HANDBOOK OF MATHEMATICAL TABLES and FORMULAS

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BURINGTON

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MATHEMATICAL TABLES AND FORMULAS

COMPILED BY

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To the Student who uses this Handbook:

As a student of mathematics you are studying the subject, either because you have a liking for it or, because it is a required subject in your professional training—happily, if for both of these reasons. The subject is one of the fundamental natural sciences and perhaps the most fundamental of all. While a mathematician may not necessarily need to know of engineering, physics, chemistry, or other natural sciences, yet to every engineer, physicist, chemist, or other scientist mathematics is a neccessity.

Unquestionably you will many times in later life need to know the formulas and the numerical data to be found in this Handbook. These, if not forgotten completely, will be recalled then so hazily as to be subject to doubt. Since mathematics is an exact science doubt can have no place in its applications.

The contents of this Handbook were carefully selected by one who has had many years of experience in the applications of mathematics and in teaching the subject to prospective scientists and engineers. You have become familiar with its contents in your own use of it. Where could you possibly find a more handy and reliable source for the information which it presents and which you need?

The subject matter is not ephemeral but everlasting—as true in the future as it has been in the past. By all means, retain this book for your own reference library. You will need it many times in years to come.

The Publishers

PREFACE TO THE THIRD EDITION

The wide and excellent reception of the earlier editions of this book and the valuable suggestions from its many users for additions and improvements have resulted in the publishers urging upon the compiler a revision and enlargement of the second edition. Because of this interest on the part of the many users of the work, the author has prepared this revision. As in the cases of the earlier editions, this book has been compiled to meet the needs of students and workers in mathematics, engineering, physics, chemistry and other fields in which mathematical computations, or processes of a mathematical nature are required.

The general arrangement of the second edition has been retained. In the first part of the book a summary of the more important formulas and theorems of algebra, trigonometry, analytic geometry, calculus and vector analysis is given. A table of derivatives and a comprehensive table of integrals are included. The second part of the book contains logarithmic and trigonometric tables to five places; tables of natural logarithms, exponential and hyperbolic functions; tables of squares, cubes, square roots and cube roots; reciprocals and other numerical quantities. A five-place table of natural secants and cosecants has been incorporated in this edition. The entire book has been gone over and many changes, revisions, and additions have been made within the text as well as in the tables themselves. Every effort has been made to arrange the tables in such a manner that the user will interpret them readily and properly.

The entire book has been reset. Every effort has been made to insure accuracy. The proofs have been checked by difference methods, and by reading several times against different sources.

The author wishes to acknowledge valuable suggestions from many of the users of the book, and to express his appreciation to his colleague, Dr. D. C. May, for helpful suggestions made during the preparation of the manuscript of this edition.

The author is especially indebted to his wife, Jennet Mae Burington, for her very considerable assistance in the preparation of the manuscript and in the reading of the proof.

R. S. B.

Arlington, Virginia, March 2, 1948.

Greek Alphabet

A	α	Alpha	N	ν	Nu
B	β	Beta	Ξ	ξ	Xi
Γ	γ	Gamma	O	o	Omicron
Δ	δ	Delta	Π	π	Pi
E	ϵ	Epsilon	P	ρ	Rho
Z	ζ	Zeta	Σ	σ	Sigma
H	η	Eta	T	τ	Tau
Θ	θ	Theta	Υ	υ	Upsilon
I	ι	Iota	Φ	ϕ	Phi
K	κ	Kappa	X	χ	Chi
Λ	λ	Lambda	Ψ	ψ	Psi
M	μ	Mu	Ω	ω	Omega

Mathematical Symbols and Abbreviations

$+$	Plus or Positive
$-$	Minus or Negative
$\pm$	Plus or minus Positive or negative
$\mp$	Minus or plus Negative or positive
$\times$ or $\cdot$	Multiplied by
$\div$ or $:$	Divided by
$=$ or $::$	Equals, as
$\neq$, $\ddagger$	Does not equal
$\cong$	Equals approximately Congruent
$>$	Greater than
$<$	Less than
$\geqq$	Greater than or equal to
$\leqq$	Less than or equal to
$\backsim$	Similar to
$\therefore$	Therefore
$\sqrt{\ }$	Square root
$\sqrt[n]{\ }$	nth root
a^n	nth power of a
log or $\log_{10}$	Common logarithm Briggsian "
ln or $\log_e$ or log	Natural logarithm Hyperbolic " Napierian "
e or ϵ	Base (2.718) of natural system of logarithms
π	Pi (3.1416)
$\angle$	Angle
$\perp$	Perpendicular to
$\parallel$	Parallel to
a°	a degrees (angle)
a'	a minutes (angle) a prime
a''	a seconds (angle) a second a double-prime
a'''	a third a triple-prime
a_n	a sub n
sin	Sine.
cos	Cosine
tan	Tangent
cot or ctn	Cotangent

sec	Secant
csc	Cosecant
vers	Versed sine
covers	Coversed sine
exsec	Exsecant
$\sin^{-1} a$ or arc sin a	Anti-sine a Angle whose sine is a Inverse sine a
sinh	Hyperbolic sine
cosh	Hyperbolic cosine
tanh	Hyperbolic tangent
$\sinh^{-1} a$ or arc sinh a	Anti-hyperbolic sine a Number whose hyperbolic sine is a
$P(x,y)$	Rect. coörd. of point P
$P(r, \theta)$	Polar coörd. of point P
$f(x)$, $F(x)$ or $\Phi(x)$	Function of x
Δy	Increment of y
$\doteq$ or $\rightarrow$	Approaches as a limit
Σ	Summation of
∞	Infinity
dy	Differential of y
$\frac{dy}{dx}$ or $f'(x)$	Derivative of $y = f(x)$ with respect to x
$\frac{d^2y}{dx^2}$ or $f''(x)$	Second deriv. of $y = f(x)$ with respect to x
$\frac{d^ny}{dx^n}$ or $f^{(n)}(x)$	nth deriv. of $y = f(x)$ with respect to x
$\frac{\partial z}{\partial x}$	Partial derivative of z with respect to x
$\frac{\partial^2 z}{\partial x\,\partial y}$	Second partial deriv. of z with respect to y and x
$\int$	Integral of
$\int_a^b$	Integral between the limits a and b
j or i	Imaginary quantity $(\sqrt{-1})$, $i^2 = -1$
$x = a + jb$	Symbolic vector notation
$n! = 1 \cdot 2 \cdot 3 \cdots n$	

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PART ONE

Formulas and Theorems from Elementary Mathematics

I. ALGEBRA

1. Fundamental Laws.

(*a*) Commutative law: $a + b = b + a, \quad ab = ba.$

(*b*) Associative law: $a + (b + c) = (a + b) + c, \quad a(bc) = (ab)c.$

(*c*) Distributive law: $a(b + c) = ab + ac.$

2. Laws of Exponents.

$$a^x \cdot a^y = a^{x+y}, \quad (ab)^x = a^x \cdot b^x, \quad (a^x)^y = a^{xy}.$$

$$a^0 = 1 \quad \text{if} \quad a \neq 0, \quad a^{-x} = \frac{1}{a^x}, \quad \frac{a^x}{a^y} = a^{x-y}.$$

$$a^{\frac{x}{y}} = \sqrt[y]{a^x}, \quad a^{\frac{1}{y}} = \sqrt[y]{a}.$$

3. Operations with Zero.

$$a - a = 0, \quad a \cdot 0 = 0 \cdot a = 0.$$

If $a \neq 0$, $\frac{0}{a} = 0$, $0^a = 0$, $a^0 = 1$. (Division by zero undefined.)

4. Complex Numbers (a number of the form $a + bi$, where a and b are real).

$$i = \sqrt{-1}, \quad i^2 = -1, \quad i^3 = -i, \quad i^4 = 1, \quad i^5 = i, \quad \text{etc.}$$

$a + bi = c + di$, if and only if $a = c, \quad b = d.$

$$(a + bi) + (c + di) = (a + c) + (b + d)i.$$

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i.$$

$$\frac{a+bi}{c+di} = \frac{(a+bi)(c-di)}{(c+di)(c-di)} = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}\, i.$$

5. Laws of Logarithms (See explanation of Table 1).

If M, N, b are positive numbers and $b \neq 1$:

$$\log_b MN = \log_b M + \log_b N, \qquad \log_b \frac{M}{N} = \log_b M - \log_b N,$$

$$\log_b M^p = p \cdot \log_b M, \qquad \log_b \sqrt[q]{M} = \frac{1}{q} \cdot \log_b M,$$

$$\log_b \left(\frac{1}{M}\right) = -\log_b M, \qquad \log_b b = 1, \qquad \log_b 1 = 0.$$

Change of base of Logarithms ($c \neq 1$):

$$\log_b M = \log_c M \cdot \log_b c = \frac{\log_c M}{\log_c b}.$$

6. Binomial Theorem (n a positive integer).

$$(a + b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \cdots + nab^{n-1} + b^n$$

where

$$n! = \underline{|n} = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot (n-1)n.$$

7. Expansions and Factors.

$$(a \pm b)^2 = a^2 \pm 2ab + b^2.$$

$$(a \pm b)^3 = a^3 \pm 3a^2b + 3ab^2 \pm b^3.$$

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc.$$

$$a^2 - b^2 = (a - b)(a + b).$$

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2).$$

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2).$$

$$a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + \cdots + b^{n-1}).$$

$a^n - b^n = (a + b)(a^{n-1} - a^{n-2}b + \cdots - b^{n-1})$, for n an even integer.

$a^n + b^n = (a + b)(a^{n-1} - a^{n-2}b + \cdots + b^{n-1})$, for n an odd integer.

$$a^4 + a^2b^2 + b^4 = (a^2 + ab + b^2)(a^2 - ab + b^2).$$

8. Ratio and Proportion.

If $a : b = c : d$ or $\frac{a}{b} = \frac{c}{d}$, then $ad = bc$, $\frac{a}{c} = \frac{b}{d}$.

If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \cdots = k$, then

$$k = \frac{a + c + e + \cdots}{b + d + f + \cdots} = \frac{pa + qc + re + \cdots}{pb + qd + rf + \cdots}.$$

9. Constant Factor of Proportionality (or Variation), k.

If y varies directly as x, or y is proportional to x,

$$y = kx.$$

If y varies inversely as x, or y is inversely proportional to x,

$$y = \frac{k}{x}.$$

If y varies jointly as x and z,

$$y = kxz.$$

If y varies directly as x and inversely as z,

$$y = \frac{kx}{z}.$$

10. Arithmetic Progression.

$$a, \quad a + d, \quad a + 2d, \quad a + 3d, \cdots.$$

If a is the first term, d the common difference, n the number of terms, l the last term and s the sum of n terms,

$$l = a + (n - 1)\,d, \quad s = \frac{n}{2}\,(a + l).$$

The *arithmetic mean* of a and b is $(a + b)/2$.

11. Geometric Progression.

$$a,\ ar,\ ar^2,\ ar^3,\ \cdots.$$

If a is the first term, r the common ratio, n the number of terms, l the last term, and S_n the sum of n terms,

$$l = ar^{n-1}, \quad S_n = a\left(\frac{r^n - 1}{r - 1}\right) = \frac{rl - a}{r - 1}.$$

If $r^2 < 1$, S_n approaches the limit S_∞ as n increases without limit,

$$S_\infty = \frac{a}{1 - r}.$$

The *geometric mean* of a and b is $\sqrt{ab}$.

12. Harmonic Progression. A sequence of numbers whose reciprocals form an arithmetic progression is called an *harmonic progression*. Thus

$$\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \cdots.$$

The *harmonic mean* of a and b is $2ab/(a+b)$.

13. Permutations. Each different arrangement of all or a part of a set of things is called a *permutation.* The number of permutations of n different things taken r at a time is

$$P(n, r) = {}_nP_r = n(n-1)(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!},$$

where

$$n! = n(n-1)(n-2)\cdots(1).$$

14. Combinations. Each of the groups or relations which can be made by taking part or all of a set of things, without regard to the arrangement of the things in a group, is called a *combination.* The number of combinations of n different things taken r at a time is

$$C(n, r) = {}_nC_r = \binom{n}{r} = \frac{{}_nP_r}{r!} = \frac{n(n-1)\cdots(n-r+1)}{r(r-1)\cdots(1)} = \frac{n!}{r!(n-r)!}.$$

15. Probability. If an event may occur in p ways, and may fail in q ways, all ways being equally likely, the *probability* of its occurrence is $p/(p+q)$, and that of its failure to occur is $q/(p+q)$. (For further details see Burington & May, *Handbook of Probability & Statistics with Tables*, Handbook Publishers, Inc., Sandusky, Ohio.)

16. Remainder Theorem (See §30). If the polynomial $f(x)$ is divided by $(x-a)$, the remainder is $f(a)$. Hence, if a is a root of the equation $f(x) = 0$, then $f(x)$ is divisible by $(x-a)$.

17. Determinants. The determinant D of order n,

$$D = \begin{vmatrix} a_{11} & a_{12} & \cdot & \cdot & \cdot & a_{1n} \\ a_{21} & a_{22} & \cdot & \cdot & \cdot & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdot & \cdot & \cdot & a_{nn} \end{vmatrix},$$

is defined to be the sum

$$\Sigma\,(\pm)\, a_{1i}\, a_{2j}\, a_{3k} \cdots a_{nl}$$

of $n!$ terms, the sign in a given term being taken plus or minus according as the number of inversions (of the numbers $1, 2, 3, \cdots, n$) in the corresponding sequence $i, j, k, \cdots, l$, is even or odd.

The *cofactor* A_{ij} of the element a_{ij} is defined to be the product of $(-1)^{i+j}$ by the determinant obtained from D by deleting the ith row and the jth column.

The following theorems are true:

(*a*) If the corresponding rows and columns of D be interchanged, D is unchanged.

(*b*) If any two rows (or columns) of D be interchanged, D is changed to $-D$.

(*c*) If any two rows (or columns) are alike, then $D = 0$.

(*d*) If each element of a row (or column) of D be multiplied by m, the new determinant is equal to mD.

(*e*) If to each element of a row (or column) is added m times the corresponding element in another row (or column), D is unchanged.

(*f*) $D = a_{1j} A_{1j} + a_{2j} A_{2j} + \cdots + a_{nj} A_{nj}, \qquad j = 1, 2, \cdots, n.$

(*g*) $0 = a_{1k} A_{1j} + a_{2k} A_{2j} + \cdots + a_{nk} A_{nj}, \quad \text{if } j \neq k.$

(*h*) The solution of the system of equations

$$a_{i1} x_1 + a_{i2} x_2 + \cdots + a_{in} x_n = c_i, \qquad i = 1, 2, \cdots, n,$$

is unique if $D \neq 0$. The solution is given by the equations

$$Dx_1 = C_1, \qquad Dx_2 = C_2, \cdots, \qquad Dx_n = C_n,$$

where C_k is what D becomes when the elements of its kth column are replaced by $c_1, c_2, \cdots, c_n$, respectively.

Example 1.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22} a_{33} - a_{32} a_{23}) - a_{12}(a_{21} a_{33} - a_{31} a_{23}) + a_{13}(a_{21} a_{32} - a_{31} a_{22}).$$

Example 2. Find the values of x_1, x_2, x_3, which satisfy the system

$$\begin{aligned} 2x_1 + x_2 - 2x_3 &= -6, \\ x_1 + x_2 + x_3 &= 2, \\ -x_1 - 2x_2 + 3x_3 &= 12. \end{aligned}$$

By 17(*h*), we find

$$x_1 = \frac{\begin{vmatrix} -6 & 1 & -2 \\ 2 & 1 & 1 \\ 12 & -2 & 3 \end{vmatrix}}{\begin{vmatrix} 2 & 1 & -2 \\ 1 & 1 & 1 \\ -1 & -2 & 3 \end{vmatrix}} = \frac{8}{8} = 1; \quad x_2 = \frac{\begin{vmatrix} 2 & -6 & -2 \\ 1 & 2 & 1 \\ -1 & 12 & 3 \end{vmatrix}}{\begin{vmatrix} 2 & 1 & -2 \\ 1 & 1 & 1 \\ -1 & -2 & 3 \end{vmatrix}} = \frac{-16}{8} = -2; \quad x_3 = \frac{\begin{vmatrix} 2 & 1 & -6 \\ 1 & 1 & 2 \\ -1 & -2 & 12 \end{vmatrix}}{\begin{vmatrix} 2 & 1 & -2 \\ 1 & 1 & 1 \\ -1 & -2 & 3 \end{vmatrix}} = 3.$$

Interest, Annuities, Sinking Funds.

In this section, n is the number of years, and r the rate of interest expressed as a decimal.

18. Amount. A principal P placed at a rate of interest r for n years accumulates to an amount A_n, as follows:

At simple interest: $A_n = P(1 + nr)$.

At interest compounded annually:* $A_n = P(1 + r)^n$.

At interest compounded q times a year: $A_n = P\left(1 + \frac{r}{q}\right)^{nq}$.

19. Nominal and Effective Rates. The rate of interest quoted in describing a given variety of compound interest is called the *nominal rate.* The rate per year at which interest is earned during each year is called the *effective rate.* The effective rate i corresponding to the nominal rate r, compounded q times a year is:

$$i = \left(1 + \frac{r}{q}\right)^q - 1.$$

20. Present or Discounted Value of a Future Amount. The present quantity P which in n years will accumulate to the amount A_n at the rate of interest r, is:

At simple interest: $P = \frac{A_n}{1 + nr}$.

At interest compounded annually:† $P = \frac{A_n}{(1 + r)^n}$.

At interest compounded q times a year: $P = \frac{A_n}{\left(1 + \frac{r}{q}\right)^{nq}}$.

P is called the *present value* of A_n due in n years at rate r.

21. True Discount. The true discount is:

$$D = A_n - P.$$

22. Annuity. A fixed sum of money paid at regular intervals is called an *annuity.*

23. Amount of an Annuity.‡ If an annuity P is deposited at the end of each successive year (beginning one year hence), and the interest at rate r, compounded annually, is paid on the accumulated deposit at the end of each year, the total amount N accumulated at the end of n years is

$$N = P \cdot \frac{(1 + r)^n - 1}{r}.$$

N is called the *amount* of an annuity P.

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* See Table XIX. † See Table XX. ‡ See Table XXI.

24. Present Value of an Annuity.* The total present amount P which will supply an annuity N at the end of each year for n years, beginning one year hence, (assuming that in successive years the amount not yet paid out earns interest at rate r, compounded annually), is:

$$P = N \cdot \frac{(1+r)^n - 1}{r(1+r)^n} = N \cdot \frac{1 - (1+r)^{-n}}{r}.$$

P is called the *present value* of an annuity.

25. Amount of a Sinking Fund.‡ If a fixed investment N is made at the end of each successive year (beginning at the end of the first year), and interest paid at rate r, compounded annually, is paid on the accumulated amount of the investment at the end of each year, the total amount S accumulated at the end of n years is:

$$S = N \cdot \frac{(1+r)^n - 1}{r}.$$

S is called the *amount of the sinking fund.*

26. Fixed Investment, or Annual Installment. The amount N that must be placed at the end of each year (beginning one year hence), with compound interest paid at rate r on the accumulated deposit, in order to accumulate a sinking fund S in n years is:

$$N = S \cdot \frac{r}{(1+r)^n - 1}.$$

N is called a *fixed investment* or *annual installment.*

Algebraic Equations†

27. Quadratic Equations. If

$$ax^2 + bx + c = 0, \quad a \neq 0,$$

then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

If a, b, c are real and
if $b^2 - 4ac > 0$, the roots are real and unequal.
if $b^2 - 4ac = 0$, the roots are real and equal.
if $b^2 - 4ac < 0$, the roots are imaginary.

28. Cubic Equations. The cubic equation

$$y^3 + py^2 + qy + r = 0,$$

may be reduced by the substitution

$$y = \left(x - \frac{p}{3}\right)$$

* See Table XXII. ‡ Compare with § 23. † For linear equations, see § 17.

to the normal form

$$x^3 + ax + b = 0,$$

where

$$a = \frac{1}{3}(3q - p^2), \quad b = \frac{1}{27}(2p^3 - 9pq + 27r),$$

which has the solutions $x_1, x_2, x_3,$

$$x_1 = A + B, \quad x_2, \ x_3 = -\frac{1}{2}(A + B) \pm \frac{i\sqrt{3}}{2}(A - B),$$

where

$$i^2 = -1, \quad A = \sqrt[3]{-\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}}, \quad B = \sqrt[3]{-\frac{b}{2} - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}}.$$

If p, q, r are real (and hence, if a and b are real), and

if $\frac{b^2}{4} + \frac{a^3}{27} > 0$, there are one real root and two conjugate imaginary roots,

if $\frac{b^2}{4} + \frac{a^3}{27} = 0$, there are three real roots of which at least two are equal,

if $\frac{b^2}{4} + \frac{a^3}{27} < 0$, there are three real and unequal roots.

If

$$\frac{b^2}{4} + \frac{a^3}{27} < 0,$$

the above formulas are impractical. The real roots are,

$$x_k = 2\sqrt{-\frac{a}{3}}\cos\left(\frac{\phi}{3} + 120^\circ\, k\right), \quad k = 0, 1, 2.$$

where

$$\cos\phi = \mp\sqrt{\frac{b^2}{4} \div \left(-\frac{a^3}{27}\right)},$$

and where the upper sign is to be used if b is positive and the lower sign if b is negative.

If

$$\frac{b^2}{4} + \frac{a^3}{27} > 0, \text{ and } a > 0,$$

the real root is,

$$x = 2\sqrt{\frac{a}{3}}\operatorname{ctn} 2\phi,$$

where ϕ and ψ are to be computed from