An Introduction to Applied Optimal Control

Greg Knowles

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Greg Knowles

Department of Mathematics
Carnegie-Mellon University
Pittsburgh, Pennsylvania

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Preface

This book began as the notes for a one-semester course at Carnegie-Mellon University. The aim of the course was to give an introduction to deterministic optimal control theory to senior level undergraduates and first-year graduate students from the mathematics, engineering, and business schools. The only prerequisite for the course was a junior level course in ordinary differential equations. Accordingly, the backgrounds of the students were widely dissimilar, and the common denominator was their interest in the applications of optimal control theory. In fact, one of the most popular aspects of the course was that students were able to study problems from areas that they would not normally cover in their respective syllabi.

This text differs from the standard ones in that we have not attempted to prove the maximum principle, since this was beyond the background and interest of most of the students in the course. Instead we have tried to show its strengths and limitations through examples.

In Chapter I we introduce the concept of optimal control by means of examples. In Chapter II necessary conditions for optimality for the linear time optimal control problem are derived geometrically, and illustrations are given. In Chapters III and IV we discuss the Pontryagin maximum principle, its relation to the calculus of variations, and its application to various problems in science, engineering, and business. Since the optimality conditions arising from the maximum principle can often be solved only numerically, numerical techniques are discussed in Chapter V. In Chapter VI the dynamic programming approach to the solution of optimal control problems and differential games is considered; in Chapter VII the controllability and observability of linear control systems are discussed, and in Chapter VIII the extension of the maximum principle to state-constrained control problems is given. Finally, for more advanced students with a background in functional analysis, we consider in Chapter IX several problems in the control of systems governed by partial differential equations. This could serve as an introduction to research in this area.

The support of my colleagues and students at Carnegie-Mellon University has been invaluable during this project; without it this text would almost certainly not have appeared.

Contents

Chapter I | Examples of Control Systems; the Control Problem

Example 1 Consider a mechanism, such as a crane or trolley, of mass m which moves along a horizontal track without friction. If $x(t)$ represents the position at time t, we assume the motion of the trolley is governed by the law

$$m\ddot{x}(t) = u(t), \qquad t > 0, \tag{1}$$

where $u(t)$ is an external controlling force that we apply to the trolley (see Fig. 1). Assume that the initial position and velocity of the trolley are given as $x(0) = x_0, \dot{x}(0) = y_0$, respectively. Then we wish to choose a function u (which is naturally enough called a control function) to bring the trolley to rest at the origin in minimum time. Physical restrictions will usually require that the controlling force be bounded in magnitude, i.e., that

$$|u(t)| \leq M. \tag{2}$$

For convenience, suppose that $m = M = 1$, and rewrite Eq. (1)

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = u(t),$$

where $x_1(t)$ and $x_2(t)$ are now the position and velocity of the body at time t. Equation (1) then becomes

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

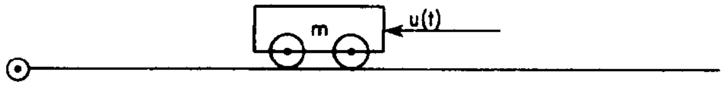

Fig. 1

or

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + \mathbf{b}u(t), \qquad \mathbf{x}(0) = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}, \tag{3}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix},$$

and the control problem is to find a function u, subject to (2), which brings the solution of (3), $\mathbf{x}(t)$, to $\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$ in minimum time t. Any control that steers us to $\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$ in minimum time is called an optimal control. Intuitively, we should expect the optimal control is first a period of maximum acceleration ($u = +1$), and then maximum braking ($u = -1$), or vice versa.

Example 2 (Bushaw [1]) A control surface on an aircraft is to be kept at rest at a fixed position. A wind gust displaces the surface from the desired position. We assume that if nothing were done, the control surface would behave as a damped harmonic oscillator. Thus if θ measures the deviation from the desired position, then the free motion of the surface satisfies the differential equation

$$\ddot{\theta} + a\dot{\theta} + \omega^2\theta = 0$$

with initial conditions $\theta(0) = \theta_0$ and $\dot{\theta}(0) = \theta'_0$. Here θ_0 is the displacement of the surface resulting from the wind gust and θ'_0 is the velocity imparted to the surface by the gust. On an aircraft the oscillation of the control surface cannot be permitted, and so we wish to design a servomechanism to apply a restoring torque and bring the surface back to rest in minimum time. The equation then becomes

$$\ddot{\theta}(t) + a\dot{\theta}(t) + \omega^2\theta(t) = u(t), \qquad \theta(0) = \theta_0, \qquad \dot{\theta}(0) = \theta'_0, \tag{4}$$

where $u(t)$ represents the restoring torque at time t. Again we must suppose that $|u(t)| \leq C$, where C is a constant, and by normalization can be taken as 1. The problem is then to find such a function u, so that the system will be brought to $\theta = 0, \dot{\theta} = 0$ in minimum time.

It is clear that if $\theta_0 > 0$ and $\theta'_0 > 0$, then the torque should be directed initially in the direction of negative θ and should have the largest possible magnitude. Thus $u(t) = -1$ initially. However, if $u(t) = -1$ is applied for too long a time, we shall overshoot the desired terminal condition $\theta = 0$, $\dot{\theta} = 0$. Therefore at some point there should be a torque reversal to $+1$ in order to brake the system.

The following questions occur:

(1) Is this strategy indeed optimal, and if so, when should the switch take place?

(2) Alternatively, is it better to remove the torque at some point, allow a small overshoot, and then apply $+1$?

(3) In this vein, we could ask whether a sequence of -1, $+1$, -1, $+1, \ldots$ of n steps is the best, and if so, what is n and where do the switches occur?

Again we are led to controls that take on (only) values ± 1; such controls are called *bang-bang* controls.

Note that as before, setting $x_1 = \theta$ and $x_2 = \dot{\theta}$, we can write the system equation (4)

$$\dot{x}_1 = x_2, \qquad x_1(0) = \theta_0,$$

$$\dot{x}_2 = -ax_2 - \omega^2 x_1 + u, \qquad x_2(0) = \theta'_0,$$

$$\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{b}u, \qquad \mathbf{x}(0) = \begin{bmatrix} \theta_0 \\ \theta'_0 \end{bmatrix},$$

where

$$A = \begin{bmatrix} 0 & 1 \\ -\omega^2 & -a \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

and u is chosen with $|u(t)| \leq 1$ and to minimize

$$C(u) = \int_0^{t_1} 1\, dt,$$

where t_1 is any time for which $x_1(t_1) = 0$ and $x_2(t_1) = 0$.

Example 3 (Isaacs [3]) Let $x(t)$ be the amount of steel produced by a mill at time t. The amount produced at time t is to be allocated to one of two uses:

(1) production of consumer products;
(2) investment.

It is assumed that the steel allocated to investment is used to increase productive capacity—by using steel to produce new steel mills, transport facilities, or whatever. Let $u(t)$, where $0 \leq u(t) \leq 1$, denote the fraction of steel produced at time t that is *allocated* to *investment*. Then $1 - u(t)$ represents the fraction allocated to consumption. The assumption that the reinvested steel is used to increase the productive capacity could be written

$$\frac{dx}{dt} = ku(t)x(t), \qquad \text{where} \quad x(0) = C - \text{initial endowment},$$

where k is an appropriate constant (i.e., rate of increase in production is proportional to the amount allocated to investment).

The problem is to choose $u(t)$ so as to maximize the total consumption over some fixed period of time $T > 0$. That is, we are to maximize

$$\int_0^T (1 - u(t))x(t)\,dt.$$

For this problem, do we consume everything produced, or do we invest some at present to increase capacity now, so that we can produce more and hence consume more later? Do we follow a bang-bang procedure of first investing everything and then consuming everything?

Example 4 Moon-Landing Problem (Fleming and Rishel [2]) Consider the problem of a spacecraft attempting to make a soft landing on the moon using the minimum amount of fuel. For a simplified model, let m denote the mass, h the height, v the vertical velocity of the spacecraft above the moon, and u the thrust of the spacecraft's engine (m, h, v, and u are functions of time). Let M denote the mass of the spacecraft without fuel, h_0 the initial height, v_0 the initial velocity, F the initial amount of fuel, α the maximum thrust of the engine, k a constant, and g the gravity acceleration of the moon (considered constant). The equations of motion are

$$\dot{h} = v,$$
$$\dot{v} = -g + m^{-1}u,$$
$$\dot{m} = -ku,$$

and the control u is restricted so that $0 \le u(t) \le \alpha$. The end conditions are

$$\begin{aligned} h(0) &= h_0, \\ v(0) &= v_0, \\ m(0) - M - F &= 0, \\ h(t_1) &= 0, \\ v(t_1) &= 0, \end{aligned}$$

where t_1 is the time taken for touchdown.

With

$$x_1 = h, \qquad x_2 = v, \qquad x_3 = m,$$
$$\mathbf{x}(0) = (h_0, v_0, M + F)^{\mathrm{T}},$$
$$\mathbf{x}(t_1) = (0, 0, \text{anything})^{\mathrm{T}},$$

this problem becomes, in matrix form,

$$\dot{\mathbf{x}} = \begin{bmatrix} x_2 \\ -g + x_3^{-1}u \\ -ku \end{bmatrix} = \mathbf{f}(t, \mathbf{x}, u),$$

and we wish to choose u so that $0 \le u(t) \le \alpha$ and

$$-M - F - \int_0^{t_1} \dot{x}_3(\tau)\, d\tau \quad (= -m(t_1))$$

is a minimum.

However, $\dot{x}_3 = -ku$, so the above becomes

$$-M - F + k \int_0^{t_1} u(\tau)\, d\tau,$$

and this is minimized at the same time as

$$C(u) = \int_0^{t_1} u(\tau)\, d\tau.$$

Note that although these problems come from seemingly completely different areas of applied mathematics, they all fit into the following general pattern.

GENERAL FORM OF THE CONTROL PROBLEM

(1) The *state equation is*

$$\dot{x}_i = f_i(t, x_1, \ldots, x_n, u_1, \ldots, u_m), \qquad i = 1, \ldots, n,$$

or in vector form

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}, \mathbf{u}),$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad \mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}.$$

(2) The *initial point* is $\mathbf{x}(0) = \mathbf{x}_0 \in R^n$, and the *final point* that we wish to reach is $\mathbf{x}_1 \in R^n$. The final point $\mathbf{x}_1$ is often called the target (point), and may or may not be given.

(3) The class Δ of *admissible controls* is the set of all those control functions u allowed by the physical limitations on the problem. (In Examples 1 and 2 we had $\Delta = \{u : |u(t)| \leq 1\}$ and $m = 1$.) Usually we shall be given a compact, convex set $\Omega \subset R^m$ (the *restraint set*) and we shall take

$$\Delta = \{\mathbf{u} = (u_1, \ldots, u_m) : u_i \text{ piecewise continuous and } \mathbf{u}(t) \in \Omega\}.$$

(4) The *cost function* or *performance index* quantitatively compares the effectiveness of various controllers. This is usually of the form

$$C(\mathbf{u}) = \int_0^{t_1} f_0(t, \mathbf{x}(t), \mathbf{u}(t))\, dt,$$

where f_0 is a given continuous real-valued function, and the above integral is to be interpreted as: we take a control $\mathbf{u} \in \Delta$, solve the state equations to obtain the corresponding $\mathbf{x}$, calculate f_0 as a function of t, and perform the integration. If a target point is given (so called *fixed-end-point* problem), then t_1 must be such that $\mathbf{x}(t_1) = \mathbf{x}_1$. In particular, if $f_0 \equiv 1$, then $C(u) = t_1$, and we have the minimum-time problem. If a target point is not given (*free-end-point* problem), then t_1 will be a fixed given time, and the integration is performed over the fixed interval $[0, t_1]$.

The optimal control problem can now be formulated: Find an admissible control $\mathbf{u}^*$ that minimizes the cost function, i.e., for which $C(\mathbf{u}^*) \leq C(\mathbf{u})$ for all $\mathbf{u} \in \Delta$. Such controls $\mathbf{u}^*$ are called *optimal controls.*

We shall first investigate in depth in Chapter 2 the linear (i.e., state equations are linear in $\mathbf{x}$ and $\mathbf{u}$) time optimal control problem, deriving a necessary condition for optimality known as Pontryagin's maximum principle [4].

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Chapter II | The General Linear Time Optimal Problem

1. INTRODUCTION

Consider a control system described by the vector differential equation

$$\dot{\mathbf{x}}(t) = A(t)\mathbf{x}(t) + B(t)\mathbf{u}(t), \qquad \mathbf{x}(0) = \mathbf{x}_0 \in R^n, \tag{1}$$

where

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}, \qquad \mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ \vdots \\ u_m(t) \end{bmatrix},$$

$$A(t) = \begin{bmatrix} a_{11}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & \cdots & a_{2n}(t) \\ \vdots & & \vdots \\ a_{n1}(t) & \cdots & a_{nn}(t) \end{bmatrix}, \qquad B(t) = \begin{bmatrix} b_{11}(t) & \cdots & b_{1m}(t) \\ b_{21}(t) & \cdots & b_{2m}(t) \\ \vdots & & \vdots \\ b_{n1}(t) & \cdots & b_{nm}(t) \end{bmatrix},$$

and we assume the elements of $A(t)$, $B(t)$ are integrable functions over any finite interval of time.

The set of admissible controls will be

$$\Delta = \{\mathbf{u} = (u_1, \ldots, u_m)^{\mathrm{T}} : |u_i(t)| \le 1, i = 1, \ldots, m\}. \tag{2}$$

A target point $\mathbf{x}_1 \in R^n$ is given, and the control problem is to minimize the time t_1 for which $\mathbf{x}(t_1) = \mathbf{x}_1$.