NORMED LINEAR SPACES

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Foreword

This book contains a compressed introduction to the study of normed linear spaces and to that part of the theory of linear topological spaces without which the main discussion could not well proceed.

Definitions of many terms which are required in passing can be found in the alphabetical index, page 134. Symbols which are used throughout all, or a significant part, of this book are indexed on page 132. Each reference to the bibliography, page 124, is made by means of the author's name, supplemented when necessary by a number in square brackets. The bibliography does not completely cover the available literature, even the most recent; each paper in it is the subject of a specific reference at some point in the text.

The writer takes this opportunity to express thanks to the University of Illinois, the National Science Foundation, and the University of Washington, each of which has contributed in some degree to the cultural, financial, or physical support of the writer, and to Mr. R. R. Phelps, who eradicated many of the errors with which the manuscript was infested.

Urbana, Illinois (USA), September 1957 — Mahlon M. Day

Contents

Chapter I

Linear spaces

§ 1. Linear Spaces and Linear Dependence

The axioms of a linear or vector space have been chosen to display some of the algebraic properties common to many classes of functions appearing frequently in analysis. Of these examples there is no doubt that the most fundamental, and earliest, examples are furnished by the n-dimensional Euclidean spaces and their vector algebras. Nearly as important, and the basic examples for most of this book, are many function spaces; for example, $C\,[0, 1]$, the space of real-valued continuous functions on the closed unit interval, $BV\,[0, 1]$, the space of functions of bounded variation on the same interval, $L^p\,[0, 1]$, the space of those Lebesgue measurable functions on the same interval which have summable p^{th} powers, and $A\,(D)$, the space of all complex-valued functions analytic in a domain D of the complex plane.

Though all these examples have further noteworthy properties, all share a common algebraic pattern which is axiomatized as follows: (BANACH, p. 26; JACOBSON).

Definition 1. A *linear space* L over a field Λ is a set of elements satisfying the following conditions:

(A) The set L is an Abelian group under an operation $+$; that is, $+$ is defined from $L \times L$ into L such that, for every x, y, z in L,

(a) $x + y = y + x,$ (commutativity)

(b) $x + (y + z) = (x + y) + z,$ (associativity)

(c) there is a w dependent on x and y such that $x + w = y$.

(B) There is an operation defined from $\Lambda \times L$ into L, symbolized by juxtaposition, such that, for λ, μ in Λ and x, y in L,

(d) $\lambda\,(x + y) = \lambda x + \lambda y,$ (distributivity)

(e) $(\lambda + \mu)\,x = \lambda x + \mu x,$ (distributivity)

(f) $\lambda(\mu x) = (\lambda\mu)\,x\,,$

(g) $1\,x = x$ (where 1 is the identity element of the field).

In this and the next section any field will do; in the rest of the book order and distance are important, so the real field R is used throughout, with remarks about the complex case when that field can be used instead.

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(1) If L is a linear space, then (a) there is a unique element 0 in L such that $x + 0 = 0 + x = x$ and $\mu 0 = 0x = 0$ for all μ in Λ and x in L; (b) $\mu x = 0$ if and only if $\mu = 0$ or $x = 0$; (c) for each x in L there is a unique y in L such that $x + y = y + x = 0$ and $(-1)x = y$; (then for z, x in L define $z - x = z + (-1)x$ and $-x = 0 - x$).

(2) It can be shown by induction on the number of terms that the commutative, associative and distributive laws hold for arbitrarily large finite sets of elements; for example, $\sum_{i \leq n} x_i$, which is defined to be $x_1 + (x_2 + (\cdots + x_n) \cdots)$, is independent of the order or grouping of terms in the process of addition.

Definition 2. A non-empty subset L' is called a *linear subspace* of L if L' is itself a linear space when the operations used in L' are those induced by the operations in L. If $x \neq y$, the *line through x and y* is the set $\{\mu x + (1-\mu) y : \mu \in \Lambda\}$. A non-empty subset E of L is *flat* if with each pair $x \neq y$ of its points E also contains the line through x and y.

(3) L' is a linear subspace of L if and only if for each x, y in L' and each λ in Λ, $x + y$ and λx are in L'.

Definition 3. If $E, F \subseteq L$ and $z \in L$, define

$E + F = \{x + y : x \in E \text{ and } y \in F\}$. $\quad -E = \{-x : x \in E\}$,

$E + z = \{x + z : x \in E\}$, $\quad E - z = E + (-z)$, $E - F = E + (-F)$

(4) (a) E is flat if and only if for each x in E the set $E - x$ is a linear subspace of L. (b) The intersection of any family of linear [flat] subsets of L is linear [either empty or flat]. (c) Hence each non-empty subset E of L is contained in a smallest linear [flat] subset of L, called the *linear* [*flat*] *hull* of E.

Definition 4. If L is a linear space and $x_1, \ldots, x_n$ are points of L, a point x is a *linear combination* of these x_i if there exist $\lambda_1, \ldots, \lambda_n$ in Λ such that $x = \sum_{i \leq n} \lambda_i x_i$. A set of points $E \subseteq L$ is called *linearly independent* if E is not θ or $\{0\}$ and[1] no point of E is a linear combination of any finite subset of the other points of E. A *vector basis* (or Hamel basis) in L is a maximal linearly independent set.

(5) (a) The set of all linear combinations of all finite subsets of a set E in L is the linear hull of E. (b) E is linearly independent if and only if for $x_1, \ldots, x_n$ distinct elements of E and $\lambda_1, \ldots, \lambda_n$ in Λ the condition $\sum_{i \leq n} \lambda_i x_i = 0$ implies that $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$.

Theorem 1. If E is a linearly independent set in L, then there is a vector basis B of L such that $B \supseteq E$.

Proof. Let $\mathfrak{S}$ be the set of all linearly independent subsets S of L such that $E \subseteq S$; let $S_1 \geq S_2$ mean that $S_1 \supseteq S_2$. Then if $\mathfrak{S}_0$ is a simply ordered subsystem of $\mathfrak{S}$ and S_0 is the union of all S in $\mathfrak{S}_0$, S_0 is also a

[1] θ is the empty set; $\{x\}$ is the set containing the single element x.

linearly independent set; indeed, $x_1, \ldots, x_n$ in S_0 imply that there exist S_i in $\mathfrak{S}_0$ with x_i in S_i. Since $\mathfrak{S}_0$ is simply ordered by inclusion, all x_i belong to the largest S_j and are, therefore, linearly independent. Hence $S_0 \in \mathfrak{S}$ and is an upper bound for $\mathfrak{S}_0$. Zorn's lemma now applies to assert that E is contained in a maximal element B of $\mathfrak{S}$. This B is the desired vector basis, for it is a linearly independent set and no linearly independent set is larger.

Corollary 1. If L_0 is a linear subspace of L and B_0 is a vector basis for L_0, then L has a vector basis $B \supseteq B_0$.

(6) If $B = \{x_s : s \in S\}$ is a vector basis in L, each x in L has a representation $x = \sum_{s \in \sigma} \lambda_s x_s$, where σ is a finite subset of S. If $x = \sum_{s \in \sigma_1} \lambda_s x_s = \sum_{s \in \sigma_2} \mu_s x_s$, then $\lambda_s = \mu_s$ for all s in $\sigma_1 \cap \sigma_2$ and $\lambda_s = 0$ for all other s in σ_1 and $\mu_s = 0$ for all other s in σ_2. Hence each $x \neq 0$ has a unique representation in which all coefficients are non-zero, and 0 has no representation in which any coefficient is non-zero. [Also see § 2, (2c).] This property characterizes bases among subsets of L.

Theorem 2. Any two vector bases S and T of a linear space L have the same cardinal number.

Proof. Symmetry of our assumptions and the Schroeder-Bernstein theorem on comparability of cardinals (KELLEY, p. 28) show that it suffices to prove that S can be matched with a subset of T. Consider the transitively ordered system of functions Φ consisting of those functions φ such that (a) the domain $D_\varphi \subseteq S$ and the range $R_\varphi \subseteq T$. (b) φ is one-to-one between D_φ and R_φ. (c) $R_\varphi \cup (S \setminus D_\varphi)$ is a linearly independent set. Order Φ by: $\varphi \geqq \varphi'$ means that φ is an extension of φ'.

Every simply ordered subsystem Φ_0 of Φ has an upper bound φ_0: Define $D_{\varphi_0} = \bigcup_{\varphi \in \Phi_0} D_\varphi$ and $\varphi_0(s) = \varphi(s)$ if $s \in D_\varphi$ and $\varphi \in \Phi_0$. This φ_0 is defined and is in Φ; it is an upper bound for Φ_0. By Zorn's lemma there is a maximal φ in Φ. We wish to show that $D_\varphi = S$.

If not, then $R_\varphi \neq T$, for each s in the complement of D_φ is dependent on T but not on R_φ. If t_0 is in $T \setminus R_\varphi$, either t_0 is linearly independent of $R_\varphi \cup (S \setminus D_\varphi)$ or is dependent on it. In the former case, for arbitrary s_0 in $S \setminus D_\varphi$ the extension φ' of φ for which $\varphi'(s_0) = t_0$ has the properties (a), (b), and (c), so φ is not maximal. In the latter case, by (c) and (6)

$$t_0 = \sum_{t \in R_\varphi} \lambda_t t + \sum_{s \notin D_\varphi} \mu_s s ,$$

where at least one μ_{s_0} is not zero, because t_0 is independent of R_φ. If φ' is the extension of φ for which $\varphi'(s_0) = t_0$, then φ' obviously satisfies (a) and (b); also $R_{\varphi'} \cup (S \setminus D_{\varphi'})$ is linearly independent, because otherwise t_0 would depend on $R_\varphi \cup (S \setminus D_{\varphi'})$, a possibility prevented by the choice of s_0, and again φ cannot be maximal.

This shows that if φ is maximal in Φ, then $D_\varphi = S$; then the cardinal number of S is not greater than that of T. The Schroeder-Bernstein theorem completes the proof of the theorem.

Definition 5. The cardinal number of a vector basis of L is called the *dimension* of L.

The linear space with no element but 0 is the only linear space with an empty vector basis; it is the unique linear space of dimension 0.

(7) (a) If K is the complex field and if L is a vector space over K, then L is also a vector space, which we shall call $L_{(r)}$, over the real field R. (b) The dimension of $L_{(r)}$ is twice that of L, for x and $i x$ are linearly independent in $L_{(r)}$.

§ 2. Linear Functions and Conjugate Spaces

In this section again the nature of the field of scalars is unimportant.

Definition 1. If L and L' are linear spaces over the same field Λ, a function F (sometimes to be called an *operator*) from L into L' is called *additive* if $F(x+y) = F(x) + F(y)$ for all x, y in L; *homogenous* if $F(\lambda x) = \lambda F(x)$ for all λ in Λ and x in L; *linear* if both additive and homogeneous. An one-to-one linear F carrying L onto L' is an *isomorphism* of L and L'.

(1) (a) Let B be a vector basis of L and for each b in B let y_b be a point of the linear space L'. Then there is a unique linear function F from L into L' such that $F(b) = y_b$ for all b in B; precisely, using § 1, (6),

$$F\left(\sum_{b\in\sigma} \lambda_b b\right) = \sum_{b\in\sigma} \lambda_b y_b .$$

(b) If T_0 is a linear function defined from a linear subspace L_0 of L into a linear space L', there is an extension T of T_0 defined from L into L'. (c) T is called *idempotent* if $T T x = T x$ for all x in L. If L_0 is a linear subspace of L, there is an idempotent linear function (a *projection*) P from L onto L_0. (d) There is an isomorphism between L and L' if and only if these spaces have the same dimension.

Linear extension problems are much simplified by the basis theorems.

Lemma 1. Let L and L' be linear spaces over Λ and let X be a subset of L, and let f be a function from X into L'. Then there is a linear function F from L into L' such that F is an extension of f if and only if whenever a linear combination of elements of X vanishes, then the same linear combination of the corresponding values of f also vanishes; i. e., if $\sum_i \lambda_i x_i = 0$, then $\sum_i \lambda_i f(x_i) = 0$.

Proof. The necessity is an immediate consequence of the linearity of F. If the condition holds, define g at any point $y = \sum_i \lambda_i x_i$ in L_0, the linear hull of X, by $g(y) = \sum_i \lambda_i f(x_i)$. If also $y = \sum_j \lambda'_j x'_j$, then

$\sum_i \lambda_i x_i - \sum_j \lambda'_j x'_j = 0$ so $\sum_i \lambda_i f(x_i) - \sum_j \lambda'_j f(x'_j) = 0$, and $g(y)$ is determined by y, not by its representations in terms of X. This shows at once that g is linear on L_0; (1b) asserts that g has a linear extension F.

Definition 2. (a) If L is a linear space, then $L\#$, *the conjugate space of* L, is the set of all linear functions from L into the field Λ. (b) Let S be a non-empty set of indices and for each s in S let L_s be a linear space over Λ. Let $\prod_{s\in S} L_s$ be the set of all functions x on S such that $x(s) \in L_s$ for all s in S; let $\sum_{s\in S} L_s$ be the subset of $\prod_{s\in S} L_s$ consisting of those functions x for which $\{s: x(s) \neq 0\}$ is finite. Then these function spaces are linear spaces under the definitions

$$(x+y)(s) = x(s) + y(s) \quad \text{and} \quad (\lambda x)(s) = \lambda(x(s))$$

for all x, y and all λ. They are called, respectively, the *direct product* and *direct sum* of the spaces L_s. (c) L^S is the special direct product in which all $L_s = L$.

(2) (a) $L\#$ is a linear subspace of Λ^L; hence $L\#$ is a linear space. (b) $(\sum_{s\in S} L_s)\#$ is isomorphic to $\prod_{s\in S}(L_s\#)$. (c) If $\{x_s: s \in S\}$ is a basis for L and if for each s in S, f_s is the unique element of $L\#$ such that $f_s(x_s) = 1$, $f_s(x_{s'}) = 0$ if $s' \neq s$, then for each x in L, $\sigma_x = \{s: f_s(x) \neq 0\}$ is a finite subset of S and for every non-empty $\sigma \supseteq \sigma_x$, $x = \sum_{s\in\sigma} f_s(x)\, x_s$. (d) If $\{x_s: s \in S\}$ is a basis in L, then L is isomorphic to $\sum_{s\in S} L_s$, where each $L_s = \Lambda$, and $L\#$ is isomorphic to Λ^S. (e) If x_i, $1 \leq i \leq n$, are linearly independent elements of L and if λ_i, $1 \leq i \leq n$, are in Λ, then there exists f in $L\#$ such that $f(x_i) = \lambda_i$, $1 \leq i \leq n$.

Definition 3. A *hyperplane* H in L is a maximal flat proper subset of L, that is, H is flat, and if $H' \supseteq H$ and H' is flat, then $H' = H$ or $H' = L$.

(3) (a) H is a hyperplane in L if and only if H is a translation $x + L_0$ of a maximal linear proper subspace L_0 of L. (b) If $f \in L\#$, if f is not 0, and if $\lambda \in \Lambda$, then $\{x: f(x) = \lambda\}$ is a hyperplane in L. (c) For each hyperplane H in L there is an $f \neq 0$, $f \in L\#$, and a λ in Λ such that $H = \{x \in L: f(x) = \lambda\}$; H is linear if and only if $\lambda = 0$. (d) If the hyperplane $H = \{x: f_1(x) = \lambda_1\} = \{x: f_2(x) = \lambda_2\}$, then there exists $\mu \neq 0$ in Λ such that $f_1 = \mu f_2$ and $\lambda_1 = \mu\lambda_2$.

Definition 4. If L_0 is a linear subspace of L, define a vector structure on the *factor space* L/L_0 of all translates, $x + L_0$, of L_0 as follows: If X and Y are translates of L_0 define $X + Y$ as in § 1, Def. 3 to be $\{x + y: x \in X \text{ and } y \in Y\}$; define λX to be $\{\lambda x: x \in X\}$ if $\lambda \neq 0$, $0X = L_0$. Let T_0 be the function carrying x in L to $x + L_0$ in L/L_0.

Theorem 1. L/L_0 is a vector space and T_0 is a linear function from L onto L/L_0.

Proof. If $x \in X$ and $y \in Y$, then $X = T_0 x = x + L_0$ and $Y = T_0 y = y + L_0$. Hence $X + Y = \{x + y + u + v : u, v \in L_0\} = \{x + y + w : w \in L_0\} = (x + y) + L_0 = T_0(x + y)$. Hence $X + Y \in L/L_0$, and T_0 is additive. Similarly $X = X + L_0$, so L_0 is the zero element of L/L_0. If $\lambda \neq 0$, then $\lambda X = \lambda T_0 x = \lambda \{x + u : u \in L_0\} = \{\lambda x + \lambda u : u \in L_0\} = \{\lambda x + v : v \in L_0\} = T_0(\lambda x)$, so $\lambda X \in L/L_0$ and $T_0(\lambda x) = \lambda T_0(x)$. If $\lambda = 0$, $0X = L_0 = T_0(0) = T_0(0x)$, so T_0 is homogeneous. Associativity, distributivity, and so on, are easily checked.

Next we improve the result of (2e).

Definition 5. A subset Γ of $L^{\#}$ is called *total over* L if $f(x) = 0$ for all f in Γ implies that $x = 0$.

Theorem 2. Let Γ be a linear subspace of $L^{\#}$ which is total over L and let x_i, $i = 1, \ldots, n$, be linearly independent elements of L; then there exist elements f_i, $i = 1, \ldots, n$, in Γ such that $f_i(x_j) = \delta_{ij}$ (Kronecker's delta) for $i, j = 1, \ldots, n$.

Proof. To prove this by induction on n, begin with $n = 1$. If x_1 is a linearly independent set, then $x_1 \neq 0$; hence, by totality there is an f in Γ with $f(x_1) \neq 0$; set $f_1 = f/f(x_1)$.

Assume the result true for $n - 1$ and let $x_1, \ldots, x_n$ be independent. Then there exist $f_1', \ldots, f_{n-1}'$ such that $f_i'(x_j) = \delta_{ij}$ for $i, j = 1, \ldots, n-1$. Let T map Γ into Λ^n by $(Tf)_j = f(x_j)$, $j = 1, \ldots, n$. We wish to show T is onto Γ^n, so we suppose, for a contradiction, that Tf is linearly dependent on the Tf_i, $i < n$, for all f in Γ. Then $Tf = \sum_{i<n} \lambda_i T f_i$ so $(Tf)_j = \sum_{i<n} \lambda_i f_i(x_j)$ for $j < n$. Then for $j < n$, $f(x_j) = (Tf)_j = \lambda_j$, so $f(x_n) = (Tf)_n = \sum_{i<n} f(x_i)\, f_i(x_n) = f(\sum_{i<n} f_i(x_n)\, x_i)$. This yields after subtraction that $f(x_n - \sum_{i<n} f_i(x_n)\, x_i) = 0$ for all f in Γ; this in turn implies that $x_n = \sum_{i<n} f_i(x_n)\, x_i$, a contradiction with linear independence of the x_i. Hence there is an f in Γ such that Tf is independent of the Tf_i, $i < n$; let $f' = f - \sum_{i<n} f(x_i)\, f_i'$ so $(Tf')_j = 0$ if $j < n$; let $f_n = f'/f'(x_n)$. Finally for $i < n$ let $f_i = f_i' - f_i'(x_n)\, f_n$. Then $f_i(x_j) = \delta_{ij}$ for $i, j \leq n$.

Corollary 1. (Solution of equations). If x_i, $i = 1, \ldots, n$, are linearly independent in L and if Γ is a linear subspace of $L^{\#}$ which is total over L, and if λ_i, $i = 1, \ldots, n$, $\in \Lambda$, then there exists f in Γ such that $f(x_i) = \lambda_i$, $i = 1, \ldots, n$.

Proof. Set $f = \sum_{i \leq n} \lambda_i f_i$, where the f_i satisfy the conclusion of the theorem.

Corollary 2. If $f_1, \ldots, f_n$ are linearly independent elements of $L^{\#}$, then there exists $x_1, \ldots, x_n$ in L such that $f_i(x_j) = \delta_{ij}$, and if $c_1, \ldots, c_n \in \Lambda$, there is an x in L such that $f_i(x) = c_i$ for $i, j, = 1, \ldots, n$.

Proof. Define Q from L into $(L^{\#})^{\#}$ by $Qx(f) = f(x)$ for all f in $L^{\#}$. Then $Q(L)$ is total over $L^{\#}$ and Theorem 2 can be applied.

Definition 6. If E is a subset of L, define $E^{\perp} = \{f \in L^{\#}: f(x) = 0$ for all x in $E\}$. If Γ is a subset of $L^{\#}$, define $\Gamma_{\perp} = \{x \in L: f(x) = 0$ for all f in $\Gamma\}$.

Corollary 3. Let φ be a finite subset of $L^{\#}$. Then the deficiency of $\varphi_{\perp}$, that is, the dimension of $L/\varphi_{\perp}$, is the number of elements in a maximal linearly independent subset of φ. Hence $(\varphi_{\perp})^{\perp}$ is the smallest linear subspace of $L^{\#}$ containing φ.

Proof. If $f_1, \ldots, f_n$ is a maximal linearly independent set in φ, take $x_1, \ldots, x_n$ by Cor. 2. Then if x is in L, $x = \sum_{i \leq n} f_i(x)\, x_i$ is in $\varphi_{\perp}$. Hence the dimension of $L/\varphi_{\perp}$ is not greater than the number n of elements x_i. But if $X_i = x_i + \varphi_{\perp}$ and if $\sum_i t_i X_i = 0$, then $x = \sum_i t_i x_i$ and $t_i = f_i(x) = 0$; hence the X_i are linearly independent and $L/\varphi_{\perp}$ has dimension precisely n.

If f vanishes on $\varphi_{\perp}$, let $c_i = f(x_i)$ and $g = f - \sum_i c_i f_i$. Then $g - f \in \varphi_{\perp}{}^{\perp}$ and $(g - f)(x_i) = 0$, so $g - f = 0$. Hence $f = g = \sum_i c_i f_i$.

Corollary 3'. Dually, if φ is a finite subset of L and if Γ is a linear subspace of $L^{\#}$ which is total over L, then the deficiency of $\varphi^{\perp} \cap \Gamma$ in Γ is the number of elements in a maximal linearly independent subset of φ. Hence $(\varphi^{\perp} \cap \Gamma)_{\perp}$ is the smallest linear subspace of L containing φ.

Corollary 4. If $H_i = \{x: f_i(x) = c_i\}$, where the f_i are linearly independent elements of $L^{\#}$, and if $H = \{x: f(x) = c\}$ contains $\bigcap_{i \leq n} H_i$, then there exists numbers t_i such that $f = \sum_{i \leq n} t_i f_i$ and $c = \sum_{i \leq n} t_i c_i$.

Proof. The equations $f_i(x) = c_i$, $f(x) = c + 1$, are inconsistent; by Corollary 1 the functions $f, f_1, \ldots, f_n$ can not be linearly independent.

(4) Let T be a linear function from a linear space L into a linear space L'; let $L_0 = T^{-1}(0)$, let $L_1 = T(L)$, and let T_0 be the natural linear map of L onto L/L_0. Then L_0 and L_1 are linear subspaces of L and L', respectively, and there is an isomorphism T_1 of L/L_0 onto L_1, defined by $T_1(x + L_0) = Tx$, such that $Tx = T_1 T_0 x$ for every x in L.

Definition 7. If T is a linear function from one linear space L into another such space L', define $T^{\#}$, the *dual function* of T, for each f' in $L'^{\#}$ by $T^{\#} f'(x) = f'(Tx)$ for each x in L.

(5) (a) For each f' in $L'^{\#}$ the function $T^{\#} f'$ is in $L^{\#}$. (b) $T^{\#}$ is a linear function from $L'^{\#}$ into $L^{\#}$. (c) $T^{\#-1}(0) = T(L)^{\perp}$, so $T^{\#}$ is an isomorphism of $L'^{\#}$ into $L^{\#}$ if and only if T carries L onto L'. (d) $T^{\#}(L'^{\#}) = T^{-1}(0)^{\perp}$, so $T^{\#}$ carries $L'^{\#}$ onto $L^{\#}$ if and only if T is an isomorphism of L into L'.

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Theorem 3. If L_0 is a linear subspace of L, then $L_0^{\#}$ is naturally isomorphic to $L^{\#}/L_0^{\perp}$ and $(L/L_0)^{\#}$ is naturally isomorphic to $L_0^{\perp}$.

Proof. If i is the identity isomorphism of L_0 into L, then by (5) $U_0 = i^{\#}$ carries $L^{\#}$ onto $L_0^{\#}$ and $U_0^{\#-1}(0) = L_0^{\perp}$. $L_0^{\#}$ and $L^{\#}/L_0^{\perp}$ are isomorphic under the U_1 associated by (4) with U_0. If T_0 is the usual mapping of L onto L/L_0, then by (5) $T_0^{\#}$ is an isomorphism of $(L/L_0)^{\#}$ onto $L_0^{\perp}$.

(6) Let L be a linear space over the complex field K and for each f in $L^{\#}$ let $f = g + i\,h$, where g and h are real-valued functions, the *real* and *imaginary parts* of f. Then: (a) If $f \in L^{\#}$, then g and $h \in L_{(r)}{}^{\#}$. (See definition in § 1, (7).) (b) $h(x) = -g(i\,x)$. (c) The correspondence between f and g is an isomorphism of $(L^{\#})_{(r)}$ and $(L_{(r)})^{\#}$.

(7) L and $L^{\#}$ (or any total linear subset Γ of $L^{\#}$) give examples of linear spaces in duality. L and M are said to be *dual linear spaces* if there is a bilinear functional $\langle\, , \rangle$ defined on $L \times M$ such that for each $x \neq 0$ in L there is a y in M such that $\langle x, y\rangle \neq 0$, and the dual condition with L and M interchanged. Then (a) If T is defined on L by $T x(y) = \langle x, y\rangle$ for all y in M, then each $T x \in M^{\#}$ and the range $T(L)$ is total over M. (b) Dually, if $U y(x) = \langle x, y\rangle$ for all x in L, then $U y \in L^{\#}$ and $U(M)$ is total over L. (c) U is dual to T in the sense that $T x(y) = U y(x)$ for all x in L and y in M.

(8) Extension problems will recur again and again throughout this book. It will pay perhaps to see how simple linear extension problems are, due to the basis theorem. Let X be a linear subspace of Y, let Z be another linear space, and let f_0 be a linear function from X into Z; the problem is to find an extension f of f_0 defined and linear from Y into Z. The question for linear functions is answered by (1 b), but further restrictions on the functions may make the problem insuperably difficult. It is to be noted for later use that there are several problems here, all equivalent in this linear case. These are: (a) The *"from" extension problem* in which X is fixed and Y and Z arbitrary. (b) The *"into" extension problem* in which Z is fixed and X, Y arbitrary. (c) The *projection problem* in which $X = Z$ and f_0 is the identity. Another problem which turns out ultimately to be distinct from these in most circumstances, is (d) the *subspace projection problem* in which Y is fixed, and $X = Z$ ranges over all subspaces of Y. The Hahn-Banach theorem of the next section solves an "into" extension problem: in it the range space is the reals, and, in addition to linearity, the functions are required to satisfy a domination condition.

Extension problems are considered in detail in V, § 4, VI, § 3, and VII, § 3.

§ 3. The Hahn-Banach Extension Theorem

Now we wish to consider convexity and order, so the real field R is assumed hereafter; an occasional application to complex fields is noted. A *functional* is a function with its values in the scalar field.

Definition 1. A functional p defined on a linear space L is *subadditive* if $p(x+y) \leqq p(x) + p(y)$ for all x, y in L; p is *positive-homogeneous* if $p(rx) = r\,p(x)$ for each $r > 0$ and each x in L; p is *sublinear* if it has both the above properties. A sublinear functional p is a *pre-norm* if $p(\lambda x) = |\lambda|\, p(x)$ for all λ in the field of scalars. A pre-norm p is a *norm* if $p(x) = 0$ if and only if $x = 0$.

(1) (a) If p is a sublinear functional, then $p(0) = 0$ and $-p(-x) \leqq p(x)$. (b) If p is a pre-norm in L, then $p(x) \geqq 0$ for all x in L and $\{x\colon p(x) = 0\}$ is a linear subspace of L.

(2) Let S be a set, let $m(S)$ be the set of all bounded real-valued functions on S, let $p_1(x) = \sup\,\{x(s)\colon s \in S\}$, and let $p_2(x) = \sup\,\{|x(s)|\colon s \in S\}$. Then $m(S)$ is a linear space, p_1 is a sublinear functional in $m(S)$, and p_2 is a norm in $m(S)$. For each s_0 in S, $p_{s_0}(X) = |x(s_0)|$ is a pre-norm in $m(S)$.

Theorem 1. (Hahn-Banach Theorem). Let p be a sublinear functional on L, let L_0 be a linear subspace of L, and let f_0 be an element of $L_0^{\#}$ which is dominated by p; that is, $f_0(x) \leqq p(x)$ for all x in L_0; then f_0 has an extension f in $L^{\#}$ which is also dominated by p.

Proof. First we prove that f_0 has a maximal extension dominated by p. Let $\mathfrak{F}$ be the family of all linear functionals f' defined on linear subspaces L' of L such that $L_0 \subseteq L' \subseteq L$ and f' is an extension of f dominated on L' by p. Define $f' \geqq f''$ to mean that f' is an extension of f''. Then $\mathfrak{F}$ is transitively ordered, and each simply-ordered subfamily $\mathfrak{F}_0$ of $\mathfrak{F}$ has an upper bound, the f defined on the union of the domains of the f' in $\mathfrak{F}_0$ to agree with each such f' in its domain. By Zorn's lemma (Kelley, p. 33) there is a maximal f in $\mathfrak{F}$; to show that this extension has all of the properties desired, it suffices to show that its domain of definition is L. Assume then that an f' in $\mathfrak{F}$ is defined on a proper subspace L' of L; we show it can be extended.

Take z not in L' and, to discover the restrictions on any possible extension, take x, y, in L'. Then

$$\begin{aligned} f'(x) - f'(y) &= f'(x-y) \leqq p(x-y) \\ &= p(x + z + (-y - z)) \leqq p(x+z) + p(-y-z), \end{aligned}$$

so

$$-p(-y-z) - f'(y) \leqq p(x+z) - f'(x)\,.$$

It follows that

$$\sup\,\{-p(-y-z) - f'(y);\, y \in L'\} \leqq \inf\,\{p(x+z) - f'(x);\, x \in L'\}\,;$$

let c be any real number between these two.

In the linear space $L_1 = \{x + rz\colon x \in L'$ and r real$\}$ define f_1 by $f_1(x + rz) = f'(x) + rc$. Since each point w in L_1 determines its x and r uniquely and linearly, this defines f_1 in $L_1^{\#}$. To show f_1 dominated by p

take $w = x + rz$. If $r = 0$, $f_1(w) = f'(x) \leq p(x) = p(w)$. If $r \neq 0$, by the choice of c we have for every y in L' that

$$-p(-y-z) - f'(y) \leq c \leq p(y+z) - f'(y).$$

Set $y = x/r$, then

$$-p(-x/r - z) - f'(x/r) \leq c \leq p(x/r + z) - f'(x/r).$$

Multiply by r and use the right (left) half of this if $r > 0$ (if $r < 0$); then

$$r c \leq p(x + r z) - f'(x)$$

or

$$f_1(w) = f'(x) + r c \leq p(w).$$

This extension shows f' not maximal if $L' \neq L$. Hence every maximal dominated extension of f_0 satisfies the conclusion of the theorem.

Corollary 1. If p is sublinear on L and $x_0 \in L$, then there is an $f \in L^{\#}$ such that $f(x) \leq p(x)$ for all x in L and $f(x_0) = p(x_0)$.

Proof. Take $L_0 = \{r x_0 \colon r \text{ real}\}$ and $f_0(r x_0) = r p(x_0)$.

Definition 2. The *core* of a subset E of L is the set $\{x\colon$ for each y in L there is an $\varepsilon(y) > 0$ such that $x + ty \in E$ if $|t| < \varepsilon(y)\}$.

Geometrically speaking, this means that every line through x meets E in a set containing an interval (disc in the complex case) about x.

Definition 3. If $x, y \in L$, the *line segment* between them is the set $\{tx + (1-t)y\colon 0 \leq t \leq 1\}$. A non-empty set E in L is *convex* if for each pair of points in E the segment between them is in E. An *open segment* is a line segment minus its end-points.

(3) (a) The intersection of a family of convex sets is either empty or convex. (b) Hence each non-empty subset of a linear space L is contained in a smallest convex set, $k(E)$, *the convex hull of* E. (c) If E is a non-empty subset of L, then $k(E) = \left\{\sum_{i \leq n} t_i x_i\colon n = 1, 2, \ldots, x_i \in E,\right.$ $\left. t_i \geq 0, \text{ and } \sum_{i \leq n} t_i = 1\right\}$.

(4) Say that a set E *lies on one side of a hyperplane* H if $k(E \setminus H)$ does not intersect H. When $f \in L^{\#}$, $f \neq 0$, and $H = \{x\colon f(x) = c\}$, then E lies on one side of H if and only if $f(x) - c$ does not change sign in E; that is, if and only if E lies in one of the two *half-spaces* $\{x\colon f(x) \leq c\}$ and $\{x\colon f(x) \geq c\}$.

Lemma 1. If p is a sublinear functional on L, if k is a positive number, and if $E = \{x\colon p(x) \leq k\}$, then E is convex and the core of E is $\{x\colon p(x) < k\}$; hence 0 is a core point of E.

Proof. If $x, y \in E$, then $p(tx + (1-t)y) \leq t p(x) + (1-t) p(y) \leq k$, so the segment from x to y is in E. If $p(x) < k$ and $y \in L$, $p(x + ty) \leq p(x) + t p(y)$. If $p(y) = 0 = p(-y)$, $\varepsilon(y)$ is arbitrary; otherwise take $\varepsilon(y) = (k - p(x))/\max[p(y), p(-y)]$.

Definition 3. Let E be a set with 0 in its core; then the ***Minkowski functional*** p_E is defined for each x in L by

$$p_E(x) = \inf\{r: x/r \in E \text{ and } r > 0\}.$$

Lemma 2. If E is convex and 0 is a core point of E, then p_E, the Minkowski functional of E, is non-negative and sublinear, and p_E is a pre-norm if and only if $rE \subseteq E$ whenever $|r| < 1$.

Proof. For each x, $x/r \in E$ if r is large enough, so $p_E(x)$, the inf of a non-empty set of positive numbers, is non-negative and finite. If $y = t\,x$, $t > 0$, then

$$\begin{aligned} p_E(y) &= \inf\{r > 0: y/r \in E\} = \inf\{r > 0: t\,x/r \in E\} \\ &= \inf\{tr' > 0: x/r' \in E\} = t \inf\{r' > 0: x/r' \in E\} = t\,p_E(x). \end{aligned}$$

If $x_1, x_2 \in L$, take $\varepsilon > 0$ and choose r_i so that $p_E(x_i) < r_i < p_E(x_i) + \varepsilon$; then $x_i/r_i \in E$. Set $r = r_1 + r_2$, then $(x + y)/r = (r_1/r)(x_1/r_1) + (r_2/r)(x_2/r_2)$ is on the segment between x_1/r_1 and x_2/r_2; by convexity, $(x + y)/r$ is in E; hence $p_E(x + y) \leq r = r_1 + r_2 < p_E(x_1) + p_E(x_2) + 2\,\varepsilon$. Letting ε tend to 0 shows that p_E is subadditive.

Bohnenblust and Sobczyk showed that the Hahn-Banach theorem holds over the complex field:

Let L be a complex-linear space, let p be a prenorm in L, let L_0 be a complex-linear subspace of L, and let f_0 be an element of $L_0^{\#}$ dominated by p, in the sense that $|f_0(x)| \leq p(x)$ for all x in L_0. Then f_0 has an extension f in $L^{\#}$ such that f is dominated by p.

Suhomlinov proved the same result for complex or quaternion scalars. Bohnenblust and Sobczyk showed that if L_0 is only real-linear, the desired conclusion may fail.

§ 4. Linear Topological Spaces

Definition 1. If a linear space L has a Hausdorff topology in which the vector operations are continuous (as functions of two variables) then L is called a ***linear topological space*** (LTS). If in addition every neighborhood of each point contains a convex open set, then L is called a ***locally convex*** linear topological space (LCS).

(1) (a) If L is an LTS and $\mathfrak{U}$ is a neighborhood basis of 0, then $\mathfrak{U}_x = \{U + x: U \in \mathfrak{U}\}$ is a neighborhood basis at x. (b) Hence every LTS has a uniform structure compatible with its topology and vector structure, and must be a completely regular space [Kelley, Chapter 6].

(2) (Von Neumann [2], Wehausen). If L is an LTS, it has a neighborhood basis $\mathfrak{U}$ at 0 such that (a) 0 is the only point common to all U in $\mathfrak{U}$; (b) if $U, V \in \mathfrak{U}$, then there is a W in $\mathfrak{U}$ such that $W \subseteq U \cap V$; (c) if $U \in \mathfrak{U}$ and $|r| \leq 1$, then[1] $rU \subseteq U$; (d) if $U \in \mathfrak{U}$ there exists $V \in \mathfrak{U}$

[1] A set U with this property is called *symmetric* if the field is real, *discoid* if the field is complex.

such that $V + V \subseteq U$; (e) 0 is a core point of each U in $\mathfrak{U}$. L is also locally convex, if and only if $\mathfrak{U}$ can also be chosen so that (f) every U in $\mathfrak{U}$ is convex. Conversely, if a neighborhood basis at 0 is chosen to satisfy (a)—(e), and neighborhoods of other points are defined [as in (1a)] by translations of the neighborhood system at 0, then L becomes an LTS, which is locally convex if (f) also holds. Finally, (c) and (d) imply (g) for every U in $\mathfrak{U}$ and $k > 0$ there exists V such that $rV \subseteq U$ if $|r| \leq k$.

(3) (a) Any linear subset of an LTS becomes an LTS under the relative topology [Kelley, p. 51] determined from L. (b) With the product-space topology [Kelley, p. 90] in which a neighborhood basis $\mathfrak{U}$ of 0 in R^S is the set of all $U(\sigma, \varepsilon) = \{x: |x(s)| < \varepsilon$ for each s in $\sigma\}$, with $\varepsilon > 0$ and σ a finite subset of S, the space R^S is an LCS. (c) If L is a linear space, then $L^{\#}$ is a closed subspace of R^L.

(4) Let L be an LTS and let X, Y be subsets of L. (a) If X is open and $r \neq 0$, then rX is open. (b) If X or Y is open, $X + Y$ is open. (c) If X is open, so is the convex hull of X. (d) The interior of a convex set is convex or empty. (e) If X is closed and Y is compact, then $X + Y$ is closed.

Lemma 1. If f from one LTS L to another L' is additive and continuous at 0, then f is uniformly continuous and real-homogeneous.

Proof. Let x_0 be a point of L; then $U' + f(x_0)$ is a neighborhood of $f(x_0)$ if and only if U' is a neighborhood of 0 in L'. Then there is a neighborhood U of 0 in L such that $f(U) \subseteq U'$. Hence $f(x_0 + U) = f(x_0) + f(U)$ is contained in $f(x_0) + U'$; i. e., f is continuous at every x_0 if it is continuous at 0. This proof gives uniform continuity as an extra bonus with no more work. In any linear space an additive function is homogeneous over the rational field; this is proved (i) by induction for integers, (ii) by change of variable for reciprocals of integers, and (iii) by combining these for arbitrary rationals. Then if r is real and (r_n) is a sequence of rationals converging to r, continuity of multiplication implies $(r_n x)$ converges to rx. Hence

$$f(rx) = \lim_{n \in \omega} f(r_n x) = \lim_n r_n f(x) = (\lim_n r_n)\, f(x) = rf(x)\,.$$

Corollary 1. An additive functional is continuous if and only if it is bounded on some open set in L.

Proof. If f is continuous, $f^{-1}((-1, 1))$ is open and f is bounded on it. If f is bounded on an open set U by a number k, and if $x_0 \in U$, then f is bounded by $2k$ on $U - x_0$, which contains a U_1 in $\mathfrak{U}$. By (2g) f is continuous at 0; the lemma asserts it is continuous everywhere.

(5) A sublinear functional p is continuous if and only if it is bounded on an open set and if and only if $\{x: p(x) < 1\}$ is open, and if and only if p is continuous at 0.

Corollary 2. A linear functional f on L is continuous if and only if there is an open set U in L and a value t which f does not take in U. Hence an f in $L^{\#}$ is continuous if and only if $L_0 = f^{-1}(0)$ is closed.

Proof. By translation of U and additivity of f it can be assumed that $0 \in U$; by (2), U contains a neighborhood V of 0 such that $rV \subseteq V$ if $|r| \leq 1$. Then $v \in V$ and $|r| \leq 1$ imply that $rf(v) \neq f(rv) \neq t$; that is $f(v) \neq t/r$ if $|r| \leq 1$. Hence $|f(v)| < |t|$ if $v \in V$; Corollary 1 asserts that f is continuous.

Lemma 2. Every line in an LTS L is uniformly homeomorphic to the real number system R; more precisely, for each $x \neq 0$ in L the mapping $f(r) = rx$ is a uniformly bicontinuous one-to-one linear function from R onto R_1, the line through 0 and x.

Proof. f is linear and one-to-one. Continuity of f and of f^{-1} follows from Corollary 2.

Definition 2. An *isomorphism* T of one LTS L *into* another LTS L' is an algebraic isomorphism (Def. 2,1) of L onto a linear subspace L_0 of L' such that T and T^{-1} are both continuous. L and L' are called *isomorphic* whenever there is an isomorphism of L *onto* L'.

(6) (VON NEUMANN [2]) (a) An open subset U of an LTS L is convex if and only if (2 f') $U + U = 2U$. (b) If a subset U of X satisfies (2 f'), then so does the interior of U. (c) An LTS is locally convex if and only if there exists a neighborhood basis at zero consisting of sets satisfying (2 a to e) and (2 f').

Lemma 3. Every one-dimensional subspace H of an LTS L is closed in L.

Proof. Suppose $(x_n, n \in \Delta)$ is a net in H such that there is an x in L for which $\lim_n x_n = x$. Then (x_n) is a Cauchy net in H; if $z \neq 0$ is in H, the (uniformly) continuous transformation $t \leftrightarrow tz$ between H and R carries $(x_n) = (t_n z)$ into a Cauchy net (t_n) in R. Since R is a complete metric space, (t_n) converges to some limit t. Then $tz = \lim_n t_n z = \lim_n x_n = x$, so $x \in H$.

Corollary 3. Let L be an LTS and let L_0 be a closed subspace. Let L_1 be a line in L which meets L_0 only at 0 and let $L_2 = L_1 + L_0$. Then (i) L_2 is closed in L, and (ii) if $0 \neq x \in L_1$, the natural correspondence $(y, r) \leftrightarrow y + rx$ between L_2 and $L_0 \times R$ is a homeomorphism.

Proof. In L_2 define f by $f(z) = f(y + rx) = r$, where $r \in R$ and $y \in L_0$; then f is linear and $f^{-1}(0) = L_0$, which is closed in L_2 since it is closed in L. By Cor. 2, f is continuous; hence for $z = y + rx$, r and y are continuous linear functions of z. Therefore, the function F defined by $F(y + rx) = (y, r)$ is a continuous function from L_2 onto $L_0 \times R$. Continuity of the vector operations asserts that F^{-1} is continuous. Hence F is a homeomorphism.

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