Grundlehren der
mathematischen Wissenschaften 277

A Series of Comprehensive Studies in Mathematics

William Fulton
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Riemann-Roch Algebra

Springer-Verlag
World Publishing Corporation

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Reprinted by World Publishing Corporation, Beijing, 1989
for distribution and sale in The People's Republic of China only

AMS Subject Classification: 14C40

Library of Congress Cataloging in Publication Data
Fulton, William
Riemann-Roch algebra.
(Grundlehren der mathematischen Wissenschaften; 277)
Bibliography: p.
Includes index.
1. Geometry, Algebraic. 2. Riemann-Roch theorems.
I. Lang, Serge, 1927- . II. Title. III. Series.
QA564.F85 1985 512'.33 84-26842

Typeset by Composition House Ltd., Salisbury, England.
Printed and bound by R. R. Donnelley & Sons, Harrisonburg, Virginia.
Printed in the United States of America.

9 8 7 6 5 4 3 2 1

ISBN 0-387-96086-4 Springer-Verlag New York Berlin Heidelberg Tokyo
ISBN 3-540-96086-4 Springer-Verlag Berlin Heidelberg New York Tokyo
ISBN 7–5062–0300–6

Introduction

In various contexts of topology, algebraic geometry, and algebra (e.g. group representations), one meets the following situation. One has two contravariant functors K and A from a certain category to the category of rings, and a natural transformation

$$\rho : K \to A$$

of contravariant functors. The Chern character being the central example, we call the homomorphisms

$$\rho_X : K(X) \to A(X)$$

characters. Given $f: X \to Y$, we denote the pull-back homomorphisms by

$$f^K : K(Y) \to K(X) \qquad \text{and} \qquad f^A : A(Y) \to A(X).$$

As functors to abelian groups, K and A may also be covariant, with push-forward homomorphisms

$$f_K : K(X) \to K(Y) \qquad \text{and} \qquad f_A : A(X) \to A(Y).$$

Usually these maps do not commute with the character, but there is an element $\tau_f \in A(X)$ such that the following diagram is commutative:

$$\begin{array}{ccc} K(X) & \xrightarrow{\tau_f \cdot \rho_X} & A(X) \\ {\scriptstyle f_K}\downarrow & & \downarrow{\scriptstyle f_A} \\ K(Y) & \xrightarrow[\rho_Y]{} & A(Y) \end{array}$$

The map in the top line is ρ_X multiplied by τ_f.

When such commutativity holds, we say that Riemann–Roch holds for f. This type of formulation was first given by Grothendieck, extending the work of Hirzebruch to such a relative, functorial setting. Since then several other theorems of this Riemann-Roch type have appeared. Underlying most of these there is a basic structure having to do only with elementary algebra, independent of the geometry. One purpose of this monograph is to describe this algebra independently of any context, so that it can serve axiomatically as the need arises.

A common feature of these Riemann-Roch theorems is that a given morphism f is factored into $p \circ i$:

$$X \xrightarrow{i} P \xrightarrow{p} Y,$$

where i is a closed imbedding and p is a bundle projection. One constructs a deformation from i to the zero-section imbedding of X in the normal bundle to X in P, suitably completed at infinity. General procedures, which we axiomatize here, allow one to deduce a general Riemann-Roch theorem from the elementary cases of imbeddings in and projections from bundles; these cases are usually handled by direct calculation.

We illustrate the formalism by giving a complete elementary account of Grothendieck's Riemann-Roch theorem in the context of schemes and local complete intersection morphisms, as first presented in [SGA 6]. Here $K(X)$ is the Grothendieck ring of locally free sheaves on X, and $A(X)$ is an associated graded group of $K(X)$, with rational coefficients. To prepare for this we include self-contained discussions of several important subjects from algebra and algebraic geometry, such as: λ-rings, Adams operations, γ-filtrations, Chern classes, algebraic K-theory, regular imbeddings and Koszul complexes, sheaves on projective bundles, and local complete intersections.

Manin's very useful notes [Man] were also written to give an accessible account of parts of [SGA 6], for the case of imbeddings of nonsingular varieties. Several developments since then allow us to give both a more elementary and more complete treatment, including a complete proof of the main theorem, as well as some conjectures left open in [SGA 6]. Most important among these developments are: (a) an understanding of deformation to the normal bundle (cf. [J], [BFM 1], [V], [BFM 2], [FM]); (b) the use of Castelnuovo-Mumford "regular" sheaves on projective bundles (cf. [Q]). Among the resulting improvements we mention:

(1) A proof that the γ-filtration on $K(X)$ is finer than the topological filtration (V, §3).
(2) A Riemann-Roch theorem for the Adams operations ψ^j without denominators (V, §6).
(3) An elementary construction of the push-forward f_K for a projective local complete intersection morphism f (V, §4).

Of these, (1) and (2) were conjectured in [SGA 6]. Other features included are:

(4) An Intersection Formula for K-theory (VI, §1).
(5) A direct proof, using a power-series calculation of R. Howe, for Grothendieck Riemann-Roch for bundle projections (II, §2).
(6) An equivalence between forms of Riemann-Roch for the Chern character and Adams operators (III, §4).

Chapter I contains an elementary treatment of λ-rings and Chern classes; the excellent exposition of Atiyah and Tall [AT] can be referred to for more on λ-rings. We include a proof of a splitting principle for abstract Chern classes; in our application in Chapter V, however, this splitting principle will be evident, so the reader can skip this proof.

In Chapter II we develop the abstract Riemann-Roch formalism. The main new feature here is an axiomatic formulation of the deformation to the normal bundle: to prove a Riemann-Roch theorem for a given imbedding, it suffices to "deform" it to an "elementary imbedding" for which one knows the theorem. We also axiomatize the dual case of an "elementary projection".

Chapter III describes the γ-filtration of Grothendieck, and constructs Chern classes in the associated graded ring.

Chapter IV is a chapter of "intermediate algebraic geometry", which could supplement a text such as Hartshorne's [H]. We establish the basic category of algebraic geometry for which we shall prove the Riemann-Roch formula, namely the category of regular morphisms. By this we mean morphisms which can be factored into a local complete intersection imbedding, and a projection from a projective bundle. We include a short proof of Micali's theorem on regular sequences, and basic facts about regular imbeddings, conormal sheaves, and blowing up. Theorem 4.5 on the residual structure of a proper transform is, we believe, new. The culmination of this chapter is a simple construction of the deformation to the normal bundle. Many of the results of Chapter IV are not needed for the proof of Riemann-Roch proper, but are included for completeness.

All these ideas come together in Chapter V, where the λ-ring $K(X)$ is shown to satisfy the abstract properties of the first three chapters. The Grothendieck Riemann-Roch theorem (including the version without denominators), and analogous theorems for the Adams operators, follow quickly.

Chapter VI contains an Intersection Formula in the context of K-theory which seems to be new in this generality, and which is analogous to the "excess intersection formula" of [FM], see also [F 2], Theorem 6.3. The formula is proved by using the general formalism of basic deformations, together with the geometric construction of the deformation to the normal bundle. This follows a pattern similar to the proof for

Riemann–Roch itself, and provides another striking application of the formalism of Chapter II.

In Chapter VI, we also discuss the relation of the Grothendieck group of locally free sheaves with the Grothendieck group of all coherent sheaves. We give an application to the calculation of an exact sequence for K of a blow up of a regularly imbedded subscheme, relying on the Intersection Formula. Finally, we discuss briefly and incompletely how Riemann–Roch can be extended beyond the case of local complete intersections. In addition, we sketch several other contexts where the formalism developed here can be applied. It would take another book to give a systematic treatment of these topics, including the relations between K-theory, the Chow group and étale cohomology in a more schemy and sheafy context than [F 2].

We have made our exposition self-contained from [H] for algebraic geometry, [L] for general algebra, and the simpler parts of [Mat] for a little more commutative algebra. Thus we have included proofs of elementary facts whenever necessary to achieve this.

At least in first reading, the reader interested only in a fast proof of Riemann–Roch is advised to skim Chapters I, IV, and the first half of Chapter V. More is included in these chapters than is strictly needed for Riemann–Roch, with the hope that this important material will be more accessible than its previous position in SGA and EGA permit. Those interested primarily in the Riemann–Roch theorem should concentrate on Chapters II, III, and V.

We have not discussed applications to the theory of group representations. For these, we refer especially to the articles by Atiyah–Tall, Evens, Kahn, Knopfmacher, Thomas, as well as Grothendieck's general discussion as listed in the Bibliography. On the other hand, the applications to group representations are not independent of those to algebraic geometry. Even though the K-groups can be defined in terms of modules, one can analyze them via considerations of topology, classifying spaces, and algebraic geometry, so there is a considerable amount of feedback.

We also do not discuss applications to topology. We refer to the lectures by Atiyah [At] and Bott [Bo] for some K-theory like that of Chapters I and III in a topological context, stopping short of Riemann–Roch theorems, however.

We hope that the simpler logical structure of the proofs which emerges in this treatise will make it easier to understand these results, and to find new situations to which this "Riemann–Roch algebra" applies.

Contents

CHAPTER I

λ-Rings and Chern Classes

This chapter describes first the basic ring structure of the objects to be encountered later in a more geometric context. The algebra involved is elementary and self-contained. We have axiomatized certain notions which originally arose in the theory of vector bundles. Actually we work with two rings, one of them usually graded. We also develop the formalism of Hirzebruch polynomials, which belongs to the basic theory of symmetric functions. We have preserved original names like Chern classes, Todd character, etc., although the algebra involved here deals only with a pair of rings and some elementary formal manipulation of power series, independently of the geometry from which they came

We now make additional comments concerning the way these notions arise in applications to algebraic geometry and group representations. These are not necessary for a logical understanding of the chapter. However, we may have at least two categories of readers: those who know some Riemann-Roch theory previously and are principally interested in a quick proof of Grothendieck Riemann-Roch; and those who have more limited knowledge in this direction and are thus directly interested in the more elementary material. Our additional comments are addressed to this second category.

A fundamental aim of algebraic geometry is to study divisor classes, or equivalently isomorphism classes of line bundles. More generally, one wishes to study vector bundles, with certain equivalence relations. The Grothendieck relations are those which to each short exact sequence

$$0 \to E' \to E \to E'' \to 0$$

gives the relation

$$[E] = [E'] + [E''].$$

The group of isomorphism classes of vector bundles over a space X modulo these relations is called the Grothendieck group $K(X)$. It has both covariant and contravariant functorial properties, although the covariant ones are much more subtle.

The addition is induced by the direct sum, and there is also a multiplication induced by the tensor product, so that $K(X)$ is in fact a ring. The class of the trivial line bundle is the unit element.

This ring has various structures. First, it has an augmentation, which to E associates its rank $\varepsilon(E)$. Then ε extends to an augmentation on $K(X)$ (algebra homomorphism into $\mathbf{Z}$). The vector bundles themselves generate a semigroup under addition. In §1, we axiomatize this structure by defining "positive elements" whose properties are modelled on those of vector bundles. The elements of augmentation 1 correspond to line bundles, and are thus called line elements.

Second, the ring $K(X)$ has another operation induced by the alternating product. To each integer $i \geqq 0$ we have $\Lambda^i E$, and therefore its class $[\Lambda^i E]$ denoted by $\lambda^i(E)$. A standard elementary formula for the direct sum $E = E' \oplus E''$ of free modules reads

$$\Lambda^n(E) \approx \bigoplus_{i=0}^{n} (\Lambda^i E' \otimes \Lambda^{n-i} E'').$$

Passing to the classes in the K-group, we get the relation

$$\lambda^n(x + y) = \sum_{i=0}^{n} \lambda^i(x)\lambda^{n-i}(y).$$

But this relation amounts to saying that the map

$$x \mapsto \sum \lambda^i(x)t^i = \lambda_t(x) \qquad \text{by definition}$$

is a homomorphism from the additive group of $K(X)$ to the multiplicative group of power series with constant term equal to 1. This gives rise to the notion of λ-ring. A great deal of the formalism of Riemann–Roch algebra can be developed for the general λ-rings. The reader should read simultaneously the beginning of Chapter I and the beginning of Chapter V to see the parallelism between the abstract algebra and the geometric construction giving rise to this algebra.

In the theory of group representations, one may start with the category of finite-dimensional vector spaces over a field k, and a representation of a (finite) group G on the space. Then again we have direct sums, tensor products of (G, k)-spaces and the analogous definition of λ-ring, formed by the isomorphism classes of such spaces modulo the relations in the Grothendieck group. The positive elements are just the classes of such spaces as distinguished from the group generated by them in the Grothendieck group.

In §2 we shall discuss a particular extension of a λ-ring, which gives an axiomatization for the extension obtained from a projective bundle.

The corresponding geometric case is discussed in Chapter V, Theorem 2.3 and Corollary 2.4. Since the existence of the extension is proved in a self-contained way by geometric means in Chapter V, the reader interested only in the geometric application can omit the existence proof of Theorem 2.1 in this chapter. The corresponding graded extension will be constructed in §3.

I §1. λ-Rings with Positive Structure

Let K be a commutative ring. For each integer $i \geqq 0$ suppose given a mapping

$$\lambda^i : K \to K$$

such that $\lambda^0(x) = 1$, $\lambda^1(x) = x$ for all $x \in K$, and if we put

$$\lambda_t(x) = \sum \lambda^i(x) t^i$$

then the map

$$x \mapsto \lambda_t(x)$$

is a homomorphism. This condition is equivalent with the conditions

$$\lambda^k(x + y) = \sum_{i=0}^{k} \lambda^i(x)\lambda^{k-i}(y) \tag{1.1}$$

for all positive integers k. A ring with such a family of maps λ^i is called a **λ-ring**.

In addition, we suppose that the λ-ring has what we shall call a **positive structure**. By this we mean:

A surjective ring homomorphism

$$\varepsilon : K \to \mathbf{Z}$$

called the **augmentation**.

A subset $\mathbf{E}$ of the additive group of K called the set of **positive elements** such that $\mathbf{E}$ together with 0 form a semigroup, satisfying the conditions

$$\mathbf{Z}^+ \subset \mathbf{E}, \qquad \mathbf{E}\mathbf{E} = \mathbf{E}, \qquad K = \mathbf{E} - \mathbf{E}$$

so every element of K is the difference of two elements of $\mathbf{E}$; furthermore for $e \in \mathbf{E}$ we have $\varepsilon(e) > 0$, and if $\varepsilon(e) = r$ then

$$\lambda^i(e) = 0 \text{ for } i > r \qquad \text{and} \qquad \lambda^r(e) \text{ is a unit in } K.$$

We define $\mathbf{L}$ to be the subset of elements $u \in \mathbf{E}$ such that $\varepsilon(u) = 1$. Since $\lambda^1 u = u$, it follows that $\mathbf{L}$ is a subgroup of the units K^*. Elements of $\mathbf{L}$ will be called **line elements**.

An **extension** K' of a λ-ring K is a λ-ring K' containing K, with λ^i and augmentation extending that of K, and with positive elements $\mathbf{E}'$ containing $\mathbf{E}$.

We shall be concerned with a class $\mathfrak{K}$ of λ-rings satisfying, in addition to the preceding conditions, the

Splitting Property. *For any $K \in \mathfrak{K}$ and positive element e in K, there is an extension K' of K in $\mathfrak{K}$ such that e splits in K', i.e.*

$$e = u_1 + \cdots + u_m,$$

with u_i line elements in K'.

It follows by induction that any finite set of positive elements can be simultaneously split in a suitable extension. The splitting property will allow us to deduce general formulas from the simple case of line elements. For example, the property that $\lambda^i(e) = 0$ for e positive and $i > r = \varepsilon(e)$ follows from the fact that $\lambda^i(u) = 0$ for all line elements u and $i > 1$.

More generally, for $u \in \mathbf{L}$ we have directly from the assumptions

$$\lambda_t(u) = 1 + ut,$$

and hence if e is split as above, then

$$\begin{aligned}\lambda_t(e) &= \prod_{i=1}^{r}(1 + u_i t)\\ &= 1 + \sum_{i=1}^{r} s_i(u_1, \ldots, u_r)t^i,\end{aligned}$$

where s_i is the i-th symmetric function. Since the coefficients $\lambda^i(e)$ are given *a priori* as elements of the λ-ring K, we see that the value of the symmetric function $s_i(u_1, \ldots, u_r)$ is independent of the splitting of e as a sum of line elements in K'.

For example, one sees from this formula that

$$\varepsilon(\lambda^i e) = \binom{\varepsilon(e)}{i}. \tag{1.2}$$

In other words, if $\mathbf{Z}$ is given a λ-ring structure by $\lambda^i(n) = \binom{n}{i}$, then the augmentation ε is a homomorphism of λ-rings.

Formulas for $\lambda^k(x \cdot y)$ and $\lambda^k(\lambda^j(x))$ can best be expressed in terms of certain universal polynomials P_k and $P_{k,j}$ as follows. Take independent variables $U_1, \ldots, U_m$ and $V_1, \ldots, V_n$. Let X_i be the i-th elementary symmetric polynomial in $U_1, \ldots, U_m$, and Y_i the i-th elementary symmetric polynomial in $V_1, \ldots, V_n$. For $m \geqq k$, $n \geqq k$, let

$$P_k(X_1, \ldots, X_k, Y_1, \ldots, Y_k) \in \mathbf{Z}[X_1, \ldots, X_k, Y_1, \ldots, Y_k]$$

be the polynomial of weight k in the variables X_i and in the variables Y_i (where X_i and Y_i are assigned weight i), determined by the identity

$$\text{(A)} \qquad \sum_{k \geqq 0} P_k(X_1, \ldots, X_k, Y_1, \ldots, Y_k) T^k = \prod_{i,j} (1 + U_i V_j T).$$

By setting some of the variables U_i or V_j equal to zero for $i, j > k$, one sees that the P_k are independent of the choice of m, $n \geqq k$. Similarly define

$$P_{k,j}(X_1, \ldots, X_{kj}) \in \mathbf{Z}[X_1, \ldots, X_{kj}]$$

of weight kj, by the identity for $m \geqq kj$:

$$\text{(B)} \qquad \sum_{k \geqq 0} P_{k,j}(X_1, \ldots, X_{kj}) T^k = \prod_{i_1 < \cdots < i_j} (1 + U_{i_1} \cdots U_{i_j} T).$$

Now if $x = \sum_{i=1}^{m} u_i$, $y = \sum_{j=1}^{n} v_j$, with u_i, v_j line elements, then

$$\lambda_t(x \cdot y) = \prod (1 + u_i v_j t).$$

From (A) this can be written

$$\text{(1.3)} \qquad \lambda^k(x \cdot y) = P_k(\lambda^1(x), \ldots, \lambda^k(x), \lambda^1(y), \ldots, \lambda^k(y))$$

For example, if x is a line element, then

$$\lambda^k(x \cdot y) = x^k \cdot \lambda^k y, \qquad \text{or} \qquad \lambda_t(xy) = \lambda_{xt}(y).$$

Similarly, if $x = \sum_{i=1}^{m} u_i$, then $\lambda^j(x) = \sum_{i_1 < \cdots < i_j} u_{i_1} \cdots u_{i_j}$, so

$$\lambda_t(\lambda^j x) = \prod_{i_1 < \cdots < i_j} (1 + u_{i_1} \cdots u_{i_j} t).$$

By (B) this can be written

$$\text{(1.4)} \qquad \lambda^k(\lambda^j(x)) = P_{k,j}(\lambda^1(x), \ldots \lambda^{kj}(x)).$$

The identities (1.1)–(1.4) say that our λ-rings are what Grothendieck calls **special** λ-rings ([SGA 6], Exp. 0). This may be reinterpreted as follows. Given a commutative ring A, define $A[[T]]^+ = TA[[T]]$, and let

$$\Lambda(A) = 1 + A[[T]]^+$$

be the set of power series in A with constant term 1. Define an addition in $\Lambda(A)$ by the multiplication of power series; a product $\cdot$ in $\Lambda(A)$ by the formula

$$(1 + \sum a_i t^i) \cdot (1 + \sum b_j t^j) = 1 + \sum P_k(a_1, \ldots, a_k, b_1, \ldots, b_k) t^k;$$

and λ-operations by

$$\lambda^j(1 + \sum a_i t^i) = 1 + \sum P_{k,j}(a_1, \ldots, a_{kj}) t^k.$$

One verifies easily that these definitions make $\Lambda(A)$ into a special λ-ring (cf. [AT] for a readable account). For any λ-ring K,

$$\lambda_t : K \to \Lambda(K)$$

is an additive homomorphism; K is special precisely when λ_t is a homomorphism of λ-rings. Note that identities (1.1)–(1.4) hold for all elements of K, not only positive elements.

Remark. An element x in a λ-ring K is said to have **λ-dimension** $= n$ if $\lambda^i(x) = 0$ for all $i > n$, and $\lambda^n(x) \neq 0$. The ring K is called **λ-finite-dimensional** if every element is a difference of two elements of finite λ-dimension. Since positive elements have finite dimension, our axioms imply that our λ-rings are all finite dimensional. Conversely, given a λ-finite-dimensional special λ-ring K, one can define **E** to be the elements of λ-finite dimension. If one assumes that all one-dimensional elements are units, then E defines a positive structure in our sense.

Let $\sum a_i t^i$ be a power series in $K[[t]]$ with $a_0 = 1$. The coefficients of the inverse series

$$\sum b_i t^i = (\sum a_i t^i)^{-1}$$

can be determined recursively from the coefficients a_i by the relation

$$\sum_{i=0}^{k} a_i b_{k-i} = 0 \qquad \text{for} \quad k > 0.$$

For $e \in \mathbf{E}$ we define the series

$$\sigma_t(e) = \lambda_{-t}(e)^{-1} = \sum_{i=0}^{\infty} \sigma^i(e)t^i.$$

Then for each i we get a map $\sigma^i : K \to K$.

If h is a homomorphism of K into some ring and $\varphi(t) \in K[[t]]$ is a power series, then we let $h(\varphi(t))$ be the power series obtained by applying h to all the coefficients of $\varphi(t)$. In particular, we have

$$\varepsilon(\sigma_t(e))\varepsilon(\lambda_{-t}(e)) = 1.$$

Lemma 1.1. *Let* $\varepsilon(e) = r + 1$. *Then*

$$\varepsilon(\lambda_{-t}(e)) = (1 - t)^{r+1} \quad \text{and} \quad \varepsilon(\sigma_t(e)) = \frac{1}{(1 - t)^{r+1}}.$$

So explicitly in terms of the coefficients,

$$\varepsilon(\lambda^i(e)) = \binom{r+1}{i} \quad \text{and} \quad \varepsilon(\sigma^j($$ [illegible]

Proof. Splitting e into $\sum u_i$, with $\varepsilon(u_i) = 1$,

$$\lambda_t(e) = \prod (1 + u_i t),$$

from which the formula for $\varepsilon(\lambda_{-t}(e))$ is clear. Since σ_t is the inverse of λ_{-t}, the formula for $\varepsilon(\sigma_t(e))$ follows. The last formula follows from the identity

$$\frac{1}{(1 - t)^{r+1}} = \sum \binom{r+j}{j} t^j.$$

I §2. An Elementary Extension of λ-Rings

Given a λ-ring K and a positive element e in K we construct a ring extension K_e of K as follows. Set $\varepsilon(e) = r + 1$,

$$p_e(T) = \sum_{i=0}^{r+1} (-1)^i \lambda^i(e) T^{r+1-i},$$

and let

$$K_e = K[T]/(p_e(T)) = K[\ell],$$

where ℓ is the image of $T \bmod p_e(T)$; we call ℓ the **canonical generator**. We have the defining relation

$$\sum_{i=0}^{r+1}(-1)^i\lambda^i(e)\ell^{r+1-i} = 0.$$

In particular, for $k \geqq r+1$, multiplying by powers of ℓ and using $\lambda^m(e) = 0$ if $m > r+1$, we get the relations

$$\sum_{i=0}^{k}(-1)^i\lambda^i(e)\ell^{k-i} = 0.$$

These relations translate into the single power series relation

$$\left(\sum(-1)^i\lambda^i(e)t^i\right)\left(\sum \ell^j t^j\right) = \sum_{k=0}^{r}\left(\sum_{i=0}^{k}(-1)^i\lambda^i(e)\ell^{k-i}\right)t^k.$$

Theorem 2.1. *There is a unique λ-ring structure on K_e, extending that on K, and satisfying*

$$\lambda_t(\ell) = 1 + \ell t.$$

Proof. First define a λ-ring structure on the polynomial ring $K[T]$, such that $\varepsilon(T) = 1$ and $\lambda^1(T) = T$, $\lambda^i(T) = 0$ for $i > 1$. From the fact that K is a special λ-ring it follows readily that $K[T]$ is also a special λ-ring. To show that this determines a λ-ring structure on K_e, it must be verified that the ideal $I = (p_e(T))$ is preserved by the λ-operations. Set $j = r+1$. Then

$$(-1)^j p_e(T) = \lambda^j(e - T).$$

Using the identity (1.3) for products, one sees that it suffices to verify that

$$\lambda^k\lambda^j(e - T) \in I = (\lambda^j(e - T))$$

for all $k \geqq 1$. From the identity

$$\lambda_t(e - T) = (1 + \lambda^1(e)t + \cdots + \lambda^j(e)t^j)\cdot(1 + Tt)^{-1}$$

it follows that

$$\lambda^k(e - T) = \pm T^{k-j}\lambda^j(e - T) \in I$$

for all $k \geqq j$. Since $\lambda^k(\lambda^j(x)) = P_{k,j}(\lambda^1(x), \ldots, \lambda^{kj}(x))$, it suffices to verify