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Transformation Groups Applied to Mathematical Physics

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EDITOR'S PREFACE

Approach your problems from the right end and begin with the answers. Then one day, perhaps you will find the final question.

'The Hermit Clad in Crane Feathers' in R.van Gulik's The Chinese Maze Murders.

It isn't that they can't see the solution. It is that they can't see the problem.

G.K. Chesterton. The Scandal of Father Brown 'The Point of a Pin'.

Growing specialization and diversification have brought a host of monographs and textbooks on increasingly specialized topics. However, the "tree" of knowledge of mathematics and related fields does not grow only by putting forth new branches. It also happens, quite often in fact, that branches which were thought to be completely disparate are suddenly seen to be related.

Further, the kind and level of sophistication of mathematics applied in various sciences has changed drastically in recent years: measure theory is used (non-trivially) in regional and theoretical economics; algebraic geometry interacts with physics; the Minkowsky lemma, coding theory and the structure of water meet one another in packing and covering theory; quantum fields, crystal defects and mathematical programming profit from homotopy theory; Lie algebras are relevant to filtering; and prediction and electrical engineering can use Stein spaces. And in addition to this there are such new emerging subdisciplines as "completely integrable systems", "chaos, synergetics and large-scale order", which are almost impossible to fit into the existing classification schemes. They draw upon widely different sections of mathematics.
This program, Mathematics and Its Applications, is devoted to such (new) interrelations as exempla gratia:

- a central concept which plays an important role in several different mathematical and/or scientific specialized areas;
- new applications of the results and ideas from one area of scientific endeavor into another;
- influences which the results, problems and concepts of one field of enquiry have and have had on the development of another.

The Mathematics and Its Applications programme tries to make available a careful selection of books which fit the philosophy outlined above. With such books, which are stimulating rather than definitive, intriguing rather than encyclopaedic, we hope to contribute something towards better communication among the practitioners in diversified fields.

Because of the wealth of scholarly research being undertaken in the Soviet Union, Eastern Europe, and Japan, it was decided to devote special attention to the work emanating from these particular regions.
Thus it was decided to start three regional series under the umbrella of the main MIA programme.

The idea of symmetry and transformation groups is, nowadays, a pervasive one in physics. Indeed a somewhat arbitrary, but probably reasonable random, sampling of mathematical physics papers in 1983 indicated that some 35 percent of the papers deal with aspects of symmetry, groups, Lie algebras, or representations in one way or another.

Next to the uses of representation theory in quantum mechanics the modern era has seen groups and symmetry ideas entering into the study of also nonlinear differential equations and e.g. the theory of critical phenomena. It is amazing how much information can be obtained from the transformational properties (under a group) of a model without actually solving the model. And this can be done - as such things go - with quite modest mathematical tools. This has become such a successful approach that it has become something of a methodological axiom: first investigate the transformational properties and then use the results to go some way to actually solving a model. This book is precisely about the first phase; a systematic account of the use of symmetry and transformation group ideas to tackle the problems of mathematical physics.

The unreasonable effectiveness of mathematics in science .....

Eugene Wigner

Well, if you knows of a better 'ole, go to it.

Bruce Bairnsfather

What is now proved was once only imagined.

William Blake

As long as algebra and geometry proceeded along separate paths, their advance was slow and their applications limited.
But when these sciences joined company they drew from each other fresh vitality and thenceforward marched on at a rapid pace towards perfection.

Joseph Louis Lagrange

Bussum, August 1984

Michiel Hazewinkel

AUTHORS' PREFACE

This book is totally concerned with investigations of the problems of mathematical physics by means of group theoretical methods. A sufficiently clear idea of what it contains can be gleaned from its table of contents. In order to ensure a selfcontained treatment and to make it accessible to specialists from applied disciplines an introductory chapter has been added.

My work has to a high degree profited from longtime contacts with Professor L. V. Ovsiannikov and our joint seminar on "group theoretical analysis" at the University of Novosibirsk. The idea to present the results obtained in the form of a separate book arose during discussions with Professor G. Birkhoff. I am most grateful to him for his critical remarks and valuable advice.

A significant part of the material has been presented in lectures (cf. [11]) which I gave at various times for students at the University of Novosibirsk. The final version of the book took form after a course of lectures at the Collège de France in the spring of 1979. I consider it a particular pleasure to thank Proffessor A. Lichnerowicz, who organized this series of lectures and who took a lively part in the discussions.

During various stages of the writing parts of the book were subjected to critical scrutiny by Professor S. P. Novikov and Professor A. B. Sabat. Dr. V. M. Tesukov gave valuable help in writing of § 4.4.. R. S. Hamitova carefully perused the whole manuscript and found a series of inaccuracies in the development of various fragments. To all these I express my deepfelt thanks. I also thank all those colleagues who sent me their (p)reprints concerning new results on the matters discussed in this book.

18 January 1982 N. H. Ibragimov.

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INTRODUCTORY CHAPTER

GROUPS AND DIFFERENTIAL EQUATIONS

§ 1. Continuous Groups

1.1. Topological groups.

A topological space G endowed with a group operation $\cdot$ is called a *topological group* if the map (a,b) $a \cdot b^{-1}$ (where b^{-1} is the inverse of the element b in the group G) of the product space $G \times G$ into G is continuous.

In the investigation of the local properties of the topological groups, we can restrict ourselves to considera-tion of a neighborhood of the identity element of the given group G, because for any fixed element $a \in G$, the map x $x \cdot a^{-1}$ is a homomorphism of the topological space G onto itself and takes a into the identity element of G.

A set $H \subset G$ is called a *subgroup* of the topological group G if it is both a closed subset of the topological space G and a subgroup of the group G. If, in addition, H is an invariant subgroup of G (i.e., $a^{-1} \cdot H \cdot a = H$ for each $a \in G$), then H is called an *invariant subgroup*, or a *normal subgroup* of the topological group G. Let H be an invariant subgroup of the topological group G, and let G/H be the family of all pairwise disjoint cosets $H \cdot a$, with $a \in G$. A topology and a group operation are naturally introduced on G/H, induced by the topology and the group operation on G. The outcome is a topological group G/H, called the *quotient* (or *factor*) *group* of the topological group G by its invariant subgroup H.

A map $f: G \to G'$ is an *isomorphism* of the topological group G onto the topological group G' if f is both an isomorphism of groups and a homeomorphism of topological spaces. If the map f is continuous and a homomorphism of the group G into the group G', then f is a *homomorphism* of the topological group G into the topological group G'.

1.2. Lie groups.

Let M, $U \subset M$, and φ be, respectively, a connected Hausdorff topological space, an open subset of M, and a homeomorphism of U onto an open subset of $\mathbb{R}^m$. The pair (U,φ) is called a (m-dimensional) *chart* on M; U is the *domain* of the chart, while the functions $\varphi^i = pr_i \circ \varphi: U \to \mathbb{R}^m$ $(i = 1,\ldots,m)$ are *local coordinates*. Here pr_i stands for the projection onto the i-th coordinate axis in m: if $x = x^1,\ldots,x^n) \in \mathbb{R}^m$, then $pr_i(x) = x^i$. Given any point $a \in U$, the m-tuple $(\varphi^1,\ldots,\varphi^m)$ is said to be a *system of local coordinates at* a, and the real numbers $x^i = \varphi^i(a)$ are the *coordinates of the point* a relative to the system of coordinates $(\varphi^1,\ldots,\varphi^m)$. A system of local coordinates will be denoted alternatively by $\{x^i\}$.

A topological space M is said to be an *m-dimensional* (topological) *manifold* if there exists a family of charts on M whose domains cover M. Whenever (U,φ) is a chart on M and $V \subset U$ is open, the restriction $\varphi|_V$ will be a homeomorphism of V onto an open subset of $\mathbb{R}^m$. Thus $(V,\varphi|_V)$ is a chart on M, called the *restriction of the chart* (U,φ) *to* V.

The following concept of a differentiable structure makes it possible to define the differential calculus on manifolds. Two charts (U,φ) and (U,ψ) having a common domain on the manifold M, are said to be C^p-*compatible* if the maps $\psi\circ\varphi^{-1} : \varphi(U) \to \psi(U)$ and $\varphi\circ\psi^{-1} : \psi(U) \to \varphi(U)$ are p-times continuously differentiable. Two arbitrary charts (U,φ) and (V,ψ) on M are C^p-*compatible* if their restrictions to $U \cap V$ are C^p-compatible, or if their domains do not intersect. A C^p-*atlas* of the m-dimensional manifold M is a family of paitwise C^p-compatible charts on M whose domains cover M. Two C^p-atlases are *equivalent* if their union is a C^p-atlas. A C^p-*differentiable structure* (p is a positive integer or ∞) on the manifold M is an equivalence class of C^p-atlases of M. Equivalently, a differentiable structure may be defined as a maximal atlas of the manifold M; such a maximal atlas is the union of all atlases belonging to the particular equivalence class under consideration.

A manifold M equipped with a C^p-differentiable structure is termed an *m-dimensional differentiable manifold of class* C^p, or an *m-dimensional* C^p-*manifold*. In order to obtain a differentiable manifold, it suffices to specify an arbitrary atlas, out of the equivalent atlases of the manifold under consideration. Replacing C^p-functions by analytic ones (in the definition of an atlas), we arrive at the notion of an *m-dimensional analytic manifold*. Henceforth, if not otherwise stipulated, all the manifolds under consideration are endowed with a C^∞-differentiable structure.

Let M and N be differentiable manifolds of dimensions m and n, respectively. A continuous mapping $f: M \to N$ is said to be *p-times continuously differentiable* (or *of class* C^p) if for arbitrary charts (U,φ) on M and (V,ψ) on N, with domains satisfying $f(U) \subset V$, the local representative of f, i.e., the mapping $\psi \circ (f|_U) \circ \varphi^{-1}$ of the open set $\varphi(U) \subset \mathbb{R}^m$ into the open set $\psi(V) \subset \mathbb{R}^n$, is p-times continuously differentiable. If M and N are analytic manifolds and the local representatives of f are all analytic, then f is said to be *analytic*. When $N = \mathbb{R}$, f is also referred to as a (real) *function* defined on the manifold M. The above notions can be similarly applied to mappings defined locally, i.e., defined on open subsets of the manifold M.

A *curve* γ on the manifold M passing through the point $x \in M$ is a continuously differentiable map $\gamma : I \to M$, with I an open interval on the real line $\mathbb{R}$, $0 \in I$, and $\gamma(0) = x$. Two curves, γ_1 and γ_2, passing through $x \in M$, are *tangent* at x if the derivatives of the local representatives of γ_1 and γ_2 at 0 coincide. It is clear that tangency at x does not depend on the choice of the chart used to calculate the local representatives, and yields an equivalence relation among the curves passing through a fixed point of the given manifold. The equivalence class of any curve γ passing through $x \in M$, i.e., the family of all curves passing through x and tangent to γ, is called a *vector tangent to* M *at the point* x and is denoted by $[\gamma]_x$. The tangent vector $[\gamma]_x$ may be viewed also as the derivative $\gamma'(0)$ of a local representative of γ at the point 0, thus explaining the use of the notation $\gamma'(0)$ along with $[\gamma]_x$. The set of all tangent vectors at the point x is called the *tangent space* to M at x, and is denoted by M_x. If $\dim M = m$ then one can endow M_x with a structure of m-dimensional vector space using the differentiable structure of M. To this end, we may choose a chart (U,φ) in a neighborhood of x, i.e., with $x \in U$, and define a mapping $\theta : M_x \to \mathbb{R}^m$ by the rule: θ takes each tangent vector $[\gamma]_x$ into the derivative of the local representative $\varphi \circ \gamma$ of the curve γ at 0. Then θ is one-to-one and onto, and can be used to introduce on M_x the required vector--space structure. Indeed, the sum of the tangent vectors $[\gamma_1]_x$ and $[\gamma_2]_x$, and multiplication of $[\gamma]_x$ by a real number λ are defined as

$$[\gamma_1]_x + [\gamma_2]_x = \theta^{-1}(h_1 + h_2),\ \lambda[\gamma]_x = \theta^{-1}(\lambda h),$$

where $h_i = \theta([\gamma_i]_x)$, $i = 1,2$ $h = ([\gamma_x])$.

The dual space M_x^* to M_x, i.e., the space of linear mappings $M_x \to \mathbb{R}$, is called the *cotangent space to* M *at* **x**. The tensors at the point $x \in M$ are defined as affine tensors over the vector space M_x, i.e., as real multilinear maps on products of copies of M_x and M_x^*. More precisely, a *tensor of contravariant order* r *and covariant order* s, or simply a *tensor of type* $\binom{r}{s}$, at the point $x \in M$, is a multilinear map $\underbrace{M_x^* \times \ldots \times M_x^*}_{r} \times \underbrace{M_x \times \ldots \times M_x}_{s} \to \mathbb{R}$. A *tensor field of type* $\binom{r}{s}$ on the manifold M is determined by associating with each point x of M a tensor of type $\binom{r}{s}$ at x. In particular, one can use the isomorphism betwee the spaces $(M_x^*)^*$ and M_x to define a *vector field* ξ on manifold x $\xi(x) \in M_x$ which takes each point $x \in M$ into a (uniquely determined) tangent vector to M at x.

An analytic manifold G with a group operation • defined on it is a *Lie group* if the mapping (a,b) $a \cdot b^{-1}$ of the product-manifold $G \times G$ into G is analytic.

1.3. Local groups

A topological space G is said to be a *local group* if there exists an element (the identity) $e \in G$ and neighborhoods U,V (with $V \subset U$) of e, such that there is a mapping $U \times U \to U$, (a,b) $a \cdot b$ (a local group operation) satisfying the following conditions:

(1) $V \cdot V \subset U$;

(2) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a,b,c \in V$;

(3) $e \cdot a = a \cdot e = a$ for all $a \in U$;

(4) for any $a \in V$ there exists the (inverse) element $a^{-1} \in U$, such that $a \cdot a^{-1} = a^{-1} \cdot a = e$;

(5) the mapping (a,b) $a \cdot b^{-1}$ is continuous on $U \times V$.

The above neighborhoods, U and V, are not uniquely determined: they can be replaced by suitably selected smaller neighborhoods of e, $U' \subset U$ and $V' \subset V$. This freedom in the definition of a local group guarantees that the notion introduced above is general enough to be useful in solving local problems and at the same time increases the applicability of group-theoretic methods. According to this definition, every topological group is also a local group.

A closed subset H of the local group G, containing the identity $e \in G$, is a *subgroup of* (*the local group*) G if H is itself a local group, relative to the local group operation in G. If, in addition, one can find an open neighborhood $U \subset G$ of e such that $a^{-1} \cdot b \cdot a \in H$ for all $a \in U$ and $b \in U \cap H$, then H is said to be an *invariant subgroup* (or a *normal subgroup*) of the local group G. The *quotient group* G/H of the local group G by its normal subgroup H is constructed as in the case of topological groups, except that here the elements of G/H are the cosets modulo H of the elements belonging to some neighborhood of the identity of G.

The family of local groups is divided into equivalence classes by employing the following notion of local isomorphism. Let G and G' be two local groups with identity elements e and e', respectively, and let U,V and U',V' be neighborhoods of e and e' satisfying the axioms 1-5. Let $f : U \to U'$ be a homeomorphism such that $f(V) \subset V'$ and $f(a \cdot b) = f(a) \cdot f(b)$ for all $a,b \in V$. Then the mapping f is referred to as a *local isomorphism*, and the groups G and G' are said to be *locally isomorphic*. The inverse mapping f^{-1} (or possibly its restriction to some neighborhood of e') is also a local isomorphism. The usual properties of the group isomorphisms hold here as well: $f(e) = e'$, and $f(a^{-1}) = (f(a))^{-1}$ for all $a \in V$.

1.4. Local Lie groups.

Suppose G is a local group, (U,φ) is an r-dimensional chart on G with $e \in U$, the local group operation in G is defined in U, and the homeomorphism φ satisfies $\varphi(0) = e$. Further, let $V \subset U$ be an open subset selected so that U and V satisfy conditions 1-5, § 1.3, and let the mapping $(a,b) \to a \cdot b$ be analytic in $V \times V$. Then we say that analytic coordinates were introduced in the local group G. That is to say, the coordinates c^i $(i = 1,\dots,r)$ of the element $c = a \cdot b \in U$ in the local chart (U,φ) are analytic functions $c^i = \psi^i(a^1,\dots,a^r, b^1,\dots,b^r)$ of the coordinates a^i, b^i of the elements $a,b \in V$ in the local chart $(V,\varphi|_V)$.