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LECTURE NOTES

Nonlinear Analysis on Manifolds: Sobolev Spaces and Inequalities

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Preface

These notes deal with the theory of Sobolev spaces on Riemannian manifolds. Though Riemannian manifolds are natural extensions of Euclidean space, the naive idea that what is valid for Euclidean space must be valid for manifolds is false. Several surprising phenomena appear when studying Sobolev spaces on manifolds. Questions that are elementary for Euclidean space become challenging and give rise to sophisticated mathematics, where the geometry of the manifold plays a central role. The reader will find many examples of this in the text.

These notes have their origin in a series of lectures given at the Courant Institute of Mathematical Sciences in 1998. For the sake of clarity, I decided to deal only with manifolds without boundary. An appendix concerning manifolds with boundary can be found at the end of these notes. To illustrate some of the results or concepts developed here, I have included some discussions of a special family of PDEs where these results and concepts are used. These PDEs are generalizations of the scalar curvature equation. As is well known, geometric problems often lead to limiting cases of known problems in analysis.

The study of Sobolev spaces on Riemannian manifolds is a field currently undergoing great development. Nevertheless, several important questions still puzzle mathematicians today. While the theory of Sobolev spaces for noncompact manifolds has its origin in the 1970s with the work of Aubin, Cantor, Hoffman, and Spruck, many of the results presented in these lecture notes have been obtained in the 1980s and 1990s. This is also the case for the applications already mentioned to scalar curvature and generalized scalar curvature equations. A substantial part of these notes is devoted to the concept of best constants. This concept appeared very early on to be crucial for solving limiting cases of some partial differential equations. A striking example of this was the major role that best constants played in the Yamabe problem.

These lecture notes are intended to be as self-contained as possible. In particular, it is not assumed that the reader is familiar with differentiable manifolds and Riemannian geometry. The present notes should be accessible to a large audience, including graduate students and specialists of other fields.

The present notes are organized into nine chapters. Chapter 1 is a quick introduction to differential and Riemannian geometry. Chapter 2 deals with the general theory of Sobolev spaces for compact manifolds, while Chapter 3 deals with the general theory of Sobolev spaces for complete, noncompact manifolds. Best constants problems for compact manifolds are discussed in Chapters 4 and 5, while Chapter 6 deals with some special type of Sobolev inequalities under constraints. Best constants problems for complete noncompact manifolds are discussed in Chapter 7. Chapter 8 deals with Euclidean-type Sobolev inequalities. The influence of symmetries on Sobolev embeddings is discussed in Chapter 9. An appendix at the end of these notes briefly discusses the case of manifolds with boundaries.

It is my pleasure to thank my friend Jalal Shatah for encouraging me to write these notes. It is also my pleasure to express my deep thanks to my friends and colleagues Tobias Colding, Zindine Djadli, Olivier Druet, Antoinette Jourdain, Michel Ledoux, Frédéric Robert, and Michel Vaugon for stimulating discussions and valuable comments about the manuscript. Finally, I wish to thank Reeva Goldsmith, Paul Monsour, and Joe Shearer for the wonderful job they did in the preparation of the manuscript.

Emmanuel Hebey
Paris, September 1998

Contents

CHAPTER 1

Elements of Riemannian Geometry

The purpose of this chapter is to recall some basic facts concerning Riemannian geometry. Needless to say, for dimension reasons, we are obliged to be succinct and partial. For those who have only a slight acquaintance with Riemannian geometry, we recommend the following books: Chavel [**45**], Do-Carmo [**70**], Gallot-Hulin-Lafontaine [**88**], Hebey [**109**], Jost [**127**], Kobayashi-Nomizu [**136**], Sakai [**171**], and Spivak [**181**]. Of course, many other excellent books on the subject do exist. We mention that Einstein's summation convention is adopted: an index occurring twice in a product is to be summed. This also holds for the rest of the book.

1.1. Smooth Manifolds

Paraphrasing a sentence of Elie Cartan, a manifold is really made of small pieces of Euclidean space. More precisely, let M be a Hausdorff topological space. We say that M is a topological manifold of dimension n if each point of M possesses an open neighborhood that is homeomorphic to some open subset of the Euclidean space $\mathbb{R}^n$. A chart of M is then a couple (Ω, φ) where Ω is an open subset of M, and φ is a homeomorphism of Ω onto some open subset of $\mathbb{R}^n$. For $y \in \Omega$, the coordinates of $\varphi(y)$ in $\mathbb{R}^n$ are said to be the coordinates of y in (Ω, φ). An atlas of M is a collection of charts (Ω_i, φ_i), $i \in I$, such that $M = \bigcup_{i\in I} \Omega_i$. Given $(\Omega_i, \varphi_i)_{i\in I}$ an atlas, the transition functions are

$$\varphi_j \circ \varphi_i^{-1} : \varphi_i(\Omega_i \cap \Omega_j) \to \varphi_j(\Omega_i \cap \Omega_j)$$

with the obvious convention that we consider $\varphi_j \circ \varphi_i^{-1}$ if and only if $\Omega_i \cap \Omega_j \neq \emptyset$. The atlas is then said to be of class C^k if the transition functions are of class C^k, and it is said to be C^k-complete if it is not contained in a (strictly) larger atlas of class C^k. As one can easily check, every atlas of class C^k is contained in a unique C^k-complete atlas.

For our purpose, we will always assume in what follows that $k = +\infty$ and that M is connected. One then gets the following definition of a smooth manifold: A smooth manifold M of dimension n is a connected topological manifold M of dimension n together with a C^∞-complete atlas.

Classical examples of smooth manifolds are the Euclidean space $\mathbb{R}^n$ itself, the torus T^n, the unit sphere S^n of $\mathbb{R}^{n+1}$, and the real projective space $\mathbb{P}^n(\mathbb{R})$.

Given M and N two smooth manifolds, and $f : M \to N$ some map from M to N, we say that f is differentiable (or of class C^k) if for any charts (Ω, φ) and $(\tilde{\Omega}, \tilde{\varphi})$ of M and N such that $f(\Omega) \subset \tilde{\Omega}$, the map

$$\tilde{\varphi} \circ f \circ \varphi^{-1} : \varphi(\Omega) \to \tilde{\varphi}(\tilde{\Omega})$$

is differentiable (or of class C^k). In particular, this allows us to define the notion of diffeomorphism and the notion of diffeomorphic manifolds. Independently, one can define the rank $R(f)_x$ of f at some point x of M as the rank of $\tilde{\varphi} \circ f \circ \varphi^{-1}$ at $\varphi(x)$, where (Ω, φ) and $(\tilde{\Omega}, \tilde{\varphi})$ are as above, with the additional property that $x \in \Omega$. This is an intrinsic definition in the sense that it does not depend on the choice of the charts. The map f is then said to be an immersion if, for any $x \in M$, $R(f)_x = m$, where m is the dimension of M, and a submersion if for any $x \in M$, $R(f)_x = n$, where n is the dimension of N. It is said to be an embedding if it is an immersion that realizes a homeomorphism onto its image.

We refer to the above definition of a manifold as the abstract definition of a smooth manifold. Looking carefully to what it says, and to the questions it raises, things appear to be less clear than they may seem at first glance. Given M a connected topological manifold, one can ask if there always exists a structure of smooth manifold on M, and if this structure is unique. Here, uniqueness has to be understood in the following sense: given M a connected topological manifold, and $\mathcal{A}$ a C^∞-complete atlas of M, the smooth structure of M is said to be unique if, for any other C^∞-complete atlas $\tilde{\mathcal{A}}$ of M, the smooth manifolds $(M, \mathcal{A})$ and $(M, \tilde{\mathcal{A}})$ are diffeomorphic. With this definition of uniqueness, the only reasonable definition for that notion, one gets surprising answers to the questions we asked above. From the works of Moïse, developed in the 1950s, one has that up to dimension 3, any topological manifold possesses one, and only one, smooth structure. But starting from dimension 4, one gets that there exist topological manifolds which do not possess smooth structures (this was shown by Freedman in the 1980s), and that there exist topological manifolds which possess many smooth structures. Coming back to the works of Milnor in the 1950s, and to the works of Kervaire and Milnor, one has that S^7 possesses 28 smooth structures, while S^{11} possesses 992 smooth structures! Perhaps more surprising are the consequences of the works of Donaldson and Taubes: While $\mathbb{R}^n$ possesses a unique smooth structure for $n \neq 4$, there exist infinitely many smooth structures on $\mathbb{R}^4$!

Up to now, we have adopted the abstract definition of a manifold. As a surface gives the idea of a two-dimensional manifold, a more concrete approach would have been to define manifolds as submanifolds of Euclidean space. Given M and N two manifolds, one will say that N is a submanifold of M if there exists a smooth embedding $f : N \to M$. According to a well-known result of Whitney, the two approaches (concrete and abstract) are equivalent, at least when dealing with paracompact manifolds, since for any paracompact manifold M of dimension n, there exists a smooth embedding $f : M \to \mathbb{R}^{2n+1}$. In other words, any paracompact (abstract) manifold of dimension n can be seen as a submanifold of some Euclidean space.

Let us now say some words about the tangent space of a manifold. Given M a smooth manifold and $x \in M$, let $\mathcal{F}_x$ be the vector space of functions $f : M \to \mathbb{R}$ which are differentiable at x. For $f \in \mathcal{F}_x$, we say that f is flat at x if for some chart (Ω, φ) of M at x, $D(f \circ \varphi^{-1})_{\varphi(x)} = 0$. Let $\mathcal{N}_x$ be the vector space of such functions. A linear form X on $\mathcal{F}_x$ is then said to be a tangent vector of M at x if $\mathcal{N}_x \subset \operatorname{Ker} X$. We let $T_x(M)$ be the vector space of such tangent vectors. Given

(Ω, φ) some chart at x, of associated coordinates x^i, we define $\left(\frac{\partial}{\partial x_i}\right)_x \in T_x(M)$ by: for any $f \in \mathcal{F}_x$,

$$\left(\frac{\partial}{\partial x_i}\right)_x \cdot (f) = D_i\left(f \circ \varphi^{-1}\right)_{\varphi(x)}$$

As a simple remark, one gets that the $\left(\frac{\partial}{\partial x_i}\right)_x$'s form a basis of $T_x(M)$. Now, one defines the tangent bundle of M as the disjoint union of the $T_x(M)$'s, $x \in M$. If M is n-dimensional, one can show that $T(M)$ possesses a natural structure of a $2n$-dimensional smooth manifold. Given (Ω, φ) a chart of M,

$$\left(\bigcup_{x \in \Omega} T_x(M), \Phi\right)$$

is a chart of $T(M)$, where for $X \in T_x(M)$, $x \in \Omega$,

$$\Phi(X) = \left(\varphi^1(x), \ldots, \varphi^n(x), X(\varphi^1), \ldots, X(\varphi^n)\right)$$

(the coordinates of x in (Ω, φ) and the components of X in (Ω, φ), that is, the coordinates of X in the basis of $T_x(M)$ associated to (Ω, φ) by the process described above). By definition, a vector field on M is a map $X : M \to T(M)$ such that for any $x \in M$, $X(x) \in T_x(M)$. Since M and $T(M)$ are smooth manifolds, the notion of a vector field of class C^k makes sense.

Given M, N two smooth manifolds, x a point of M, and $f : M \to N$ differentiable at x, the tangent linear map of f at x (or the differential map of f at x), denoted by $f_\star(x)$, is the linear map from $T_x(M)$ to $T_{f(x)}(N)$ defined by: For $X \in T_x(M)$ and $g : N \to \mathbb{R}$ differentiable at $f(x)$,

$$\left(f_\star(x) \cdot (X)\right) \cdot (g) = X(g \circ f)$$

By extension, if f is differentiable on M, one gets the tangent linear map of f, denoted by $f_\star$. That is the map $f_\star : T(M) \to T(N)$ defined by: For $X \in T_x(M)$, $f_\star(X) = f_\star(x).(X)$. As one can easily check, $f_\star$ is C^{k-1} if f is C^k. For $f : M_1 \to M_2$, $g : M_2 \to M_3$, and $x \in M_1$, $(g \circ f)_\star(x) = g_\star(f(x)) \circ f_\star(x)$.

Similar to the construction of the tangent bundle, one can define the cotangent bundle of a smooth manifold M. For $x \in M$, let $T_x(M)^\star$ be the dual space of $T_x(M)$. If (Ω, φ) is a chart of M at x of associated coordinates x^i, one gets a basis of $T_x(M)^\star$ by considering the dx^i_x's defined by $dx^i_x \cdot \left(\frac{\partial}{\partial x_j}\right)_x = \delta^i_j$. As for the tangent bundle, the cotangent bundle of M, denoted by $T^\star(M)$, is the disjoint union of the $T_x(M)^\star$'s, $x \in M$. Here again, if M is n-dimensional, $T^\star(M)$ possesses a natural structure of $2n$-dimensional smooth manifold. Given (Ω, φ) a chart of M,

$$\left(\bigcup_{x \in \Omega} T_x(M)^\star, \Phi\right)$$

is a chart of $T(M)$, where for $\eta \in T_x(M)^\star$, $x \in \Omega$,

$$\Phi(\eta) = \left(\varphi^1(x), \ldots, \varphi^n(x), \eta\left(\frac{\partial}{\partial x_1}\right)_x, \ldots, \eta\left(\frac{\partial}{\partial x_n}\right)_x\right)$$

(the coordinates of x in (Ω, φ) and the components of η in (Ω, φ), that is, the coordinates of η in the basis of $T_x(M)^\star$ associated to (Ω, φ) by the process described above). By definition, a 1-form on M is a map $\eta : M \to T^\star(M)$ such that for any $x \in M$, $\eta(x) \in T_x(M)^\star$. Here again, since M and $T^\star(M)$ are smooth manifolds, the notion of a 1-form of class C^k makes sense. For f a function of class C^k on M, let df be defined by: For $x \in M$ and $X \in T_x(M)$, $df(x) \cdot X = X(f)$. Then df is a 1-form of class C^{k-1}.

Given M a smooth n-manifold, and $1 \leq q \leq n$ an integer, let $\bigwedge^q T_x(M)^\star$ be the space of skew-symmetric q-linear forms on $T_x(M)$. If (Ω, φ) is a chart of M at x, of associated coordinates x^i, $\left\{dx_x^{i_1} \wedge \cdots \wedge dx_x^{i_q}\right\}_{i_1 < \cdots < i_q}$ is a basis of $\bigwedge^q T_x(M)^\star$. With similar constructions to the ones made above, one gets that $\bigwedge^q(M)$, the disjoint union of the $\bigwedge^q T_x(M)^\star$'s, possesses a natural structure of a smooth manifold. Its dimension is $n + C_n^q$, where $C_n^q = n!/(q!(n-q)!)$. Some map $\eta : M \to \bigwedge^q(M)$ is then said to be an exterior form of degree q, or just an exterior q-form, if for any $x \in M$, $\eta(x) \in \bigwedge^q T_x(M)^\star$. Here again, the notion of an exterior q-form of class C^k makes sense. Given (Ω, φ) some chart of M, and η a q-form of class C^k whose expression in (Ω, φ) is

$$\eta = \sum_{i_1 < \cdots < i_q} \eta_{i_1 \dots i_q} dx^{i_1} \wedge \cdots \wedge dx^{i_q}$$

the exterior derivative of η, denoted by $d\eta$, is the exterior $(q+1)$-form of class C^{k-1} whose expression in (Ω, φ) is

$$d\eta = \sum_{i_1 < \cdots < i_q} d\eta_{i_1 \dots i_q} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_q}$$

One then gets that for any exterior q-form η, $d(d\eta) = 0$. Conversely, by the Poincaré lemma, if η is an exterior q-form such that $d\eta = 0$, that is, a closed exterior q-form, around any point in M, there exists an exterior $(q-1)$-form $\tilde{\eta}$ such that $d\tilde{\eta} = \eta$. One says that a closed exterior form is locally exact.

As another generalization, given M a smooth n-manifold, x some point of M, and p, q two integers, one can define $T_p^q(T_x(M))$ as the space of (p,q)-tensors on $T_x(M)$, that is, the space of $(p+q)$-linear forms

$$\eta : \underbrace{T_x(M) \times \cdots \times T_x(M)}_{p} \times \underbrace{T_x(M)^\star \times \cdots \times T_x(M)^\star}_{q} \longrightarrow \mathbb{R}$$

An element of $T_p^q\big(T_x(M)\big)$ is said to be p-times covariant and q-times contravariant. If (Ω, φ) is a chart of M at x, of associated coordinates x^i, the family

$$\left\{dx_x^{i_1} \otimes \cdots \otimes dx_x^{i_p} \otimes \left(\frac{\partial}{\partial x_{j_1}}\right)_x \otimes \cdots \otimes \left(\frac{\partial}{\partial x_{j_q}}\right)_x\right\}_{i_1,\dots,i_p,j_1,\dots,j_q}$$

is a basis of $T_p^q\big(T_x(M)\big)$. Here again, one gets that the disjoint union $T_p^q(M)$ of the $T_p^q\big(T_x(M)\big)$'s possesses a natural structure of a smooth manifold. Its dimension is $n\big(1 + n^{p+q-1}\big)$. A map $T : M \to T_p^q(M)$ is then said to be a (p,q)-tensor field on M if for any $x \in M$, $T(x) \in T_p^q\big(T_x(M)\big)$. It is said to be of class C^k if it is of class C^k from the manifold M to the manifold $T_p^q(M)$. Given (Ω, φ) and (Ω, ψ) two charts of M of associated coordinates x^i and y^i, and T a (p,q)-tensor field, let us denote by $T_{i_1\ldots i_p}^{j_1\ldots j_q}$ and $\tilde{T}_{i_1\ldots i_p}^{j_1\ldots j_q}$ its components in (Ω, φ) and (Ω, ψ). Then, for any $i_1, \ldots, i_p, j_1, \ldots, j_q$, and any $x \in \Omega$,

$$\tilde{T}_{i_1\ldots i_p}^{j_1\ldots j_q}(x) = T_{\alpha_1\ldots\alpha_p}^{\beta_1\ldots\beta_q}(x)\left(\frac{\partial x^{\alpha_1}}{\partial y_{i_1}}\right)_x \cdots \left(\frac{\partial x^{\alpha_p}}{\partial y_{i_p}}\right)_x \left(\frac{\partial y^{j_1}}{\partial x_{\beta_1}}\right)_x \cdots \left(\frac{\partial y^{j_q}}{\partial x_{\beta_q}}\right)_x \tag{1.1}$$

As a remark, given M and N two manifolds, $f : M \to N$ a map of class C^{k+1}, and T a $(p, 0)$-tensor field of class C^k on N, one can define the pullback $f^\star T$ of T by f, that is, the $(p, 0)$-tensor field of class C^k on M defined by: For $x \in M$ and $X_1, \ldots, X_p \in T_x(M)$,

$$(f^\star T)(x) \cdot (X_1, \ldots, X_p) = T(f(x)) \cdot (f_\star(x) \cdot X_1, \ldots, f_\star(x) \cdot X_p)$$

As one can easily check, for $f : M_1 \to M_2$ and $g : M_2 \to M_3$, $(g \circ f)^\star = f^\star \circ g^\star$.

Let us now define the notion of a linear connection. Denote by $\Gamma(M)$ the space of differentiable vector fields on M. A linear connection D on M is a map

$$D : T(M) \times \Gamma(M) \to T(M)$$

such that

1. $\forall x \in M, \forall X \in T_x(M), \forall Y \in \Gamma(M)$, $D(X, Y) \in T_x(M)$,
2. $\forall x \in M$, $D : T_x(M) \times \Gamma(M) \to T_x(M)$ is bilinear,
3. $\forall x \in M$, $\forall X \in T_x(M)$, $\forall f : M \to \mathbb{R}$ differentiable, $\forall Y \in \Gamma(M)$, $D(X, fY) = X(f)Y(x) + f(x)D(X, Y)$, and
4. $\forall X, Y \in \Gamma(M)$, and $\forall k$ integer, if X is of class C^k and Y is of class C^{k+1}, then $D(X, Y)$ is of class C^k, where $D(X, Y)$ is the vector field $x \to D(X(x), Y)$.

Given D a linear connection, the usual notation for $D(X, Y)$ is $D_X(Y)$. One says that $D_X(Y)$ is the covariant derivative of Y with respect to X. Let (Ω, φ) be a chart of M of associated coordinates x^i. Set

$$\nabla_i = D_{(\frac{\partial}{\partial x_i})}$$

As one can easily check, there exist n^3 smooth functions $\Gamma_{ij}^k : \Omega \to \mathbb{R}$ such that for any i, j, and any $x \in \Omega$,

$$\nabla_i\left(\frac{\partial}{\partial x_j}\right)(x) = \Gamma_{ij}^k(x)\left(\frac{\partial}{\partial x_k}\right)_x$$

Such functions, the Christoffel symbols of D in (Ω, φ), characterize the connection in the sense that for $X \in T_x(M)$, $x \in \Omega$, and $Y \in \Gamma(M)$,

$$D_X(Y) = X^i(\nabla_i Y)(x) = X^i\left(\left(\frac{\partial Y^j}{\partial x_i}\right)_x + \Gamma_{i\alpha}^j(x)Y^\alpha(x)\right)\left(\frac{\partial}{\partial x_j}\right)_x$$

where the X^i's and Y^i's denote the components of X and Y in the chart (Ω, φ), and for $f : M \to \mathbb{R}$ differentiable at x,

$$\left(\frac{\partial f}{\partial x_i}\right)_x = D_i(f \circ \varphi^{-1})_{\varphi(x)}$$

As one can easily check, since (1.1) is not satisfied by the Γ_{ij}^k's, the Γ_{ij}^k's are not the components of a (2, 1)-tensor field. An important remark is that linear connections have natural extensions to differentiable tensor fields. Given T a differentiable (p,q)-tensor field, x a point of M, $X \in T_x(M)$, and (Ω, φ) a chart of M at x, $D_X(T)$ is the (p,q)-tensor on $T_x(M)$ defined by $D_X(T) = X^i(\nabla_i T)(x)$, where

$$(\nabla_i T)(x)^{j_1\dots j_q}_{i_1\dots i_p} = \left(\frac{\partial T^{j_1\dots j_q}_{i_1\dots i_p}}{\partial x_i}\right)_x - \sum_{k=1}^{p} \Gamma^{\alpha}_{i i_k}(x) T(x)^{j_1\dots j_q}_{i_1\dots i_{k-1}\alpha i_{k+1}\dots i_p}$$
$$+ \sum_{k=1}^{q} \Gamma^{j_k}_{i\alpha}(x) T(x)^{j_1\dots j_{k-1}\alpha j_{k+1}\dots j_q}_{i_1\dots i_p}$$

The covariant derivative commutes with the contraction in the sense that

$$D_X\left(C^{k_2}_{k_1} T\right) = C^{k_2}_{k_1} D_X(T)$$

where $C^{k_2}_{k_1} T$ stands for the contraction of T of order (k_1, k_2). More, for $X \in T_x(M)$, and T and $\tilde{T}$ two differentiable tensor fields, one has that

$$D_X\left(T \otimes \tilde{T}\right) = \left(D_X(T)\right) \otimes \tilde{T}(x) + T(x) \otimes \left(D_X(\tilde{T})\right)$$

Given T a (p,q)-tensor field of class C^{k+1}, we let ∇T be the $(p+1,q)$-tensor field of class C^k whose components in a chart are given by

$$\left(\nabla T\right)^{j_1\dots j_q}_{i_1\dots i_{p+1}} = \left(\nabla_{i_1} T\right)^{j_1\dots j_q}_{i_2\dots i_{p+1}}$$

By extension, one can then define $\nabla^2 T$, $\nabla^3 T$, and so on. For $f : M \to \mathbb{R}$ a smooth function, one has that $\nabla f = df$ and, in any chart (Ω, φ) of M,

$$\left(\nabla^2 f\right)(x)_{ij} = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)_x - \Gamma^k_{ij}(x)\left(\frac{\partial f}{\partial x_k}\right)_x$$

where

$$\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)_x = D^2_{ij}\left(f \circ \varphi^{-1}\right)_{\varphi(x)}$$

In the Riemannian context, $\nabla^2 f$ is called the Hessian of f and is sometimes denoted by Hess(f).

Finally, let us define the torsion and the curvature of a linear connection D. The torsion T of D can be seen as the smooth $(2,1)$-tensor field on M whose components in any chart are given by the relation $T^k_{ij} = \Gamma^k_{ij} - \Gamma^k_{ji}$. One says that the connection is torsion-free if $T \equiv 0$. The curvature R of D can be seen as the smooth $(3,1)$-tensor field on M whose components in any chart are given by the relation

$$R^l_{ijk} = \frac{\partial \Gamma^l_{ki}}{\partial x_j} - \frac{\partial \Gamma^l_{ji}}{\partial x_k} + \Gamma^l_{j\alpha}\Gamma^{\alpha}_{ki} - \Gamma^l_{k\alpha}\Gamma^{\alpha}_{ji}$$

As one can easily check, $R^l_{ijk} = -R^l_{ikj}$. Moreover, when the connection is torsion-free, one has that

$$R^l_{ijk} + R^l_{kij} + R^l_{jki} = 0$$
$$(\nabla_i R)^l_{mjk} + (\nabla_k R)^l_{mij} + (\nabla_j R)^l_{mki} = 0$$

Such relations are referred to as the first Bianchi's identity, and the second Bianchi's identity.

1.2. Riemannian Manifolds

Let M be a smooth manifold. A Riemannian metric g on M is a smooth $(2,0)$-tensor field on M such that for any $x \in M$, $g(x)$ is a scalar product on $T_x(M)$. A smooth Riemannian manifold is a pair (M, g) where M is a smooth manifold and g a Riemannian metric on M. According to Whitney, for any paracompact smooth n-manifold there exists a smooth embedding $f : M \to \mathbb{R}^{2n+1}$. One then gets that any smooth paracompact manifold possesses a Riemannian metric. Just think to $g = f^\star e$, e the Euclidean metric. Two Riemannian manifolds (M_1, g_1) and (M_2, g_2) are said to be isometric if there exists a diffeomorphism $f : M_1 \to M_2$ such that $f^\star g_2 = g_1$.

Given (M, g) a smooth Riemannian manifold, and $\gamma : [a, b] \to M$ a curve of class C^1, the length of γ is

$$L(\gamma) = \int_a^b \sqrt{g(\gamma(t)) \cdot \left(\left(\frac{d\gamma}{dt}\right)_t, \left(\frac{d\gamma}{dt}\right)_t \right)}\, dt$$

where $(\frac{d\gamma}{dt})_t \in T_{\gamma(t)}(M)$ is such that $(\frac{d\gamma}{dt})_t \cdot f = \left(f \circ \gamma\right)'(t)$ for any $f : M \to \mathbb{R}$ differentiable at $\gamma(t)$. If γ is piecewise C^1, the length of γ may be defined as the sum of the lengths of its C^1 pieces. For x and y in M, let $\mathcal{C}_{xy}$ be the space of piecewise C^1 curves $\gamma : [a, b] \to M$ such that $\gamma(a) = x$ and $\gamma(b) = y$. Then

$$d_g(x, y) = \inf_{\gamma \in \mathcal{C}_{xy}} L(\gamma)$$

defines a distance on M whose topology coincides with the original one of M. In particular, by Stone's theorem, a smooth Riemannian manifold is paracompact. By definition, d_g is the distance associated to g.

Let (M, g) be a smooth Riemannian manifold. There exists a unique torsion-free connection on M having the property that $\nabla g = 0$. Such a connection is the Levi-Civita connection of g. In any chart (Ω, φ) of M, of associated coordinates x^i, and for any $x \in \Omega$, its Christoffel symbols are given by the relations

$$\Gamma_{ij}^k(x) = \frac{1}{2}\left(\left(\frac{\partial g_{mj}}{\partial x_i}\right)_x + \left(\frac{\partial g_{mi}}{\partial x_j}\right)_x - \left(\frac{\partial g_{ij}}{\partial x_m}\right)_x \right) g(x)^{mk}$$

where the g^{ij}'s are such that $g_{im} g^{mj} = \delta_i^j$. Let R be the curvature of the Levi-Civita connection as introduced above. One defines:

1. the *Riemann curvature* $\mathrm{Rm}_{(M,g)}$ of g as the smooth $(4,0)$-tensor field on M whose components in a chart are $R_{ijkl} = g_{i\alpha} R_{jkl}^\alpha$,
2. the *Ricci curvature* $\mathrm{Rc}_{(M,g)}$ of g as the smooth $(2,0)$-tensor field on M whose components in a chart are $R_{ij} = R_{\alpha i \beta j} g^{\alpha\beta}$, and
3. the *scalar curvature* $\mathrm{Scal}_{(M,g)}$ of g as the smooth real-valued function on M whose expression in a chart is $\mathrm{Scal}_{(M,g)} = R_{ij} g^{ij}$.

As one can check, in any chart,

$$R_{ijkl} = -R_{jikl} = -R_{ijlk} = R_{klij}$$

and the two Bianchi identities are

$$\begin{aligned}
&R_{ijkl} + R_{iljk} + R_{iklj} = 0\,, \\
&\left(\nabla_i \operatorname{Rm}_{(M,g)}\right)_{jklm} + \left(\nabla_m \operatorname{Rm}_{(M,g)}\right)_{jkil} + \left(\nabla_l \operatorname{Rm}_{(M,g)}\right)_{jkmi} = 0\,.
\end{aligned}$$

In particular, the Ricci curvature $\operatorname{Rc}_{(M,g)}$ of g is symmetric, so that in any chart $R_{ij} = R_{ji}$. For $x \in M$, let $G_x^2(M)$ be the 2-Grassmannian of $T_x(M)$. The sectional curvature $K_{(M,g)}$ of g is the real-valued function defined on $\bigcup_{x\in M} G_x^2(M)$ by: For $P \in G_x^2(M)$,

$$K_{(M,g)}(P) = \frac{\operatorname{Rm}_{(M,g)}(x)(X,Y,X,Y)}{g(x)(X,X)g(x)(Y,Y) - g(x)(X,Y)^2}$$

where (X, Y) is a basis of P. As one can easily check, such a definition does not depend on the choice of the basis. Moreover, one can prove that the sectional curvature determines the Riemann curvature.

Given (M, g) a smooth Riemannian manifold, and D its Levi-Civita connection, a smooth curve $\gamma : [a, b] \to M$ is said to be a geodesic if for all t,

$$D_{\left(\frac{d\gamma}{dt}\right)_t}\left(\frac{d\gamma}{dt}\right) = 0$$

This means again that in any chart, and for all k,

$$\left(\gamma^k\right)''(t) + \Gamma_{ij}^k\left(\gamma(t)\right)\left(\gamma^i\right)'(t)\left(\gamma^j\right)'(t) = 0$$

For any $x \in M$, and any $X \in T_x(M)$, there exists a unique geodesic $\gamma : [0, \varepsilon] \to M$ such that $\gamma(0) = x$ and $(\frac{d\gamma}{dt})_0 = X$. Let $\gamma_{x,X}$ be this geodesic. For $\lambda > 0$ real, $\gamma_{x,\lambda X}(t) = \gamma_{x,X}(\lambda t)$. Hence, for $\|X\|$ sufficiently small, where $\|\cdot\|$ stands for the norm in $T_x(M)$ associated to $g(x)$, one has that $\gamma_{x,X}$ is defined on $[0, 1]$. The exponential map at x is the map from a neighborhood of 0 in $T_x(M)$, with values in M, defined by $\exp_x(X) = \gamma_{x,X}(1)$. If M is n-dimensional and up to the assimilation of $T_x(M)$ to $\mathbb{R}^n$ via the choice of an orthonormal basis, one gets a chart $\left(\Omega, \exp_x^{-1}\right)$ of M at x. This chart is normal at x in the sense that the components g_{ij} of g in this chart are such that $g_{ij}(x) = \delta_{ij}$, with the additional property that the Christoffel symbols Γ_{ij}^k of the Levi-Civita connection in this chart are such that $\Gamma_{ij}^k(x) = 0$. The coordinates associated to this chart are referred to as geodesic normal coordinates.

Let (M, g) be a smooth Riemannian manifold. The Hopf-Rinow theorem states that the following assertions are equivalent:

1. the metric space (M, d_g) is complete,
2. any closed-bounded subset of M is compact,
3. there exists $x \in M$ for which $\exp_x$ is defined on the whole of $T_x(M)$, and
4. for any $x \in M$, $\exp_x$ is defined on the whole of $T_x(M)$.

Moreover, one gets that any of the above assertions implies that any two points in M can be joined by a minimizing geodesic. Here, a curve γ from x to y is said to be minimizing if $L(\gamma) = d_g(x, y)$.

Given (M, g) a smooth Riemannian n-manifold, one can define a natural positive Radon measure on M. In particular, the theory of the Lebesgue integral can be applied. For $(\Omega_i, \varphi_i)_{i\in I}$ some atlas of M, we shall say that a family $(\Omega_j, \varphi_j, \alpha_j)_{j\in J}$ is a partition of unity subordinate to $(\Omega_i, \varphi_i)_{i\in I}$ if the following holds:

1. $(\alpha_j)_j$ is a smooth partition of unity subordinate to the covering $(\Omega_i)_i$,
2. $(\Omega_j, \varphi_j)_j$ is an atlas of M, and
3. for any j, $\operatorname{supp} \alpha_j \subset \Omega_j$.

As one can easily check, for any atlas $(\Omega_i, \varphi_i)_{i\in I}$ of M, there exists a partition of unity $(\Omega_j, \varphi_j, \alpha_j)_{j\in J}$ subordinate to $(\Omega_i, \varphi_i)_{i\in I}$. One can then define the Riemannian measure as follows: Given $f : M \to \mathbb{R}$ continuous with compact support, and given $(\Omega_i, \varphi_i)_{i\in I}$ an atlas of M,

$$\int_M f\, dv(g) = \sum_{j\in J} \int_{\varphi_j(\Omega_j)} \left(\alpha_j \sqrt{|g|} f\right) \circ \varphi_j^{-1}\, dx$$

where $(\Omega_j, \varphi_j, \alpha_j)_{j\in J}$ is a partition of unity subordinate to $(\Omega_i, \varphi_i)_{i\in I}$, $|g|$ stands for the determinant of the matrix whose elements are the components of g in (Ω_j, φ_j), and dx stands for the Lebesgue volume element of $\mathbb{R}^n$. One can prove that such a construction does not depend on the choice of the atlas $(\Omega_i, \varphi_i)_{i\in I}$ and the partition of unity $(\Omega_j, \varphi_j, \alpha_j)_{j\in J}$.

The Laplacian acting on functions of a smooth Riemannian manifold (M, g) is the operator Δ_g whose expression in a local chart of associated coordinates x^i is

$$\Delta_g u = -g^{ij}\left(\frac{\partial^2 u}{\partial x_i \partial x_j} - \Gamma_{ij}^k \frac{\partial u}{\partial x_k}\right)$$

For u and v of class C^2 on M, on then has the following integration by parts formula

$$\int_M (\Delta_g u) v\, dv(g) = \int_M \langle \nabla u, \nabla v\rangle dv(g) = \int_M u(\Delta_g v) dv(g)$$

where $\langle \cdot, \cdot \rangle$ is the scalar product associated with g for 1-forms.

Coming back to geodesics, one can define the injectivity radius of (M, g) at some point x, denoted by $\operatorname{inj}_{(M,g)}(x)$, as the largest positive real number r for which any geodesic starting from x and of length less than r is minimizing. One can then define the (global) injectivity radius by

$$\operatorname{inj}_{(M,g)} = \inf_{x\in M} \operatorname{inj}_{(M,g)}(x)$$

One has that $\operatorname{inj}_{(M,g)} > 0$ for a compact manifold, but it may be zero for a complete noncompact manifold. More generally, one can define the cut locus $\operatorname{Cut}(x)$ of x as a subset of M and prove that $\operatorname{Cut}(x)$ has measure zero, that $\operatorname{inj}_{(M,g)}(x) = d_g(x, \operatorname{Cut}(x))$, and that $\exp_x$ is a diffeomorphism from some star-shaped domain of $T_x(M)$ at 0 onto $M \setminus \operatorname{Cut}(x)$. In particular, one gets that the distance function r

to a given point is differentiable almost everywhere, with the additional property that $|\nabla r| = 1$ almost everywhere.

1.3. Curvature and Topology

As is well-known, curvature assumptions may give topological and diffeomorphic information on the manifold. A striking example of the relationship that exists between curvature and topology is given by the Gauss-Bonnet theorem, whose present form is actually due to the works of Allendoerfer [**2**], Allendoerfer-Weil [**3**], Chern [**49**], and Fenchel [**81**]. One has here that the Euler-Poincaré characteristic $\chi(M)$ of a compact manifold can be expressed as the integral of a universal polynomial in the curvature. For instance, when the dimension of M is 2,

$$\chi(M) = \frac{1}{4\pi} \int_M \mathrm{Scal}_{(M,g)}\, dv(g)$$

and when the dimension of M is 4, as shown by Avez [**15**],

$$\chi(M) = \frac{1}{16\pi^2} \int_M \left(\frac{1}{2} |\mathrm{Weyl}_{(M,g)}|^2 + \frac{1}{12} \mathrm{Scal}^2_{(M,g)} - |E_{(M,g)}|^2 \right) dv(g)$$

where $|\cdot|$ stands for the norm associated to g for tensors, and where $\mathrm{Weyl}_{(M,g)}$ and $E_{(M,g)}$ are, respectively, the Weyl tensor of g and the traceless Ricci tensor of g. In a local chart, the components of $\mathrm{Weyl}_{(M,g)}$ are

$$\begin{aligned} W_{ijkl} = R_{ijkl} &- \frac{1}{n-2} \big(R_{ik} g_{jl} + R_{jl} g_{ik} - R_{il} g_{jk} - R_{jk} g_{il} \big) \\ &+ \frac{\mathrm{Scal}_{(M,g)}}{(n-1)(n-2)} \big(g_{ik} g_{jl} - g_{il} g_{jk} \big) \end{aligned}$$

where n stands for the dimension of the manifold. As another striking example of the relationship that exists between curvature and topology, one can refer to Myer's theorem (see, for instance, [**88**]). This theorem states that a smooth, complete Riemannian n-manifold (M, g) whose Ricci curvature satisfies

$$\mathrm{Rc}_{(M,g)} \geq (n-1) k^2 g$$

as bilinear forms, and for some $k > 0$ real, must be compact, with the additional property that its diameter $\mathrm{diam}_{(M,g)}$ is less than or equal to $\frac{\pi}{k}$. Moreover, by Hamilton's work [**99**], any 3-dimensional, compact, simply connected Riemannian manifold of positive Ricci curvature must be diffeomorphic to the unit sphere S^3 of $\mathbb{R}^4$. Conversely, by recent results of Lohkamp [**153**], negative sign assumptions on the Ricci curvature have no effect on the topology, since any compact manifold possesses a Riemannian metric of negative Ricci curvature. This does not hold anymore when dealing with sectional curvature. By the Cartan-Hadamard theorem (see, for instance, [**88**]), one has that any complete, simply connected, n-dimensional Riemannian manifold of nonpositive sectional curvature is diffeomorphic to $\mathbb{R}^n$.

As other examples of the relationship that exists between curvature and topology, let us mention the well-known sphere theorem of Berger [**26**], Klingenberg