}{\sim} \exp(-x/\xi) \qquad (1\text{-}1)$$

then we define the asymptotic behaviour of the correlation length ξ, as a function of the temperature by:

$$\xi = \begin{matrix} f_+(T - T_c)^{-\nu} & T > T_c \\ f_-(T - T_c)^{-\nu'} & T < T_c \end{matrix} \qquad (1\text{-}2)$$

For magnets $s(x)$ is the local magnetic moment, and δs is its value relative to its average. For antiferromagnets $s(x)$ is the local sum of spins added with a positive sign if on one lattice, and with a negative sign if on the other. For a liquid $s(x)$ is the difference between the local density at x and the mean density at the critical point. For systems like superfluids the fluctuating field is not directly observable. It is a complex order parameter. The behaviour of the correlation length can be deduced using Josephson's[1] relation which states that the exponent of the superfluid density is equal to ν.

But despite the great variety of physical systems one can use a unified language, based on some order-parameter field, whose identification may at times require much ingenuity. This order parameter may have one component, as in the Ising model, or 3 as in the Heisenberg model, or perhaps 18 as in the He^3 superfluid transition.

Within this unified language we define a susceptibility

$$\chi \sim \begin{matrix} C_+ |T - T_c|^{-\gamma} & T > T_c \\ C_- |T - T_c|^{-\gamma'} & T < T_c \end{matrix} \qquad (1\text{-}3)$$

χ is the response of the system – the change in the average order parameter – when an infinitesimal external field which couples linearly to the order parameter, is applied.

Again there may be a variety of situations. In the liquid χ is the compressibility; in the antiferromagnet it is the response to a staggered field; in a superfluid it is not a physical quantity; in a Heisenberg system there may be a susceptibility tensor, with longitudinal and transverse components, etc.

If one is right at the critical temperature, $T = T_c$, then the correlation function generally decreases as a power. Thus we define

$$\langle \delta s(x)\delta s(0)\rangle \underset{|x|\to\infty}{\sim} |x|^{-(d-2+\eta)} \qquad (1\text{-}4)$$

where d is the number of space dimensions.

The specific heat is described asymptotically by:

$$C \sim \begin{matrix} A_+ |T - T_c|^{-\alpha} & T > T_c \\ A_- |T - T_c|^{-\alpha'} & T < T_c \end{matrix} \qquad (1\text{-}5)$$

where $\alpha = 0$ implies at times a discontinuity, and more recently a logarithmic behaviour.

In many systems one can measure a coexistence curve, the term being borrowed from the liquid–gas system, where it describes the thermodynamic subspace in which gas and liquid coexist in equilibrium. In a magnetic system it is the behaviour of the magnetization as a function of temperature at zero external field. One uses the magnetic notation, writing

$$M \sim (T_c - T)^{\beta} \qquad (1\text{-}6)$$

Another exponent is defined by considering the approach to the transition at $T = T_c$, but with external field, $h \neq 0$. The magnetization can be described as

$$M \sim h^{1/\delta} \qquad (1\text{-}7)$$

Similarly, this will describe the approach to the liquid critical point at $T = T_c$, but the pressure $p \neq p_c$.

The various quantities and exponents have to be interpreted anew for every system. We have used the simplest illustrations but the concepts have been applied successfully to a whole variety of transitions – including polymers, percolation problem, liquid crystals, helical magnets, ferroelectrics, etc., . . . and four pages of etc., . . . [2]

1-2 SCALING AND HOMOGENEITY

The idea of scaling, as first conjectured by Widom,[3] consists of writing the asymptotic, sometimes called singular, part of the free energy, or the equation of state as a homogeneous function of the variables. For example, the equation of state describes a relation between the magnetization, the temperature and the magnetic field. In general we could write:

$$h = M^{\delta} f(M, t) \qquad t = |T - T_c| \qquad (1\text{-}8)$$

Instead, Widom proposed that f should depend on a single variable. Thus we can write

$$h = M^{\delta} f(t/M^{1/\beta}) \qquad (1\text{-}9)$$

This type of relation seemed to be obeyed quite well experimentally. It was also verified in various approximations and models, such as the mean-field approximations, the droplet model, and the spherical model. Similarly, one writes for the singular part of the free energy

$$F(t,h) = t^{2-\alpha}\varphi(t/h^{1/\beta\delta}) \qquad (1\text{-}10)$$

and for the correlation function, at $M = h = 0$, for example,

$$G(r,t) = |r|^{-(d-2+\eta)} g(r/t^{-\nu}) \qquad (1\text{-}11)$$

From these homogeneous relations one can derive what are called "scaling laws," or relations among exponents. Thus, one finds easily that:

$$\begin{aligned} 2\beta + \gamma &= 2 - \alpha \\ 2\beta\delta - \gamma &= 2 - \alpha \\ \gamma &= \nu(2 - \eta) \\ \nu d &= 2 - \alpha \end{aligned} \qquad (1\text{-}12)$$

Furthermore, one finds that the exponents are symmetrical about the transition. Consequently, if scaling really holds, one needs to know only two exponents in order to know them all. These relations are exactly obeyed by the two-dimensional Ising model, and by the spherical model. Both experimental results and numerical studies of various models support them strongly. The idea of scaling was injected with a very creative intuitive insight by Kadanoff[4] and formulated by Patashinskii and Pokrovskii.[5] The phenomenological aspect of this idea has been developed extensively by Fisher,[6] and brought to its logical and aesthetic extreme by Griffiths.[7]

1-3 COMPARISON OF VARIOUS RESULTS FOR CRITICAL EXPONENTS

In several cases self-consistent approximations were developed for the description of critical phenomena. For example, the Van der Waals equation of state was devised for the gas–liquid transition. The Weiss theory was developed for ferromagnetism, and the Curie–Weiss theory for antiferromagnetism, etc. Landau, in 1937, unified all theories of this type under what has since been called Landau theory.[8] Basically, all these theories assume that the interacting system can be replaced by a system in an external field, if only that field is properly chosen. A non-interacting system in an external field can be exactly solved, and