ENCYCLOPEDIA OF COMPUTER SCIENCE AND TECHNOLOGY

EXECUTIVE EDITORS

Jack Belzer   *Albert G. Holzman*   *Allen Kent*

VOLUME 10

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UNIVERSITY OF PITTSBURGH
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VOLUME 10

Linear and Matrix Algebra to Microorganisms

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CONTRIBUTORS TO VOLUME 10

TILAK AGERWALA, Department of Electrical Engineering, The University of Texas, Austin, Texas: *Microcomputers*

HSU CHANG, International Business Machines, T. J. Watson Research Center, Yorktown Heights, New York: *Memory Technology, Magnetic-Bubble*

DONALD A. DUNN, Department of Engineering-Economic Systems, Stanford University, Stanford, California: *Message Switching*

MICHAEL GRIESGRABER, Control Data Corporation, Minneapolis, Minnesota: *Mass Data Storage*

ALBERT G. HOLZMAN, Department of Industrial Engineering, University of Pittsburgh, Pittsburgh, Pennsylvania: *Linear Programming*

ROBERT G. HOUSE, The Ohio State University, Columbus, Ohio: *Logistics*

PIERRE R. JANIN, Analytab Products, Division of Ayerst Laboratories, Plainview, New York: *Microorganisms, Computer-Assisted Identification*

ANDREW J. KASARDA, Center for Information Science, Lehigh University, Bethlehem, Pennsylvania: *Man-Machine Systems*

WINFRED LEHMANN, Department of Linguistics and Germanic Languages, The University of Texas, Austin, Texas: *Machine Translation*

G. JACK LIPOVSKI, Department of Electrical Engineering, The University of Texas, Austin, Texas: *Microcomputers*

CHARLES T. MEADOW, Graduate School of Library Science, Drexel University, Philadelphia, Pennsylvania: *Man-Machine Communication*

WILLIAM F. MORGAN, President, William F. Morgan Associates, Inc., Minneapolis, Minnesota.

PIERRE J. NAUDO, Analytab Products, Division of Ayerst Laboratories, Plainview, New York: *Microorganisms, Computer-Assisted Identification*

MARVIN PERLMAN, Jet Propulsion Laboratory, California Institute of Technology, Pasadena, California: *Linear and Matrix Algebra*

VAUGHAN PRATT, Laboratory for Computer Science, Massachusetts Institute of Technology, Cambridge, Massachusetts: *LISP*

STEVEN R. RUTH, Former Director Material Distribution Analysis Group, Department of Defense, Washington, D.C.: *Logistics*

CLAIRE K. SCHULTZ, Professor of Information Science and Director of Libraries, The Medical College of Pennsylvania, Philadelphia, Pennsylvania: *Luhn, H. P.*

WILLIAM A. SMITH, JR., Department of Industrial Engineering, North Carolina State University, Raleigh, North Carolina: *Management Information Systems*

JAMES A. TOMPKINS, Department of Industrial Engineering, North Carolina State University, Raleigh, North Carolina: *Material Handling Systems*

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LINEAR AND MATRIX ALGEBRA

THE CONCEPT OF A VECTOR

Quantities with direction as well as magnitude are encountered in physics and engineering. Both a magnitude and a direction are used to represent force, velocity, and displacement, for example. These quantities are termed vector quantities or simply vectors.

Vectors may be represented geometrically as directed line segments; i.e., arrows. The arrow's length and orientation correspond to magnitude and direction, respectively. Three properties of vectors are illustrated in Fig. 1.

Two vectors are equal if and only if they have the same magnitude and direction. See Fig. 1(a).

Let X and Y in Fig. 1(b) represent displacements. The combined effect of the two displacements applied sequentially results in displacement X + Y represented by an arrow directed from the origin of X to the terminus of Y. The vectors X and Y are two sides of a parallelogram which has X + Y as its diagonal. This is known as the parallelogram law for the addition of vectors.

In Fig. 1(c), X represents a displacement. The displacement $2 \cdot X$ has twice the magnitude of X and the same direction as X. The displacement $-1 \cdot X$ has the same magnitude as X but it is in the opposite direction. In general, X may be multiplied by any real number c. If $c > 0$, $c \cdot X$ has the same direction as X, whereas if $c < 0$, the direction is opposite that of X. The magnitude of $c \cdot X$ is $|c| \cdot X$, where $|c|$ denotes the absolute value of c. If $c = 0$, the magnitude of $c \cdot X$ is zero and its direction is undefined. Hereafter, multiplication is denoted by juxtaposition; i.e., cX. The number c is called a scalar and cX is known as scalar multiplication.

The geometric representation of vectors may readily be transformed into an algebraic context. In Fig. 1(b), the terminus of vector X and vector Y are representable by ordered pairs of real numbers (x_1, x_2) and (y_1, y_2), respectively. The origin of each is denoted by (0, 0). The ordered pairs of real numbers represent the coordinates of points in a two-dimensional space. A distinction is made between coordinates of points and vectors represented by directed line segments. Let

$$X = [x_1, x_2]$$

denote a vector X whose origin and terminus are the coordinates (0, 0) and (x_1, x_2), respectively. Vector addition and scalar multiplication are conveniently computed on a coordinate-by-coordinate basis under the following rules.

$$X + Y = [x_1, x_2] + [y_1 + y_2] = [x_1 + y_1, x_2 + y_2]$$

$$cX = c[x_1, x_2] = [cx_1, cx_2]$$

Note that the origin $\underline{0} = [0, 0]$ is a vector of zero magnitude and undefined direction.

The vectors $X = [x_1, x_2]$ and $Y = [y_1, y_2]$ are equal if and only if

$$x_1 = y_1 \quad \text{and} \quad x_2 = y_2$$

The magnitude of X is the length of the line segment between (0, 0) and (x_1, x_2). The length r is computed in accordance with the Pythagorean theorem

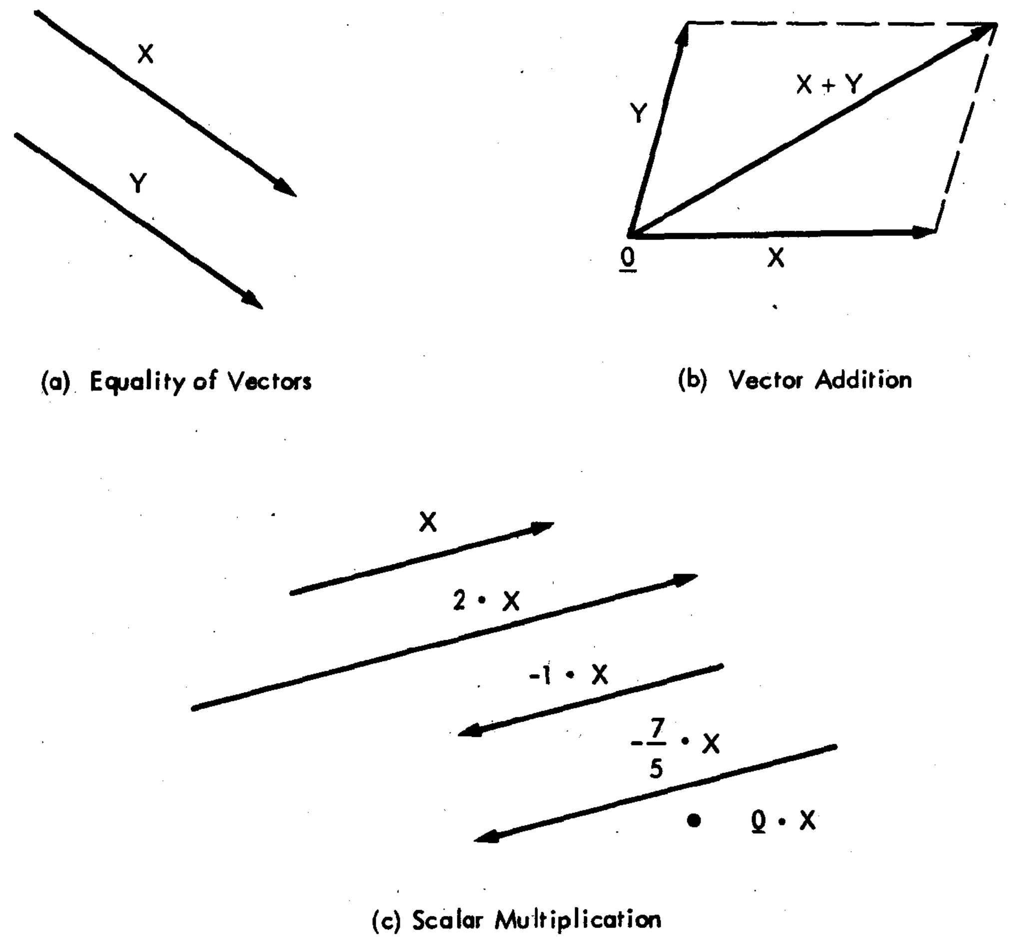

FIG. 1. Properties of vectors.

$$r = |\{(x_1 - 0)^2 + (x_2 - 0)^2\}^{\frac{1}{2}}| = |(x_1^2 + x_2^2)^{\frac{1}{2}}|$$

The magnitude of X (i.e., r) is nonnegative.

The vectors in Fig. 1 are 2-dimensional. The number of dimensions can be extended arbitrarily. Force, velocity, and displacement in a 3-dimensional space, for example, are modeled by 3-dimensional or 3-component vectors.

Forces encountered in the theory of statics may be viewed as 6-dimensional when applied to a rigid solid. Three components pull at the center of gravity along three mutually perpendicular axes. Each of the remaining three components represents a torque which results in a rotation about one of the three perpendicular axes. As in the 2-dimensional case, two forces are added component-by-component (i.e., coordinate-by-coordinate), and scalar multiplication results from multiplying each of the components by a given scalar.

Heretofore, vectors with components and scalars taken from a field of real numbers were discussed. Such vectors having a finite number of components, say n, are members of the set

$$\overset{n}{\times} R = \{[x_1, x_2, \ldots, x_n] \mid x_i \in R \text{ for } i = 1, 2, \ldots, n\}$$

The set may be denoted by $V_n(R)$, the n-dimensional vector space over the field of real numbers R. Every member of $V_n(R)$ is an ordered n-tuple of real numbers (i.e., a n-th Cartesian product of the set R). That is

$$X \in V_n(R)$$

where

$$X = [x_1, x_2, \ldots, x_n]$$

The concepts of equality of vectors, vector addition, and scalar multiplication apply to $V_n(R)$ as follows.

Equality of Vectors

Given $X = [x_1, x_2, \ldots, x_n]$ and $Y = [y_1, y_2, \ldots, y_n]$, then $X = Y$ if and only if

$$x_i = y_i, \quad \text{for } i = 1, 2, \ldots, n$$

Vector Addition

$$X + Y = [x_1, x_2, \ldots, x_n] + [y_1, y_2, \ldots, y_n]$$
$$= [x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n]$$

Scalar Multiplication

$$cX = c[x_1, x_2, \ldots, x_n]$$
$$= [cx_1, cx_2, \ldots, cx_n]$$

Since $\langle R, +, \cdot \rangle$ is a field and $c \in R$

$$x_i + y_i \quad \text{and} \quad cx_i \in R, \quad \text{for } i = 1, 2, \ldots, n$$

The magnitude of X is

$$r = |(x_1^2 + x_2^2 + \cdots + x_n^2)^{\frac{1}{2}}|$$

Example 1

$$X = [-3, 0, -1.5, 2]$$
$$Y = [-4, 1, 0.5, -1]$$
$$X - 3Y = X + (-3Y)$$
$$= [-3, 0, -1.5, 2] + [12, -3, -1.5, 3]$$
$$= [9, -3, -3, 5]$$

X, Y, $-3Y$, and $X - 3Y \in V_4(R)$.

VECTOR SPACES AND SUBSPACES

Vector Spaces V over a Field F

The components of vectors and the scalars need not be taken from the field of real numbers. Algebraic properties of equality of vectors, vector addition, and scalar multiplication hold for vectors whose components are elements of any given field. The scalars are also elements of the given field. Additional algebraic laws are satisfied by vectors V over a field F. All the algebraic properties are summarized in the definition of an algebraic system called a vector space.

Consider the following algebraic systems and operation:

1. An additive Abelian group $\langle V, +\rangle$ whose elements are vectors
2. A field $\langle F, +, \cdot\rangle$ whose elements are scalars
3. An operation * defined as scalar multiplication

Note that the symbol "*" is introduced merely to distinguish the operation of scalar multiplication from the field operation "·". Thus

$$c*X = [c\cdot x_1, c\cdot x_2, \ldots, c\cdot x_n]$$

Juxtaposition is used to represent both as discussed in the previous section.

Scalar multiplication may be viewed as the mapping

$$F \times V \to V$$

which connects the field and Abelian group.

The set of elements V is called a vector space (or linear space) over a field F if it satisfies the following axioms.

Axiom 1. V is an Abelian group under addition.

Axiom 2. For any vector $X \in V$ and any scalar (i.e., field element) $c \in F$, the scalar product $cX \in V$ is defined.

Axiom 3. If $X, Y \in V$ and $c \in F$:

$$c(X + Y) = cX + cY$$

Axiom 4. If $X \in V$ and $c, d \in F$:

$$(c + d)X = cX + dX$$

Axiom 5. If $X \in V$ and $c, d \in F$:

$$(cd)X + c(dX)$$

Also, $1\cdot X = X$, where 1 is the multiplicative identity of F.

Scalar multiplication distributes over vector addition (Axiom 3), and scalar multiplication is associative (Axiom 5). Multiplication (of a scalar) by a vector distributes over scalar addition (Axiom 4).

From Axiom 1, if $X, Y, Z \in V$,

$$(X + Y) + Z = X + (Y + Z)$$

and

$$X + Y = Y + X$$

The zero vector $\underline{0}$ is the additive identity vector, and $-1 \cdot X$ is the additive inverse of the vector X.

The zero vector $\underline{0}$ and the zero scalar 0 can be related through the distributive laws as follows:

$$0X = c\underline{0} = \underline{0}, \quad \text{for all } X \in V \text{ and all } c \in F$$

An accurate symbolic notation for a vector space V over a field F is

$$\langle\langle V, +\rangle, \langle F, +, \cdot\rangle, *\rangle$$

However, the simpler notation V(F) is commonly used.

Example 2. Let U be the set of all functions f(x) of a real variable x which are single-valued and continuous on the closed interval $a \leq x \leq b$ where $a < b$. Two such functions, say f(x) and g(x), when added result in

$$f(x) + g(x) = h(x) \in U$$

Furthermore

$$cf(x) \in U$$

where $c \in R$ (field of real numbers).

U is an infinite-dimensional vector space over the field R which arises in mathematical analysis. Vectors in U do not have a geometric interpretation. However, a vector in U may be viewed as having one component for each point x on the line $a \leq x \leq b$. A component is the value of the function at a given point.

Example 3. Let S be the set of all functions f(x) of a real variable x which are a solution to the linear homogeneous differential equation

$$f''(x) - 3f'(x) - 10f(x) = 0$$

where

$$f''(x) = d^2f(x)/dx^2 \quad \text{and} \quad f'(x) = df(x)/dx$$

Two solutions are

$$f_1(x) = e^{5x} \quad \text{and} \quad f_2(x) = e^{-2x}$$

and the set of all solutions is of the form

$$f(x) = c_1 e^{5x} + c_2 e^{-2x} \in S$$

where $c_1, c_2 \in R$.

S is a 2-dimensional vector space over R. Its dimension is best explained after concepts of linear independent vectors and a basis of a finite-dimensional vector space are discussed.

Example 4. The set of complex numbers

$$C = \{a + bi \mid (a, b) \in R \quad \text{and} \quad i = \sqrt{-1}\}$$

are elements of the field of infinite order $\langle C, +, \cdot \rangle$.

The set P of polynomials

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n$$

where $a_i \in C$ and n is a nonnegative integer, is an (n + 1)-dimensional vector space P over the field C of complex numbers.

Vector Subspaces

Given V(F) and a nonempty subset $W \subseteq V$. If W(F) satisfies the axioms of a vector space with respect to the operations of vector addition and scalar multiplication in V(F), then W(F) is defined to be a subspace of V(F). Thus W(F) is a subspace of V(F) if and only if it satisfies the two following conditions:

1. $\langle W, + \rangle$ is a subgroup of $\langle V, + \rangle$
2. W is closed under scalar multiplication defined in V(F)

Conditions 1 and 2 correspond to Axioms 1 and 2, respectively, of a vector space. The algebraic structure of W(F) is inherited from V(F).

Example 5. Given the vectors

$$[x_1, x_2, x_3] \in V_3(R)$$

If the vectors W over the field R are of the form

$$[x_1, 0, x_3]$$

then

$$[x_1, 0, x_3] + [y_1, 0, y_3] = [x_1 + y_1, 0, x_3 + y_3] \in W$$

and

$$c[x_1, 0, x_3] = [cx_1, 0, cx_3] \in W$$

Therefore W(R) is a subspace of $V_3(R)$. Geometrically, the vectors W are projections of V onto the plane containing the vectors $[x_1, 0, 0]$ and $[0, 0, x_3]$.

Example 6. For a given set of vectors $X_1, X_2, \ldots, X_r$ in V(F), the vectors S over the field F of the form

$$c_1 X_1 + c_2 X_2 + \cdots + c_r X_r$$

where each $c_i \in F$, form a subspace S(F) of V(F).

Note that

$$(c_1X_1 + c_2X_2 + \cdots + c_rX_r) + (\hat{c}_1X_1 + \hat{c}_2X_2 + \cdots + \hat{c}_rX_r)$$
$$= (c_1 + \hat{c}_1)X_1 + (c_2 + \hat{c}_2)X_2 + \cdots + (c_1 + \hat{c}_2)X_r \in S$$

and

$$a(c_1X_1 + c_2X_2 + \cdots + c_rX_r) = (ac_1)X_1 + (ac_2)X_2 + \cdots + (ac_r)X_r \in S$$

hold for all the given vectors X_i and all scalars c_i, $\hat{c}_i$, and $a \in F$.

The null vector $\underline{0}$ is a subspace of any given subspace. A vector space V(F) is by definition a subspace of itself.

Example 7. A system of m linear homogeneous equations in n unknowns, namely, $x_1, x_2, \ldots, x_n$, over a field F is the set of m equations

$$\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0 \\ \vdots \qquad \vdots \qquad \ddots \qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0 \end{array}$$

where each coefficient $a_{ij} \in F$.

If $x_1 = c_1, x_2 = c_2, \ldots, x_n = c_n$ is a solution of the system, then

$$[c_1, c_2, \ldots, c_n] \in V_n(F)$$

is called a solution vector. Furthermore, it can be proven that the set U of all solution vectors is a subspace of $V_n(F)$. The set U is termed a solution space of the system of equations. The solution space is nonempty since

$$[c_1, c_2, \ldots, c_n] = [0, 0, \ldots, 0] \in U$$

LINEAR INDEPENDENCE AND DIMENSION

Linearly Independent Vectors

The vectors $X_1, X_2, \ldots, X_n$ are said to be linearly independent over F if and only if, for all scalars $c_i \in F$,

$$c_1X_1 + c_2X_2 + \cdots + c_nX_n = \underline{0}$$

implies that $c_i = 0$ for $i = 1, 2, \ldots, n$.

Vectors which are not linearly independent are said to be linearly dependent.

A vector of the form

$$c_1X_1 + c_2X_2 + \cdots + c_nX_n$$

where each $X_i \in V(F)$ and each $c_i \in F$ is called a linear combination of X_1, X_2, ..., X_r. See Example 6.

The nonzero vectors $X_1, X_2, \ldots, X_n \in V(F)$ are linearly dependent if and only if some one of the vectors, say X_k, is a linear combination of the preceding ones. If

$$X_k = c_1X_1 + c_2X_2 + \cdots + c_{k-1}X_{k-1}$$

then

$$c_1X_1 + c_2X_2 + \cdots + c_{k-1}X_{k-1} + (-1)X_k = 0$$

where at least -1 is one nonzero coefficient. Thus the nonzero vectors X_1, X_2, ..., X_n are dependent.

Conversely, suppose the vectors are dependent. Then

$$d_1X_1 + d_2X_2 + \cdots + d_nX_n = \underline{0}$$

If k is the last subscript for which $d_k \neq 0$, then .

$$X_k = (-d_k^{-1}d_1)X_1 + (-d_k^{-1}d_2)X_2 + \cdots + (-d_k^{-1}d_{k-1})X_{k-1} = \underline{0}$$

is an expression for X_k as a linear combination or preceding vectors except for $k = 1$. For $k = 1$,

$$d_1X_1 = \underline{0} \quad \text{and} \quad X_1 = \underline{0}$$

since $d_1 \neq 0$ leads to a contradiction since X_1 was assumed to be nonzero.

Example 8. The vectors

$$X_1 = [1,\ 0,\ 0], \quad X_2 = [1,\ 0,\ 1], \quad X_3 = [0,\ 0,\ 1]$$

are taken from the subspace W(F) in Example 5. Since

$$X_1 + (-1)X_2 + X_3 = \underline{0}$$

X_1, X_2, and X_3 are linearly dependent. However,

$$c_1X_1 + c_2X_2 = [c_1 + c_2,\ 0,\ c_2] = \underline{0}$$

implies that $c_1 = c_2 = 0$. Thus X_1 and X_2 are linearly independent.

Given a vector space (or a vector subspace) V(F) such that every vector $X \in V(F)$ can be expressed as a linear combination of a fixed subset of vectors $X_1, X_2, \ldots, X_r \in V(F)$. That is

$$X = c_1X_1 + c_2X_2 + \cdots + c_rX_r$$

The vector space V(F) is said to be generated or spanned by the vectors $X_1, X_2, \ldots, X_r$, called a set of generators.

In Example 8, X_3 as well as every $X \in W(F)$ is expressible as a linear combination of X_1 and X_2. Therefore, X_1 and X_2 are a set of generators of W(F).

The dimension of a vector space V is equal to the maximum number of linearly independent vectors contained in V. Consider the vector spaces (or subspaces which are also vector spaces) over R.

The vector space consisting of the null vector $\underline{0}$ is 0-dimensional since

$c\underline{0} = \underline{0}$ does not imply $c = 0$

and the vector space contains no linearly independent vectors.

The vector subspace W of $V_2(R)$ containing vectors $[x_1, x_2]$ with a slope of b (i.e., $x_2/x_1 = b$) is 1-dimensional. Since

$c[1, b] = \underline{0}$ implies $c = 0$

the vector $X = [1, b]$ is linearly independent. So is $Y = [2, 2b]$. However, $2X - Y = \underline{0}$. Therefore X and Y are linearly dependent. Each generator of W contains one and only one linearly independent vector Thus W over R is denoted as $W_1(R)$.

In Example 5, a maximum of two linearly independent vectors is contained in the subspace W over R. All vectors in W lie on a plane and W has dimension 2.

Any set of a maximum number of linearly independent vectors in a vector space spans the vector space.

Theorem 1. If n vectors span a vector space V containing r linearly independent vectors, then $n \geq r$.

Proof: Let $S_0 = \{X_1, X_2, \ldots, X_n\}$, a sequence of n vectors that span V which contains $A = \{A_1, A_2, \ldots, A_r\}$, a sequence of r linearly independent vectors.

The vector A_1 may be expressed as a linear combination of the vectors $\in S_0$. The sequence $T_1 = \{A_1, X_1, X_2, \ldots, X_n\}$ is linearly dependent. Since S_0 spans V, every vector in V is expressible as

$$0A_1 + c_1X_1 + c_2X_2 + \cdots + c_nX_n$$

and T_1 also spans V. Some vector in T_1 must be dependent on its predecessors by arguments given at the beginning of this section. The vector cannot be $A_1 \neq 0$, which is from the sequence A of linearly independent vectors. Hence some vector X_i is dependent on its predecessors $A_1, X_1, X_2, \ldots, X_{i-1}$. Deleting X_i from the sequence T_1 results in the sequence

$$S_1 = \{A_1, X_1, X_2, \ldots, X_{i-1}, X_{i-2}, \ldots, X_n\}$$

which also spans V.

Repeat the argument with the sequence $T_2 = \{A_2, S_1\}$ or

$$T_2 = \{A_2, A_1, X_1, X_2, \ldots, X_{j-1}, X_{j+1}, \ldots, X_n\}$$

which is linearly independent and spans V. Some vector in T_2 is linearly dependent on its predecessors. The vector cannot be A_2 or A_1 and must be some X_j where $j \neq i$. Deleting X_j results in a new sequence of n vectors

$$S_2 = \{A_2, A_1, X_1, X_2, \ldots, X_{i-1}, X_{i+1}, \ldots, X_{j-1}, X_{j+1}, \ldots, X_n\}$$

which span V. After r times the r vectors in A will have been exhausted and r of the vectors contained in S_0 will have been eliminated. This proves that S_0 must have contained at least r vectors so that $n \geq r$.

Example 9. Given U a subspace of $V_3(R)$ consisting of vectors whose coordinates sum to zero. The vectors

$$X_1 = [1, 0, -1], \quad X_2 = [0, 1, -1], \quad \text{and} \quad X_3 = [1, -1, 0]$$

span U. However, X_3 is the only generating vector that can be expressed as a linear combination of its predecessors. Namely

$$X_3 = [1, 0, -1] - [0, 1, -1]$$

Therefore X_1 and X_2 are linearly independent and are contained in the sequence $\{X_1, X_2, X_3\}$ which spans U. The vector subspace U is 2-dimensional and X_1 and X_2 alone span U.

The dimension of a finite-dimensional vector space is not to be confused with the dimension of a finite-dimensional vector. The dimension of a vector space is the maximum number of linearly independent vectors it contains or, equivalently, the minimum number of generating vectors required to span the space. The number of components a vector has is the vector's dimension.

BASES OF VECTOR SPACES

A basis of a vector space is defined as a linearly independent subset of vectors which spans the vector space. The vector space $V_n(F)$ contains the n linearly independent unit vectors

$$E_1 = [1, 0, 0, \ldots, 0, 0], \quad E_2 = [0, 1, 0, \ldots, 0, 0], \quad \ldots, \quad E_n = [0, 0, 0, \ldots, 0, 1]$$

Each vector $X \in V$ can be expressed as a linear combination of the n unit vectors

$$X = [x_1, x_2, \ldots, x_n] = x_1E_1 + x_2E_2 + \cdots + x_nE_n$$

The n unit vectors are a basis of $V_n(F)$. Often they are referred to as a natural basis.

A vector space is finite-dimensional if and only if it can be formed with a finite basis.

Theorem 2. Every basis of a finite-dimensional vector space consists of the same number of vectors.

Proof: Assume

$$X = \{X_1, X_2, \ldots, X_n\} \quad \text{and} \quad Y = \{Y_1, Y_2, \ldots, Y_r\}$$

are bases of vector space V. In accordance with Theorem 1, since X spans V and the r vectors in set Y are linearly independent, $n \geq r$. Similarly, since Y spans V and the n vectors in set X are linearly independent, $r \geq n$. Then $n = r$.

The number of vectors in any basis of a finite-dimensional space is the dimension of the vector. Consistent with previous comments and stated without proof is Corollary 1.

Corollary 1: A set of n vectors of an n-dimensional vector space V is a basis of V only if V is spanned by the n vectors or the n vectors are linearly independent.

In Example 3, a basis for vector space S over R is $\{f_1(x), f_2(x)\}$. Therefore S is 2-dimensional.

Theorem 3. Every vector in $V_n(F)$ can be expressed uniquely as a linear combination of $\{X_1, X_2, \ldots, X_n\}$, a basis of V.

Proof: Assume an $X \in V$ has two expressions as follows:

$$X = c_1X_1 + c_2X_2 + \cdots + c_nX_n$$

and

$$X = \hat{c}_1X_1 + \hat{c}_2X_2 + \cdots + \hat{c}_nX_n$$

Then

$$X - X = (c_1 - \hat{c}_1)X_1 + (c_2 - \hat{c}_2)X_2 + \cdots + (c_n - \hat{c}_n)X_n = \underline{0}$$

Since $X_1, X_2, \ldots, X_n$ are linearly independent, $c_i = \hat{c}_i$ for $i = 1, 2, \ldots, n$, proving the expression for X is unique.

SIMULTANEOUS LINEAR EQUATIONS

Given the n m-component vectors

$$\begin{aligned} A_1 &= [a_{11}, a_{21}, \ldots, a_{m1}] \\ A_2 &= [a_{12}, a_{22}, \ldots, a_{m2}] \\ &\vdots \\ A_n &= [a_{1n}, a_{2n}, \ldots, a_{mn}] \end{aligned}$$

If $A_1, A_2, \ldots, A_n$ are linearly dependent

$$c_1A_1 + c_2A_2 + \cdots + c_nA_n = \underline{0}$$

for a set of c_i's not all of which are 0. Equivalently, the corresponding simultaneous linear homogeneous equations has a nontrivial solution:

$$\begin{aligned} a_{11}c_1 + a_{12}c_2 + \cdots + a_{1n}c_n &= 0 \\ a_{21}c_1 + a_{22}c_2 + \cdots + a_{2n}c_n &= 0 \\ &\vdots \\ a_{m1}c_1 + a_{m2}c_2 + \cdots + a_{mn}c_n &= 0 \end{aligned}$$

See Example 7 where each c_i (scalar multiplier of a respective A_i) is a value of a corresponding x_i.

Efficient methods for solving simultaneous linear equations involve matrices. The concept of a matrix is therefore introduced here.

A matrix is defined as a rectangular array of elements of a field F. If a matrix contains m rows and n columns, it is termed an $m \times n$ matrix over the field F. The vector

$$X = [x_1, x_2, \ldots, x_n]$$

may be viewed as a $1 \times n$ matrix which is called a row vector. Thus the $m \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

may be regarded as m n-component row vectors.

The $m \times 1$ matrix

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

is called a column vector. Matrix A may also be regarded as n m-component column vectors.

Matrix A represents the coefficients of the system of equations in Example 7. The matrix can be simplified by applying a sequence of elementary row operations which belong to one of the following three types.

1. r_{ij} which denotes the interchange of rows i and j
2. $r_i(k)$ which denotes the multiplication of row i by $k \in F$ where $k \neq 0$
3. $r_{ij}(k)$ which denotes the replacement of row i by row i added to k times row j

Elementary row operations provide a means for testing for linear dependence and determining the solution(s) of simultaneous linear equations.

Given the $m \times n$ matrix A. A finite sequence of elementary row operations on A will result in the $m \times n$ matrix B. Matrix B is row-equivalent to matrix A. Row-equivalence of matrices is an equivalence relation on the set of all $m \times n$ matrices. (See the article entitled Abstract Algebra.)

The subspace of $V_n(F)$ spanned by the row vectors of a $m \times n$ matrix A is defined as the row space of A. It can be proven that row-equivalent matrices have the same row space. Also, any sequence of elementary row operations on matrix A resulting in the row-equivalent matrix B yields explicit expressions for the rows of B as linear combinations of the rows of A.