THEORY OF OPERATORS

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Preface

The second edition of this book contains significant changes and additions intended to bring its content closer to the current curriculum of required courses in functional analysis offered in many universities of the USSR.

A certain part of the material has been rearranged. Appendix I (the theory of measure, measurable functions, and the integral) and Appendix II (distributions and the Fourier transform) have been enlarged and placed in separate chapters.

Chapter 2 on vector spaces has been revised. It now contains a detailed exposition of contemporary material on convex sets in vector spaces and topological vector spaces.

Chapter 4, devoted to the spectral theory of operators, has been enlarged. A section has been added on unbounded operators and the spectral theory of self-adjoint unbounded operators; material on completely continuous operators has also been included. Proofs have been carried out for certain propositions that were contained in the first edition as exercises and more detailed proofs have been given for many propositions.

Chapter 5, The Trace of an Operator, has been enlarged and some new results of the author relating to the traces of discrete operators have been included. These new investigations can be used for finding the regularized traces of partial differential operators. This material can be covered as a separate course. A significant number of examples illustrating the contents have been added, and new exercises have been included to promote better mastery of the material.

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V. Sadovnichii

Contents

Chapter 1

Metric and Topological Spaces

1. ELEMENTARY CONCEPTS OF SET THEORY

1.1. Elementary Properties of Sets. Mappings. Cartesian Product of Sets

A *set* is a collection of objects having a given property.

Every set is defined by some property P and consists of those objects and only those objects that have the property.

In what follows we shall agree to consider only sets belonging to some "universal" set E and to denote the sets under consideration by capital letters $A, B, C, \dots$ or $X, Y, Z \dots$. The set A consisting of elements $x, y, z, \dots$ is often denoted as follows: $A = \{x, y, z, \dots\}$.

If the elements a and b coincide, we write $a = b$. If the elements a and b are distinct, we write $a \neq b$. The condition that an element a belongs to the set A is written as follows: $a \in A$ and the notation $a \notin A$ means that the element a does not belong to the set A (does not have property P).

If it is necessary to emphasize that the set A is comprised of elements belonging to the universal set E and having property P, we often apply the notation

$$A = \{a \in E : P\}.$$

This notation is read as follows: "The set A consists of the elements of E having property P."

1.1.1. Set Inclusion

Let A and B be two sets in E. The set B is said to be *contained* in the set A (or *included** in the set A) if each element of the set B is also an element of the set A. The inclusion of the set B in the set A is denoted by the symbol "$\subset$" and written as follows: $B \subset A$. The set B is not contained in A ($B \not\subset A$) if there exists at least one element $b \in B$ such that $b \notin A$.

Two sets A and B are said to be *coincident* (or *equal*) if they consist of the same elements; in this case we write $A = B$.

The inclusion relation of two sets has the following properties:

1^0) $A \subset A$;
2^0) if $A \subset B$ and $B \subset A$, then $A = B$;
3^0) if $B \subset A$ and $A \subset C$, then $B \subset C$.

1.1.2. The Concept of the Empty Set

Consider the set $\{a\}$ of elements of E for which $a \neq a$. Such a set does not contain any elements; it is called the *empty set* and is denoted $\varnothing$:

$$\varnothing = \{a \in E \,:\, a \neq a\}.$$

If a set $A \neq \varnothing$, then A contains at least one element.

The sets A and $\varnothing$ are called the *improper* subsets of the set A. The remaining subsets of A are called *proper* subsets. The following two properties are obvious:

4^0) $\varnothing \subset A$ for any A in E;
5^0) $A \subset E$ for any A in E.

1.1.3. Operations on Sets

Let A and B be two sets in E. The *union* (or *sum*) of the sets A and B is defined to be the set C consisting of the elements belonging to at least one of the sets A and B. The union C of the two sets A and B is denoted as follows: $C = A \cup B$.

Similarly $C = \bigcup\limits_{\alpha} A_\alpha$ denotes the union of any number of sets A_α, where the index α in turn belongs to some set.

The *intersection* of the sets A and B is defined to be the set C consisting of the elements that belong to both the sets A and B. The intersection of the two sets A and B is denoted as follows: $C = A \cap B$.

In exactly the same way $C = \bigcap\limits_{\alpha} A_\alpha$ denotes the intersection of any number of sets A_α.

The operations just introduced have the following properties, whose verification is immediate:

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*It is a *subset*.

6^0) $A \cup B = B \cup A$ (commutativity of union);
7^0) $A \cap B = B \cap A$ (commutativity of intersection);
8^0) $A \cup (B \cup C) = (A \cup B) \cup C$ (associativity of union);
9^0) $A \cap (B \cap C) = (A \cap B) \cap C$ (associativity of intersection);
10^0) $A \cup A = A$;
11^0) $A \cap A = A$;
12^0) $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$ (distributivity of intersection), and

$$\left(\bigcup_\alpha A_\alpha\right) \cap B = \bigcup_\alpha \left(A_\alpha \cap B\right);$$

13^0) $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$ (distributivity of union), and

$$\left(\bigcap_\alpha A_\alpha\right) \cup C = \bigcap_\alpha \left(A_\alpha \cup C\right);$$

14^0) $A \cup \varnothing = A$;
15^0) $A \cap \varnothing = \varnothing$;
16^0) $A \cup E = E$;
17^0) $A \cap E = A$;
18^0) $A \subset B$ is equivalent to $A \cup B = B$ and to $A \cap B = A$.

Properties 1^0)–18^0) possess duality in the sense that if the symbols $\subset$, $\cup$, and $\varnothing$ are replaced by $\supset$, $\cap$, and E respectively in any one of them, the result is another formula from the same list of 18 formulas. Thus to each theorem whose proof is based on one of the properties 1^0)–18^0) there corresponds a dual theorem.

The *difference* of the sets A and B is the set of elements of A that do not belong to B. The difference of the sets A and B is denoted as follows: $A \setminus B$. Thus $A \setminus B = \{x \in E : x \in A \text{ and } x \notin B\}$. In this definition it is not assumed that $A \supset B$.

The *complement* of the set A, denoted A', is defined to be the set of elements of E not belonging to A:

$$A' = \{x \in E : x \notin A\} = E \setminus A.$$

The following properties are obvious:

19^0) $A \cup A' = E$;
20^0) $A \cap A' = \varnothing$;
21^0) $\varnothing' = E$;
22^0) $E' = \varnothing$;
23^0) $(A')' = A$;

24^0) The relation $A \subset B$ is equivalent to $A' \supset B'$;

25^0) $(A \cup B)' = A' \cap B'$ (the complement of a union is the intersection of the complements), $\left(\bigcup_\alpha A_\alpha\right)' = \bigcap_\alpha A'_\alpha$;

26^0) $(A \cap B)' = A' \cup B'$ (the complement of an intersection is the union of the complements), $\left(\bigcap_\alpha A_\alpha\right)' = \bigcup_\alpha A'_\alpha$.

Properties 19^0)–26^0) also possess duality, just like properties 1^0)–18^0).

The *symmetric difference* of two sets A and B is the set C defined as follows: $C = (A \cup B) \setminus (A \cap B)$. The symmetric difference of the sets A and B is denoted $A \Delta B$. It is easy to see that $A \Delta B = (A \setminus B) \cup (B \setminus A)$.

1.1.4. Mappings. The Cartesian Product of Sets

The most important concept in analysis is the concept of a mapping of one set into another.

Let A and B be two sets. Suppose that *with each element a of the set A there is associated a definite element $b = g(a)$ belonging to the set B.* In this case a *mapping* g is defined from the set A into the set B and can be concisely denoted as follows:

$$g : A \to B.$$

The element b is called the *image* of the element a under the mapping g, and the element a is called the *preimage* of the element b. The element $a \in A$ is often called a *variable* and the element $g(a) \in B$ the *value* of g at the element a.

If each element b of the set B has at least one preimage a under the mapping g, we say that the mapping g is a mapping of A *onto* B.

Let $M \subset A$. Then $g(M)$ denotes the set of those elements of B that are the images of elements $a \in M$. The set $g(M)$ is called the *image* of the set M under the mapping g.

Thus if $g : A \to B$ and $g(A) = B$, then g is a mapping of A onto B. If $g(A) \subset B$, we say that g is a mapping of A *into* B.

If $N \subset B$, we denote by $g^{-1}(N)$ the set of elements of A whose images under the mapping g lie in N. The set $g^{-1}(N)$ is called the *complete preimage* of the set N under the mapping g.

It is sometimes convenient to call a mapping $g : A \to B$ a *function* with domain of definition A and range of values contained in B. In some areas of mathematics, depending on the nature of the sets A and B and the properties of g, the mapping g is called an *operator*, a *functional*, etc.

A mapping g of the set A onto the set B is said to be *one-to-one* (or a *bijection*) if each element of the set B has only one preimage under the mapping g.

If $g : A \to B$ and if it follows from the relation $a \neq a'$ that $g(a) \neq g(a')$, the mapping g is called an *injection*. Thus in this case for any $b \in B$ the equation $g(a) = b$ has at most one solution. An injection is a one-to-one mapping of A *into* B.

If $g : A \to B$ and if for each $b \in B$ the equation $g(a) = b$ has at least one solution, then the mapping g is a *surjection*. A surjection is a mapping of A *onto* B.

According to what has been said a bijection is simultaneously an injection and a surjection, i.e., for any $b \in B$ the equation $g(a) = b$ has one and only one solution.

Obviously if g is a one-to-one mapping of a set A onto a set B or a one-to-one correspondence between the elements of these two sets, it is possible to define the mapping g^{-1} *inverse* to g, i.e., knowing the element b it is possible to determine the element a uniquely from the equation $g(a) = b$ and then set $a = g^{-1}(b)$.

Let A be a set. Consider a subset R of the set of all ordered pairs (a, b) of elements of this set. If $(a, b) \in R$, we say that a and b are connected by the relation $\varphi = \varphi_R$ and denote this fact $a \underset{\varphi}{\sim} b$. The relation φ is called an *equivalence relation* if it is *reflexive* (i.e., $a \underset{\varphi}{\sim} a$ for any element $a \in A$), *symmetric* (i.e., if $a \underset{\varphi}{\sim} b$, then $b \underset{\varphi}{\sim} a$), and *transitive* (i.e., if $a \underset{\varphi}{\sim} b$ and $b \underset{\varphi}{\sim} c$, then $a \underset{\varphi}{\sim} c$).

It is not difficult to verify that these conditions are necessary and sufficient for the relation φ to partition the set A into disjoint classes.

Indeed, a partition of the set into classes defines a certain equivalence relation. In this situation $a \underset{\varphi}{\sim} b$ means that a and b belong to the same class.

Conversely, if φ is some equivalence relation on the set A and K_a is the class of elements $x \in A$ equivalent to A, then by reflexivity $a \in K_a$. We shall show that two such classes either do not intersect or coincide. Let $c \in A$ and $c \in K_a$ and $c \in K_b$, i.e., $c \underset{\varphi}{\sim} a$, and $c \underset{\varphi}{\sim} b$. Then by symmetry $a \underset{\varphi}{\sim} c$ and by transitivity $a \underset{\varphi}{\sim} b$. By this relation if $x \in K_a$, i.e., then $x \underset{\varphi}{\sim} a \underset{\varphi}{\sim} b$, and therefore $x \underset{\varphi}{\sim} b$, i.e., $x \in K_b$. In exactly the same way it is proved that each element $y \in K_b$ belongs to K_a. Thus two classes K_a and K_b having a common element must coincide.

If g is a mapping of the set A into B, then the elements of the set A whose images coincide form disjoint classes in the set A, i.e., partitioning into classes is closely connected with the concept of a mapping.

We now pass to the study of an important concept—the Cartesian product of sets. Let $\Omega = \{1, 2, \ldots, n\}$, and let $A_1, A_2, \ldots, A_n$ be subsets of some set A. The *Cartesian product* of the sets A_k, denoted $\prod_{k=1}^{n} A_k$, is defined as the set of functions f mapping Ω into A such that $f(k) \in A_k$ for all $k = 1, \ldots, n$. Obviously $\prod_{k=1}^{n} A_k$ can be regarded as the set of all possible collections $(a_1, a_2, \ldots, a_n)$ with $a_k \in A_k$. Similarly if $\Omega = \{1, 2, 3, \ldots\}$, then $\prod_{k=1}^{\infty} A_k$ is the set of all sequences $\{a_1, a_2, a_3, \ldots,\}$, with $a_k \in A_k$ for any k.

In exactly the same way if Ω is an arbitrary set and a subset A_α of the set A is defined for each $\alpha \in \Omega$, the *Cartesian product* $\prod_\alpha A_\alpha$ of the sets A_α is defined as the set of functions f mapping Ω into A for which $f(\alpha) \in A_\alpha$ for all $\alpha \in \Omega$.

If $\Omega = \{1, 2, \ldots, n\}$, then $\prod_{k=1}^{n} A_k$ is also denoted $A_1 \times A_2 \times \cdots \times A_n$; if $A = A_i = A_j$ for any $i, j = 1, \ldots, n$, the notation $A \times A \times \cdots A = A^n$ is used.

The concept of the upper limit of a sequence of sets is also of interest. Suppose some infinite sequence of sets $\{A_n\}$ is given. The set A consisting of the points belonging to an infinite number of the sets A_n is called the *upper limit* of the sequence of sets A_n and denoted as follows:

$$A = \overline{\lim} A_n .$$

The *lower limit* of the sequence of sets $\{A_n\}$ is defined as the set A consisting of the elements belonging to all but a finite number of the sets A_n. For the lower limit of a sequence of sets we use the following notation:

$$A = \underline{\lim} A_n .$$

If a sequence of sets is monotonically increasing, i.e., $A_1 \subset A_2 \subset A_3 \subset \ldots$, then

$$\overline{\lim} A_n = \underline{\lim} A_n = \bigcup_{i=1}^{\infty} A_i .$$

Similarly if a sequence of sets is monotonically decreasing, then

$$\overline{\lim} A_n = \underline{\lim} A_n = \bigcap_{i=1}^{\infty} A_i .$$

1.2. The Cardinality of a Set

Two sets are said to be *equivalent* if a one-to-one correspondence exists between them. We shall say that equivalent sets have the same *cardinality* or *cardinal number.* Thus with each set a certain object is associated—its cardinality—and the same cardinality is associated with equivalent sets.

A set is called *finite* if it is equivalent to the set of natural numbers $\{1,2,\ldots,n\}$ for some n. It is natural to denote the cardinality of such a set by the same letter n.

The first infinite cardinal is the cardinality of the set of all natural numbers $\{1,2,\ldots,\}$. Sets of this cardinality are called *countable*, and we shall denote their cardinality by the letter a.

The cardinality of the set of points of the interval $[0,1]$ is called the *cardinality of the continuum.* This cardinality is denoted by the letter c.

The cardinality of an arbitrary set X will be denoted by the symbol $m(X)$.

EXAMPLES

1. The set of points of a sphere in three-dimensional space is equivalent to the set of points of the extended plane. A one-to-one correspondence can be established using stereographic projection, for example.

2. The set of rational numbers is countable. Let $r = p/q$, where $q > 0$ and p and q are integers and the fraction is in lowest terms. We call the number $|p| + q$ the *height* of the rational number f. It is clear that the number of fractions having a given height is finite. It then remains only to enumerate all the rational numbers having heights 1, 2, ... Then every rational number will receive one index—a natural number.

3. The set of points of an interval $[a,b]$, $a \neq b$, is uncountable. Indeed suppose, to the contrary, that the set of points of the interval can be arranged in a sequence

$$x_1, x_2, \ldots, x_n, \ldots.$$

Divide the interval $[a,b]$ into three equal parts. Choose a part not containing the point x_1 in either its interior or on its boundary. We denote the interval chosen by λ_1. We then denote by λ_2 one of the three equal parts of the interval λ_1 not containing the point x_2, etc. The infinite sequence of intervals $\lambda_1 \supset \lambda_2 \supset \cdots \supset \lambda_n \supset \cdots$ has one common point γ by a well-known theorem of analysis. The point γ belongs to each of the intervals λ_k and consequently cannot coincide with any of the points x_k. Thus the sequence $x_1, x_2, \ldots, x_n, \ldots$ cannot contain all the points of an interval.