Contemporary Concepts in Physics
Volume 4

R.Z. Sagdeev
D.A. Usikov
G.M. Zaslavsky

Nonlinear Physics

From the Pendulum to Turbulence and Chaos

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by
R. Z. Sagdeev
D. A. Usikov
G. M. Zaslavsky

Space Research Institute
USSR Academy of Sciences
Moscow

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Igor R. Sagdeev
Solid State Physics Institute, Moscow

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Preface to the Series

The series of volumes, *Concepts in Contemporary Physics*, is addressed to the professional physicist and to the serious graduate student of physics. The subjects to be covered will include those at the forefront of current research. It is anticipated that the various volumes in the series will be rigorous and complete in their treatment, supplying the intellectual tools necessary for the appreciation of the present status of the areas under consideration and providing the framework upon which future developments may be based.

Preface

Physics, as we know it today, began with nonlinear laws of particle dynamics. The classical example of the integrable Kepler's problem clearly shows the fundamental properties of nonlinear systems: periodic orbits with a large number of harmonics and a period of the oscillations depending on their amplitude. Meanwhile, the famous three-body problem helped us to encounter the most formidable difficulties of nonlinear dynamics: nonintegrability, the appearance of small denominators in perturbation theory series, etc. What is more, it is now clear that in typical nonlinear cases it is impossible to predict for an arbitrarily long time the dynamic properties of even weakly perturbed systems. In the present state of affairs we could not answer many important questions, among them the problem of the eternal (i.e. unbounded in time) stability of dynamic systems. In practice, these difficulties become even more pronounced in the case of considerably more complex physical objects, such as fluids, nonlinear fields (such as Einstein's gravitational fields), etc. As a sort of compensation, physicists made great advances in constructing some purely linear theories: electromagnetic-field theory and quantum theory. The success achieved in these fields somewhat distracted scientists' attention from the genuine nonlinearities. Also, quantum mechanical techniques were quite successfully applied to a variety of classical problems by using the idea of linearizing the initial equations and constructing convenient perturbation theory series for nonlinear problems.

Some time elapsed before it became clear that the old problems remained unsolved, and that the old difficulties could not be removed by linearization. By that time, all branches of physics could boast "nonlinear" problems of their own. Nonlinear optics, nonlinear acoustics and nonlinear radiophysics had appeared. However, it was plasma that turned out to be the object richest in various nonlinear problems. The combination of problems involving particle dynamics and those involving dynamics of nonlinear media in the absence of collisions brought about a sort of "physical laboratory", in which it proved possible to demonstrate a real physical analogue of virtually any process from any other domain of physics. It is sufficient to mention such vivid analogies as adiabatic invariants in mechanics and the conservation of the magnetic moment of a charged particle in a magnetic trap, waves on "shallow water" and magnetosonic waves in a plasma, the dynamics of a rigid rotator and parametric decay instabilities of waves, and many others.

Perhaps, it was these features of collision-free plasma as a nonlinear medium that promoted the development of new and, in a sense, unexpected methods for investigating it. On the one hand, there were techniques which introduced a stochastic element into the dynamics of the medium, due to complex nonlinear interactions in the absence of explicitly random forces (quasi-linear theory, weak turbulence, etc.), and on the other, there were methods for the exact integration of complex nonlinear equations.

It was also about that time that radical changes took place in the rigorous methods of analyzing nonlinear systems. A universal technique for the approximate averaging of nonlinear systems (the Krylov–Bogolyubov–Mitropol'sky method) appeared, the theorem of conservation of invariants (Kolmogorov–Arnol'd–Moser) was proved, and, finally, a definition appeared of a newly discovered property of nonlinear systems: Kolmogorov–Sinai dynamic entropy. This entropy, being a new invariant of the system, expressed quantitatively the ability of nonlinear systems to execute motion with mixing — a property which was studied even earlier in the works of E. Hopf and N.S. Krylov. It now turned out that mixing, or chaos, may set in even in a system with as few as two degrees of freedom, and that its presence or absence depends only on the values of the parameters or initial conditions of the problem. Thus, a qualitatively new element of motion was introduced into nonlinear dynamics, which demanded a revision of a number of approximate results obtained earlier.

The development of new ideas in the field of nonlinear dynamics was greatly assisted by the advent of computers. Their use for analyzing nonlinear systems was begun by E. Fermi and S. Ulam and has now reached such a scale, that a good notion of the nature of a process cannot usually be formed without watching it on a computer display screen, even in those cases where formal results are quite readily obtainable.

As a consequence of all the above-mentioned advances, as well as many other results, a certain notion of the nonlinear dynamics of various processes, irrespective of the domain of physics to which they belong, is now being formed. Some common physical concepts have appeared, independent of the specific domain of application, as well as the most representative types of solutions in the simplest physical situations, corresponding to these concepts. These considerations encouraged the authors to try to help physicists obtain a coherent idea of the special features of contemporary nonlinear problems in physics. Taking into account the abundance of literature in this field, as well as the absence of final results in many problems, it is easy to understand the difficulties the authors have encountered in selecting the material to be included in this book. All of it, "from the pendulum to turbulence and chaos", is presented in the same informal style, primarily based on a qualitative analysis and physical estimates. Lengthy calculations are fairly rare, and their use is essentially restricted to the examples. The work is divided into 4 quite natural parts:

1. Particles
2. Waves
3. Examples
4. Numerical Simulation.

Part 4 — Numerical Simulation — requires special comment.

We will list a few new concepts, indispensable for any discussion of contemporary nonlinear physics: a singular change of a periodic trajectory into a chaotic one, triggered by a seemingly insignificant variation of the parameters; the fractal properties of individual trajectories and the fractal structures of regions in phase space; Cantori with a porous structure of a Cantor-set type; conversion of a conventional attractor into a stochastic one; and quasi-attractors. This list could be considerably extended. These concepts appeared comparatively recently, and getting used to them takes some time. An understanding of these concepts is assisted by an informal view of the new properties of nonlinear systems, promoted by the material comprising Part 4 of the book. This part is divided into three sections and contains results of a numerical simulation of a few mappings, characteristic of some typical physical situations.

Part 4A contains photographs taken from the colour monitor screen illustrating various physical phenomena in the phase plane. The use of colour enables us to discern many subtle properties of these phenomena. Part 4B is comprised of diskettes on which computer graphics are recorded. These graphics consist of specially chosen static and dynamic scenarios. The duration of the dynamic scenarios is arbitrary and is determined by the user. If the user wishes, he may turn on a special "audio accompaniment" to a dynamic scenario. These visuals develop in real time, just as the corresponding mappings are iteratively applied.

Finally, Part 4C is intended for the user wishing to carry out his own investigation of nonlinear phenomena using a PC. It consists of an ATRS diskette, on which the program used in the first two sections is recorded; a detailed manual is included. The diskette may be viewed as a new kind of "computer game" in which the "player" investigates very complex and subtle physical processes. Not only does the ATRS diskette enable the user to obtain all the results presented in the two preceding sections, but it also lets him study new properties of the mappings included. The results of this simulation may also be recorded on disk as a sequence of static and dynamic scenarios. In other words, the ATRS diskette enables the user to produce his own visuals. Included in this section is a description of the ATRS program, which allows a user familiar with BASIC computer programming language to supplement the program with some missing elements, as well as to change any of the subroutines.

The main body of the book, i.e., the first three parts, devoted to an analysis of specific nonlinear physical models, was written by R.Z. Sagdeev and G.M. Zaslavsky. For a more detailed coverage of the physical picture outlined in this book, the reader is referred to the earlier monographs: *Plasma Physics for Physicists*, by L.A. Artsimovich and R.Z. Sagdeev (Atomizdat, Moscow, 1979) and *Chaos in Dynamic Systems*, by G.M. Zaslavsky (Harwood Academic Publishers, Chur, 1985). The ATRS program and Part 4C were written by D.A. Usikov. All the authors took part in creating the illustrative material of Parts 4A and B.

The authors are also deeply grateful to A.A. Chernikov and M.Yu. Zakharov, who helped with Part 4, and to I.R. Sagdeev for the English version of the book.

Notation

Double enumeration is used for formulas within each chapter: the first number is that of the section; the second one is that of the formula proper.

Triple enumeration is used in several cases for formulas from other chapters: the first number for the chapter; the second one for the section; and the third for the formula.

Figures have double enumeration: the first number is for the chapter; the second for the figure proper.

References to sections use double enumeration: the first number for the chapter; the second one for the section.

Each chapter is followed by numbered Notes. These are referred to in the text in parentheses.

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