Stochastic models, estimation, and control

VOLUME 3

PETER S. MAYBECK

Stochastic models, estimation, and control

VOLUME 3

PETER S. MAYBECK

DEPARTMENT OF ELECTRICAL ENGINEERING
AIR FORCE INSTITUTE OF TECHNOLOGY
WRIGHT-PATTERSON AIR FORCE BASE
OHIO

1982

ACADEMIC PRESS
A Subsidiary of Harcourt Brace Jovanovich, Publishers
New York London
Paris San Diego San Francisco São Paulo Sydney Tokyo Toronto

DR36/10

ACADEMIC PRESS, INC.
111 Fifth Avenue, New York, New York 10003

United Kingdom Edition published by
ACADEMIC PRESS, INC. (LONDON) LTD.
24/28 Oval Road, London NW1 7DX

Library of Congress Cataloging in Publication Data

Maybeck, Peter S.
Stochastic models, estimation, and control.

(Mathematics in science and engineering)
Includes bibliographies and indexes.
1. System analysis. 2. Control theory. 3. Estimation theory. I. Title. II. Series.
QA402.M37 519.2 78-8836
ISBN 0-12-480703-8 (v.3) AACR2

PRINTED IN THE UNITED STATES OF AMERICA

82 83 84 85 9 8 7 6 5 4 3 2 1

Preface

As was true of Volumes 1 and 2, the purpose of this book is twofold. First, it attempts to develop a thorough understanding of the fundamental concepts incorporated in stochastic processes, estimation, and control. Second, and of equal importance, it provides experience and insights into applying the theory to realistic practical problems. Basically, it investigates the theory and derives from it the tools required to reach the ultimate objective of systematically generating effective designs for estimators and stochastic controllers for operational implementation.

Perhaps most importantly, the entire text follows the basic principles of Volumes 1 and 2 and concentrates on presenting material in the most lucid, best motivated, and most easily grasped manner. It is oriented toward an engineer or an engineering student, and it is intended both to be a textbook from which a reader can *learn* about estimation and stochastic control and to provide a good reference source for those who are deeply immersed in these areas. As a result, considerable effort is expended to provide graphical representations, physical interpretations and justifications, geometrical insights, and practical implications of important concepts, as well as precise and mathematically rigorous development of ideas. With an eye to practicality and eventual implementation of algorithms in a digital computer, emphasis is maintained on the case of continuous-time dynamic systems with sampled-data measurements available; nevertheless, corresponding results for discrete-time dynamics or for continuous-time measurements are also presented. These algorithms are developed in detail to the point where the various design trade-offs and performance evaluations involved in achieving an efficient, practical configuration can be understood. Many examples and problems are used throughout the text to aid comprehension of important concepts. Furthermore, there is an extensive set of references in each chapter to allow pursuit of ideas in the open literature once an understanding of both theoretical concepts and practical implementation issues has been established through the text.

This volume builds upon the foundations set in Volumes 1 and 2. Chapter 13 introduces the basic concepts of stochastic control and dynamic programming as the fundamental means of synthesizing optimal stochastic control laws. Subsequently, Chapter 14 concentrates attention on the important LQG synthesis of controllers, based upon linear system models, quadratic cost criteria for defining optimality, and Gaussian noise models. This chapter and much of Chapter 13 can be understood solely on the basis of modeling and estimation concepts from Volume 1. It covers the important topics of stability and robustness, and synthesis and realistic performance analysis of digital (and analog) controllers, including many practically useful controller forms above and beyond the basic LQG regulator. Finally, Chapter 15 develops practical nonlinear controllers, exploiting not only the linear control insights from the preceding two chapters and Volume 1, but also the nonlinear stochastic system modeling and both adaptive and nonlinear filtering of Volume 2.

Thus, these three volumes form a self-contained and integrated source for studying stochastic models, estimation, and control. In fact, they are an outgrowth of a three-quarter sequence of graduate courses taught at the Air Force Institute of Technology; and thus the text and problems have received thorough class testing. Students had previously taken a basic course in applied probability theory, and many had also taken a first control theory course, linear algebra, and linear system theory, but the required aspects of these disciplines have also been developed in Volume 1. The reader is assumed to have been exposed to advanced calculus, differential equations, and some vector and matrix analysis on an engineering level. Any more advanced mathematical concepts are developed within the text itself, requiring only a willingness on the part of the reader to deal with new means of conceiving a problem and its solution. Although the mathematics becomes relatively sophisticated at times, efforts are made to motivate the need for, and to stress the underlying basis of, this sophistication.

The author wishes to express his gratitude to the many students who have contributed significantly to the writing of this book through their feedback to me—in the form of suggestions, questions, encouragement, and their own personal growth. I regard it as one of God's many blessings that I have had the privilege to interact with these individuals and to contribute to their growth. The stimulation of technical discussions and association with Professors Michael Athans, John Deyst, Nils Sandell, Wallace Vander Velde, William Widnall, and Alan Willsky of the Massachusetts Institute of Technology, Professor David Kleinman of the University of Connecticut, and Professors Jurgen Gobien, James Negro, J. B. Peterson, and Stanley Robinson of the Air Force Institute of Technology has also had a profound effect on this work. I deeply appreciate the continual support provided by Dr. Robert Fontana, Chairman of the Department of Electrical Engineering at AFIT,

and the painstaking care with which many of my associates have reviewed the manuscript. Finally, I wish to thank my wife, Beverly, and my children, Kristen and Keryn, without whose constant love and support this effort could not have been fruitful.

Notation

Vectors, Matrices

Scalars are denoted by upper or lower case letters in italic type.

Vectors are denoted by lower case letters in boldface type, as the vector $\mathbf{x}$ made up of components x_i.

Matrices are denoted by upper case letters in boldface type, as the matrix $\mathbf{A}$ made up of elements A_{ij} (ith row, jth column).

Random Vectors (Stochastic Processes), Realizations (Samples), and Dummy Variables

Random vectors are set in boldface sans serif type, as $\mathsf{x}(\cdot)$ or frequently just as x made up of scalar components x_i; $\mathsf{x}(\cdot)$ is a mapping from the sample space Ω into real Euclidean n-space R^n: for each $\omega_k \in \Omega$, $\mathsf{x}(\omega_k) \in R^n$.

Realizations of the random vector are set in boldface roman type, as $\mathbf{x}$: $\mathsf{x}(\omega_k) = \mathbf{x}$.

Dummy variables (for arguments of density or distribution functions, integrations, etc.) are denoted by the equivalent Greek letter, such as $\boldsymbol{\xi}$ being associated with x: e.g., the density function $f_{\mathsf{x}}(\boldsymbol{\xi})$. The correspondences are $(\mathsf{x}, \boldsymbol{\xi})$, $(\mathsf{y}, \boldsymbol{\rho})$, $(\mathsf{z}, \boldsymbol{\zeta})$, $(\mathsf{Z}, \mathscr{Z})$.

Stochastic processes are set in boldface sans serif type, just as random vectors are. The n-vector stochastic process $\mathsf{x}(\cdot,\cdot)$ is a mapping from the product space $T \times \Omega$ into R^n, where T is some time set of interest: for each $t_j \in T$ and $\omega_k \in \Omega$, $\mathsf{x}(t_j, \omega_k) \in R^n$. Moreover, for each $t_j \in T$, $\mathsf{x}(t_j, \cdot)$ is a random vector, and for each $\omega_k \in \Omega$, $\mathsf{x}(\cdot, \omega_k)$ can be thought of as a particular time function and is called a *sample* out of the process. In analogy with random vector realizations, such samples are set in boldface roman type: $\mathsf{x}(\cdot, \omega_k) = \mathbf{x}(\cdot)$ and $\mathsf{x}(t_j, \omega_k) = \mathbf{x}(t_j)$. Often the second argument of a stochastic process is suppressed: $\mathsf{x}(t, \cdot)$ is often written as $\mathsf{x}(t)$, and this stochastic process evaluated at time t is to be distinguished from a process sample $\mathbf{x}(t)$ at that same time.

Subscripts

a:	augmented	I:	ideal
c:	continuous-time; or controller	m:	model (command generator model)
d:	discrete-time; or desired	n:	nominal; or noise disturbance
e:	error	r:	reference variable
f:	final time; or filter (shaping filter)	ss:	steady state
		t:	truth model
		0:	initial time

Superscripts

$^{\mathrm{T}}$:	transpose (matrix)	$^{\mathrm{R}}$:	right inverse
*:	complex conjugate transpose; or optimal	$^{\#}$:	pseudoinverse
$^{-1}$:	inverse (matrix)	$\hat{}$:	estimate
$^{\mathrm{L}}$:	left inverse	$\bar{}$:	Fourier transform; or steady state solution

Matrix and Vector Relationships

$\mathbf{A} > \mathbf{0}$: **A** is positive definite.
$\mathbf{A} \geq \mathbf{0}$: **A** is positive semidefinite.
$\mathbf{x} \leq \mathbf{a}$: componentwise, $x_1 \leq a_1, x_2 \leq a_2, \ldots$, and $x_n \leq a_n$.

Commonly Used Abbreviations and Symbols

$E\{\cdot\}$	expectation	w.p.1	with probability of one
$E\{\cdot\|\cdot\}$	conditional expectation	$\|\cdot\|$	determinant of
exp(·)	exponential	$\\|\cdot\\|$	norm of
lim.	limit	$\sqrt{\cdot}$	matrix square root of (see Volume 1)
l.i.m.	limit in mean (square)	$\in$	element of
ln(·)	natural log	$\subset$	subset of
m.s.	mean square	$\{\cdot\,\vdots\,\cdot\}$	set of; such as $\{\mathbf{x} \in \mathbf{X} : \mathbf{x} \leq \mathbf{a}\}$, i.e., the set of $\mathbf{x} \in \mathbf{X}$ such that $x_i \leq a_i$ for all i
max.	maximum		
min.	minimum		
R^n	Euclidean n-space		
sgn(·)	signum (sign of)		
tr(·)	trace		

List of symbols and pages where they are defined or first used

$\mathbf{A}_{ij}$	153
$\mathbf{a}$	247
$\mathscr{A}$	105
$\mathbf{B}$	6; 88
$\mathbf{B}_{cy}$	167
$\mathbf{B}_{cz}$	167
$\mathbf{B}_{d}$	6; 76
$\bar{\mathbf{B}}$	74; 76
$\mathscr{B}$	105
$\mathbf{C}$	69; 231
$\mathbf{c}$	223
$\mathscr{C}$	13; 26
$\mathscr{C}^*$	13; 15; 47; 48
$\mathbf{D}_y$	81; 89; 231
$\mathbf{D}_z$	81
$\mathbf{d}$	131; 132
$\mathbf{dx}$	5
$\mathbf{d\beta}$	5
$\mathbf{d\beta}_m$	5
$\mathbf{E}$	137
$\mathbf{E}_x$	151
$\mathbf{E}_y$	151
$E\{\cdot\}$	9
$E\{\cdot\|\cdot\}$	13
$\mathbf{e}$	50; 100; 115
$\exp\{\cdot\}$	24
$\mathbf{F}$	6; 88
$\mathbf{f}$	5
$f_{\mathbf{x}}$	31
$f_{\mathbf{x}}(\xi, t \| \rho, t')$	15
$f_{\mathbf{x},\mathbf{y}}$	54
$f_{\mathbf{x}\|\mathbf{y}}$	15; 24
$\mathbf{G}$	6
$\mathbf{G}[\mathbf{x}(t), t]$	5
$\mathbf{G}_{L}$	104
$\mathbf{G}_{c}$	96
$\mathbf{G}_{c}^*$	16; 69; 73; 186
$\bar{\mathbf{G}}_{c}^*$	71; 78; 95; 242
$\mathbf{G}_{cx}$	167
$\mathbf{G}_{cy}$	167
$\mathbf{G}_{cz}$	167
$\mathbf{G}_{d}$	6
$\mathbf{G}_{d}[\mathbf{x}(t_i), t_i]$	6
g	16; 188
$\mathscr{G}$	148
$\mathscr{G}_{L}$	105
$\mathbf{H}$	6; 184
$\mathbf{h}$	5
$\mathbf{I}$	45
$\mathbf{I}$	76
$\mathbf{I}_i$	45
I	105
J	10
J_c	74
J_r	75
$\bar{J}_r$	76
$\mathbf{K}$	16; 19; 185
$\bar{\mathbf{K}}$	100
$\tilde{\mathbf{K}}$	101
$\mathbf{K}_c$	15; 70; 73; 186
$\bar{\mathbf{K}}_c$	71; 95; 188
$\mathbf{K}_{ss}$	79
$\mathbf{K}_x$	144; 145
$\mathbf{K}_y$	144
$\mathbf{K}_\xi$	144; 145
$\mathscr{K}_{i}$	148
$\mathscr{K}_{p}$	148
L_r	75
L	26; 40
$\bar{L}$	46; 47
$\bar{L}'$	48
L_f	21
$\bar{L}_f$	46; 47
$\bar{L}_f'$	48
$\mathbf{M}$	90
$\mathbf{M}_D$	92
$\mathbf{M}_{DTI}$	93
$\mathbf{M}_{TI}$	93

$\mathbf{m}_x$	174
$\mathbf{m}_{xa}$	174
$\mathbf{m}_{ya}$	171
m	5
$\mathbf{n}_d$	127
n	5
$\mathbf{P} = \mathbf{P}(t/t)$	185
$\mathbf{P}(t_i^-)$	19
$\mathbf{P}(t_i^+)$	15; 19
$\bar{\mathbf{P}}$	100; 188
$\bar{\mathbf{P}}^-$	100; 243
$\mathbf{P}_{xa}$	174
$\mathbf{P}_{xx}(t)$	174
$\mathbf{P}_{ya}$	171
$\mathbf{P}_0$	10
$\mathbf{p}(t_i)$	40; 204
p	2
$p_k(t_i)$	40; 254
$\mathbf{Q}$	5
$\bar{\mathbf{Q}}$	75; 76
$\mathbf{Q}_d$	6; 9; 76
$\mathbf{q}$	133
$\mathbf{q}_k$	171
$\mathbf{q}_0$	137
q_k	171
$\mathbf{R}$	5; 10
$\mathbf{R}_c$	5; 184
R^n	13
$\mathbf{r}$	254
r	2
$\mathbf{S}$	48
$\mathbf{S}$	73; 76
$\bar{\mathbf{S}}$	76
$\mathbf{S}_i$	47
$\mathbf{S}_y$	115
$\mathbf{s}$	52
s	5
T	xiii
t_{N+1}	10
t_f	186
t_i	6
$\mathbf{U}$	10; 76
$\bar{\mathbf{U}}$	76
$\mathbf{u}$	14
$\mathbf{u}$	2
$\mathbf{u}_I$	153
$\mathbf{u}_k$	253
$\mathbf{u}_0$	123; 230
$\mathbf{u}^{ACE}$	241
$\mathbf{u}^{CL}$	226
$\mathbf{u}^{DO}$	227
$\mathbf{u}^{F}$	226
$\mathbf{u}^{MF}$	226
$\mathbf{u}^{OL}$	225
$\mathbf{u}^*$	13; 16; 17; 47; 48
$\mathscr{U}$	26
$\mathscr{u}_0$	230
V	85
$\mathbf{v}$	5
$\mathbf{v}_c$	5; 185
$\mathbf{W}$	90; 100
$\mathbf{W}_D$	91
$\mathbf{W}_{DTI}$	92
$\mathbf{W}_{TI}$	92
$\mathbf{W}_{uu}$	74
$\mathbf{W}_{xu}$	74
$\mathbf{W}_{xx}$	74
$\mathbf{w}$	74
$\mathbf{w}_d$	6
$\mathbf{X}$	10; 76
$\bar{\mathbf{X}}$	76
$\mathbf{X}_f$	10
$\mathbf{x}$	3
$\hat{\mathbf{x}}(t)$	185
$\hat{\mathbf{x}}(t_i^-)$	18
$\hat{\mathbf{x}}(t_i^+)$	15; 19
$\hat{\mathbf{x}}(t/t_{i-1})$	243
$\tilde{\mathbf{x}}$	101
$\mathbf{x}_I$	153
$\mathbf{x}_n$	83
$\mathbf{x}_0$	123; 230
$\hat{\mathbf{x}}_k$	254
$\hat{\mathbf{x}}_0$	52
$\mathbf{x}^d$	99
$\mathscr{x}_0$	230
$\mathbf{Y}$	69

Contents

VOLUME 3

Chapter 13 Dynamic programming and stochastic control

Chapter 14 Linear stochastic controller design and performance analysis

Chapter 15 Nonlinear stochastic controllers

CHAPTER 13

Dynamic programming and stochastic control

13.1 INTRODUCTION

Up to this point, we have considered *estimation* of uncertain quantities on the basis of both sampled-data (or continuous) measurements from a dynamic system and mathematical models describing that system's characteristics. Now we wish to consider exerting appropriate *control* over such a system, so as to make it behave in a manner that we desire. This chapter will formulate the *optimal stochastic control* problem, provide the theoretical foundation for its solution, and discuss the potential structure for such solutions. The two subsequent chapters will then investigate the practical design of stochastic controllers for problems adequately described by the "LQG" assumptions (*L*inear system model, *Q*uadratic cost criterion for optimality, and *G*aussian noise inputs; see Chapter 14) and for problems requiring nonlinear models (see Chapter 15).

Within this chapter, Sections 13.2 and 13.3 are meant to be a basic overview of the optimal stochastic control problem, providing insights into important concepts before they are developed in detail later. *Stochastic dynamic programming* is the fundamental tool for solving this problem, and this algorithm is developed in the remainder of the chapter. It will turn out to be inherently a backward recursion in time, and the backward Kolmogorov equation useful for reverse-time propagations is presented in Section 13.4. Dynamic programming itself is then used to generate the optimal stochastic control function, first in Section 13.5 assuming that perfect knowledge of the entire state is available from the system at each sample time, and then in Section 13.6 under the typically more realistic assumption that only incomplete, noise-corrupted measurements are available.

13.2 BASIC PROBLEM FORMULATION

Fundamentally, control problems of interest can be described by the block diagram in Fig. 13.1. There is some *dynamic system* of interest, whose behavior is to be affected by r applied *control* inputs $\mathbf{u}(t)$ in such a way that p specified *controlled variables* $\mathbf{y}_c(t)$ exhibit desirable characteristics. These characteristics are prescribed, in part, as the controlled variables $\mathbf{y}_c(t)$ matching a desired p-dimensional *reference signal* $\mathbf{y}_d(t)$ as closely, quickly, and stably as possible, either over time intervals (as in tracking a dynamic $\mathbf{y}_d(t)$ or maintaining a piecewise constant setpoint $\mathbf{y}_d$) or at specific time instants (as at some given terminal time in a problem). Stability of the controlled system is an essential requirement, after which additional performance measures can be considered. However, the system responds not only to the control inputs, but also *dynamics disturbances* $\mathbf{n}(t)$ from the environment, over which we typically cannot exert any control. These usually cause the system to behave in an other-than-desired fashion. In order to observe certain aspects of the actual system behavior, sensor devices are constructed to output *measurements* $\mathbf{z}(t)$, which unfortunately may not correspond directly to the controlled variables, and which generally are not perfect due to *measurement corruptions* $\mathbf{n}_m(t)$. These typically incomplete and imperfect measurements are then provided as inputs to a controller, to assist in its generation of appropriate controls for the dynamic system.

EXAMPLE 13.1 For instance, the dynamic system might be the internal environment of a building, with controlled variables of temperature and humidity at various locations to be maintained at desired values through the control inputs to a furnace, air conditioner, and flow control dampers in individual ducts. Environmental disturbances would include heat transfer with external and internal sources and sinks, such as atmospheric conditions and human beings. Direct measurements of temperature and humidity at all locations may not be available, and additional measurements such as duct flow rates might be generated, and all such indications are subject to sensor dynamics, biases, imprecision, readout quantization, and other errors.

The case of matching the controlled variables to desired values at a single time might be illustrated by an orbital rendezvous between two space vehicles. On the other hand, continuously

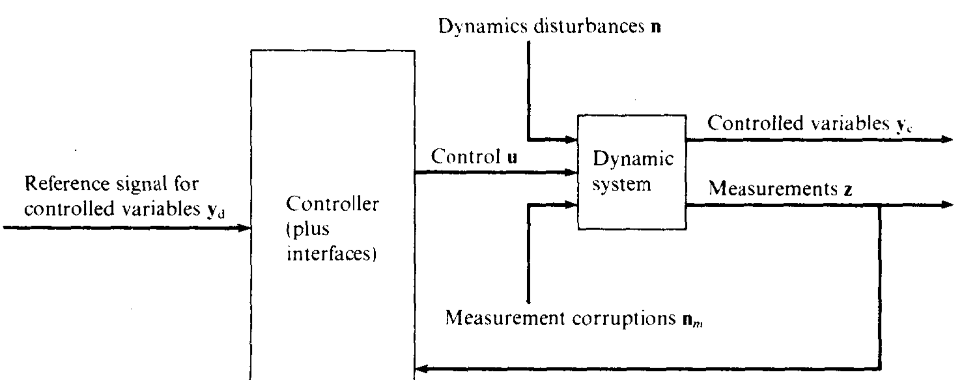

FIG. 13.1 Controlled system configuration.

matching $\mathbf{y}_c(t)$ to a dynamic $\mathbf{y}_d(t)$ is displayed in the problem of tracking airborne targets with communication antennas, cameras, or weapon systems. ■

Let us consider the "dynamic system" block in Fig. 13.1 in more detail. The first task in generating a controller for such a system is to develop a mathematical model that adequately represents the important aspects of the actual system behavior. Figure 13.2 presents a decomposition of the "dynamic system" block assuming that an adequate model can be generated in the form of a stochastic *state* model, with associated controlled variable and measurement output relations expressed in terms of that state. As shown, the controller outputs, **u**, command actuators to impact the continuous-time state dynamics, which are also driven by disturbances, **n**. The sensors generate measurements that are functions of that state, corruptions $\mathbf{n}_m$, and possibly of the controls as well. Similarly, the controlled variables are some function of the system state; the dashed line around "controlled variable output function" denotes the fact that it does not necessarily correspond to any physical characteristic or hardware, but simply a functional relationship.

A number of different types of measurements might be available. In the trivial case, nothing is measured, so that the controller in Fig. 13.1 would be an *open-loop* controller. However, in most cases of interest, there are enough disturbances, parameter variations, modeling inadequacies, and other uncertainties associated with the dynamic system that it is highly desirable to feed back observed values of some quantities relating to *actual* system response. One might conceive of a system structure in which *all states* could be measured *perfectly*; although this is not typically possible in practice, such a structure will be of use to gain insights into the properties of more physically realistic *feedback* control systems. Most typically, there are fewer measuring devices than states, and each of these sensors produces a signal as a nonlinear or linear function of the states, corruptions, and in some cases, controls. Physical sensors are often analog devices, so that continuous-time measurements from continuous-time systems are physically realistic. However, in most of the applications of interest to us, the controller itself will be implemented as software in a digital computer, so we will concentrate on *continuous-time state* descriptions with *sampled-data measurements*, compatible as inputs to the computer. We will also consider continuous-time measurements, and discrete-time measurements from systems described naturally only in discrete time, but the sampled-data case will be emphasized. On the other hand, it will be of importance to consider the controlled variables not only at the sampling instants, but throughout the entire interval of time of interest: reasonable behavior at these discrete times is not sufficient to preclude highly oscillatory, highly undesirable performance between sample times.

The noises **n** and $\mathbf{n}_m$ in Fig. 13.2 correspond to disturbances, corruptions, and sources of error that can be physically observed. Since physical continuous-time noises cannot truly be white, shaping filter models driven by fictitious

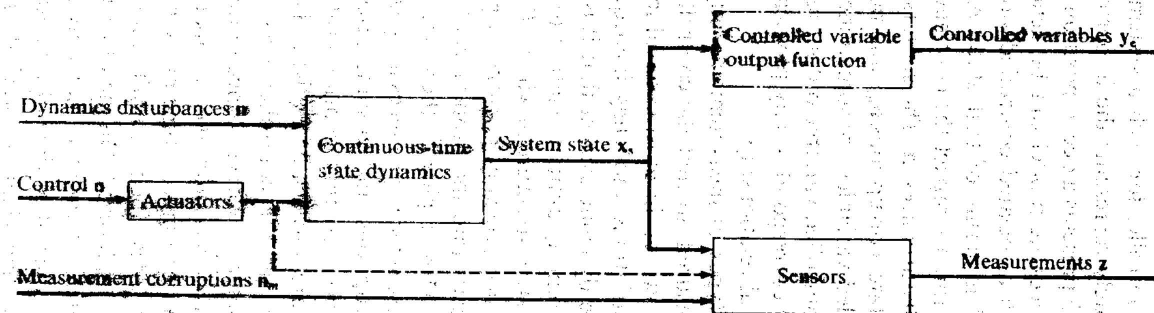

FIG. 13.2 Dynamic system block diagram.