M. VIDYASAGAR

Nonlinear Systems Analysis

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Preface

This book is intended as a text for a one-term or one-quarter course in nonlinear systems, at either the first-year graduate or senior-graduate level; it is almost self-contained and hence suitable for self-study. The only prerequisite for using the book is a course in ordinary differential equations. It is generally not necessary for the reader to have had a course in linear systems, though it is perhaps helpful to have an understanding of the concept of the *state* of a system. The contents of the book should be of interest to engineers from all branches who are interested in the *systems* approach, as well as to applied mathematicians, mathematical economists, biologists, *et cetra*. The results developed in the book are of a sufficiently general nature as to be applicable to all of these disciplines. Most of the important techniques for the analysis of nonlinear systems are covered in the book, though the coverage is by no means encyclopaedic. One of the novel features of the book is a chapter on input-output stability, presented at an elementary level for the first time.

The first version of this book was written in 1973, while I was visiting UCLA. Subsequent drafts were classroom-tested at both Concordia University and UCLA. In addition, portions of the book were also used at Berkeley for one quarter. Generally, the classes consisted of graduate students in both engineering and mathematics. This experience revealed that the entire book can be covered in about fifty classroom hours, while most of it can be covered in forty hours.

The book contains five chapters besides the introduction. Chapter 2 contains a discussion of various phase-plane techniques for the analysis of second-order systems. In chapter 3, the reader is introduced to some basic mathematical tools such as normed spaces, contraction mapping theorem, etc; this is followed by statements and proofs of the basic existence and uniqueness theorems for nonlinear differential equations, and some useful solution estimates. Chapter 4 consists of an introduction to several commonly used *approximate* analysis techniques. Chapter 5 contains a thorough treatment of Liapunov stability, including the Lur'e problem. Finally, chapter 6 comprises an elementary discussion of input-output stability, including the Nyquist, circle, and Popov criteria for feedback systems. There are numerous examples and exercises throughout. Two appendixes close the book.

It is now my pleasure to acknowledge all those who helped me in the writing of this book. I would like, first of all, to thank my wife Shakunthala for her encouragement and complete moral support throughout this project. Thanks are also due to Professor Charles A. Desoer for his thorough review of the manuscript and numerous constructive comments, as well as to Professor E. I. Jury for class-testing the manuscript and for several useful suggestions. I thank Professor M. N. S. Swamy and Dean J. C. Callaghan, both of Concordia University, for providing excellent logistic support as well as for their moral support. Professor A. V. Balakrishnan is to be thanked for making possible my visit to UCLA, during which this project was started. Finally, thanks to Veronica Markowitz and June Anderson for their excellent typing.

Montreal M. VIDYASAGAR

Notes to the Reader

1. All items within each section of each chapter (equations, theorems, examples, etc.) are numbered consecutively. A reference such as "Theorem (17)" refers to the 17th item *within the same section*. If a reference is made to an item in another section, the full number is given, e.g. example [5.1(13)] means example (13) in Sec. 5.1.
2. In some places, we write, e.g.

   $$\phi = \tan^{-1}\frac{x_2}{x_1}$$

   This means that ϕ is the unique number in $[0, 2\pi)$ such that

   $$\sin\phi = \frac{x_2}{(x_1^2 + x_2^2)^{1/2}}, \qquad \cos\phi = \frac{x_1}{(x_1^2 + x_2^2)^{1/2}}$$

   Thus $\tan^{-1}$ is a function of *both* variables x_1 and x_2, and not just of the ratio x_2/x_1. Note that $\tan^{-1}$ is well-defined everywhere in R^2 except at (0, 0).

Contents

1 Introduction

1.1 GENERAL CONSIDERATIONS

Nonlinear physical systems, that is, systems that are not necessarily linear, differ from linear systems in two important respects:

1. Generally speaking, one can usually obtain *closed-form* expressions for solutions of linear systems, whereas this is not always possible in the case of nonlinear systems. More often, one is forced to be content with obtaining sequences of approximating functions that converge to the true solution or with generating estimates for the true solution. As a result, one may not have a good "feel" for what makes a nonlinear system "tick," compared with a linear system.
2. The analysis of nonlinear systems generally involves mathematics that is more advanced in concept and more messy in detail than is the case with linear systems.

A mathematical model that describes a wide variety of physical nonlinear systems is an nth-order ordinary differential equation of the type

1 $$\frac{d^n y(t)}{dt^n} = h\left[t, y(t), \dot{y}(t), \ldots, \frac{d^{n-1}y(t)}{dt^{n-1}}, u(t)\right], \qquad t \geq 0$$

where t is the time parameter, $u(\cdot)$ is the input function (the terms *control function* and *forcing function* are also used), and $y(\cdot)$ is the output function (or *response function*). If we define the auxiliary functions

2 $$x_1(t) = y(t)$$

3 $$x_2(t) = \dot{y}(t)$$

$$\vdots$$

4 $$x_n(t) = \frac{d^{n-1}y(t)}{dt^{n-1}}$$

then the single nth-order equation (1) can be equivalently expressed as a system of n first-order equations:

5 $$\dot{x}_1(t) = x_2(t)$$

6 $$\dot{x}_2(t) = x_3(t)$$

$$\vdots$$

7 $$\dot{x}_{n-1}(t) = x_n(t)$$

8 $$\dot{x}_n(t) = h[t, x_1(t), x_2(t), \ldots, x_n(t), u(t)]$$

Finally, if we define n-vector-valued functions $\mathbf{x}(\cdot): R_+ \longrightarrow R^n$ and $\mathbf{f}: R_+ \times R^n \times R \longrightarrow R^n$ by

9 $$\mathbf{x}(t) = [x_1(t), x_2(t), \ldots, x_n(t)]'$$

10 $$\mathbf{f}(t, \mathbf{x}, u) = [x_2, x_3, \ldots, x_n, h(t, x_1, \ldots, x_n, u)]'$$

then the n first-order equations (5)–(8) can be combined into a first-order vector differential equation, namely,

11 $$\dot{\mathbf{x}}(t) = \mathbf{f}[t, \mathbf{x}(t), u(t)], \qquad t \geq 0$$

For the system described by (1), the n quantities x_1 through x_n constitute a set of *state variables*, and the vector $\mathbf{x}$ constitutes a *state vector*.

Similarly, suppose a system with p inputs and k outputs is described by a set of k ordinary differential equations of the form

12 $$\frac{d^{m_i}y_i(t)}{dt^{m_i}} = h_i[t, y_1(t), \dot{y}_1(t), \ldots, y_1^{(m_1-1)}(t), y_2(t), \ldots, y_2^{(m_2-1)}(t), \ldots, y_k(t), \ldots, y_k^{(m_k-1)}(t), u_1(t), \ldots, u_p(t)], \qquad i = 1, \ldots, k$$

where $u_1(\cdot), \ldots, u_p(\cdot)$ are the input functions and $y_1(\cdot), \ldots, y_k(\cdot)$ are the output functions. As before, define

13 $$x_{m_i+j}(t) = \frac{d^{j-1}y_{i+1}(t)}{dt^{j-1}}, \qquad j = 1, \ldots, m_{i+1}, i = 0, \ldots, k-1$$

14 $$\mathbf{x}(t) = [x_1(t) \ldots x_n(t)]'$$

15 $$\mathbf{u}(t) = [u_1(t) \ldots u_p(t)]'$$

where we take $m_0 = 0$, and define

16 $$n = m_1 + \ldots + m_k$$

Then $\mathbf{x}(t)$ is a state vector for the system described by (12), and the system of equations (12) can once again be equivalently represented by a single first-order vector differential equation of the form

17 $$\dot{\mathbf{x}}(t) = \mathbf{f}[t, \mathbf{x}(t), \mathbf{u}(t)], \qquad t \geq 0$$

With this background in mind, we shall devote much of this book to the study of systems described by an equation of the form (17).[1] For (17) to truly represent a physical system, we would expect that, corresponding to each input $\mathbf{u}(\cdot)$,

1. (17) has at least one solution (existence).
2. (17) has exactly one solution (uniqueness).
3. (17) has exactly one solution that is defined over the entire half-line $[0, \infty)$.
4. (17) has exactly one solution over $[0, \infty)$, and this solution depends continuously on the initial condition $\mathbf{x}(0)$.

Statements 1–4 are progressively stronger. Unfortunately, without some restrictions on the nature of the function $\mathbf{f}$, none of these statements may be true, as illustrated by the following examples.

18 **Example.** Consider the scalar differential equation

19 $$\dot{x}(t) = -\text{sign } x(t), \quad t \geq 0; \qquad x(0) = 0$$

where the "sign" function is defined by

20 $$x(t) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$$

It is easy to verify that no continuously differentiable function $x(\cdot)$ exists such that (19) is satisfied. Thus statement 1 does not hold for this system.

21 **Example.** Consider the scalar differential equation

22 $$\dot{x}(t) = \frac{1}{2x(t)}, \quad t \geq 0; \qquad x(0) = 0$$

This equation admits two solutions, namely,

---

[1]The exception is Chap. 6, where we shall study distributed systems, e.g., systems containing time delays.

23 $$x_1(t) = t^{1/2}$$

24 $$x_2(t) = -t^{1/2}$$

Thus statement 1 is true, but 2 is false.

25 **Example.** Consider the scalar differential equation

26 $$\dot{x}(t) = 1 + x^2(t), \quad t \geq 0; \qquad x(0) = 0$$

Then over the interval [0, 1), this equation has the unique solution

27 $$x(t) = \tan t$$

but there is no continuously differentiable function $x(\cdot)$ *defined over all of* $[0, \infty)$ such that (26) holds. Thus for this system statements 1 and 2 are true, but 3 fails.

It is therefore clear that the questions of existence and uniqueness of solutions to (17), and their continuous dependence on the initial condition, are very important. These questions are studied in Chap. 3.

In the last two examples, it was possible to derive the closed-form solutions to the equations under study, because they were of an extremely simple nature. However, in most cases, one cannot obtain an exact solution to the differential equation describing the system behavior. In such cases, one must be content either with generating "approximate" solutions or with solution *bounds*, which tell us that the solution at any time lies in a certain region of the state space. Both of these are studied in Chaps. 3 and 4.

An important question is that of the *well-behavedness*, in some suitable sense, of the solutions to (17). This is usually called the question of stability. Ideally, we would like to know whether or not the solutions to (17) are well behaved *without actually solving the system equations* (17). The stability question is studied in depth in Chaps. 5 and 6.

Finally, as a prelude to these more advanced subjects, we shall study second-order systems in Chap. 2. As we shall see there, a "geometric" approach to second-order systems yields much intuition and insight.

Problem 1.1. Determine whether or not each of the following differential equations has a unique solution over $[0, \infty)$, and if so, whether this solution depends continuously on the initial condition.

(a) $\dot{x}(t) = [x(t)]^{1/3}; \qquad x(0) = 0$

(b) $\dot{x}(t) = -x^2(t), \qquad x(0) = -1$

(c) $\dot{x}(t) = \begin{cases} -x(t) & \text{if } x(t) < 0 \\ x^2(t) & \text{if } x(t) < 0 \end{cases}; \qquad x(0) = 0$

1.2 AUTONOMY, EQUILIBRIUM POINTS

In this section, we shall introduce two definitions that are frequently used in the sequel. Before proceeding to these definitions, we shall clear up one small point. Many of the definitions, theorems, etc., that follow are stated for differential equations of the type

1 $$\dot{\mathbf{x}}(t) = \mathbf{f}[t, \mathbf{x}(t)]$$

Comparing (1) with (17) of Sec. 1.1,[2] we see that in [1.1(17)] the dependence of the right-hand side on an input $\mathbf{u}(\cdot)$ is explicitly identified, whereas this dependence, if any, is suppressed in (1). This might mislead one into thinking that [1.1(17)] describes a "forced" system, whereas (1) describes an "unforced" system. However, this is not necessarily the case. In problems of system *analysis*, as opposed to optimal control problems, one is generally concerned with the behavior of a system of the form [1.1(17)] under a fixed known input. Thus suppose that, in [1.1(17)], $\mathbf{u}(\cdot)$ is a *known fixed* function, and define $\mathbf{f}_{\mathbf{u}}: R_+ \times R^n \longrightarrow R^n$ by

2 $$\mathbf{f}_{\mathbf{u}}(t, \mathbf{x}) = \mathbf{f}[t, \mathbf{x}, \mathbf{u}(t)]$$

Then [1.1(17)] can be rewritten as

3 $$\mathbf{x}(t) = \mathbf{f}_{\mathbf{u}}[t, \mathbf{x}(t)]$$

which is of the form (1). Therefore (1) can represent either an "unforced" system or a system with a fixed input.

We shall now introduce two concepts.

4 *[definition]* The system described by (1) is said to be *autonomous* if $\mathbf{f}(t, \mathbf{x})$ is independent of t and is said to be *nonautonomous* otherwise.

5 *[definition]* A vector $\mathbf{x}_0 \in R^n$ is said to be an *equilibrium point* at time $t_0 \in R_+$ of (1) if

6 $$\mathbf{f}(t, \mathbf{x}_0) = \mathbf{0}, \qquad \forall\, t \geq t_0$$

If $\mathbf{x}_0$ is an equilibrium point of (1) at time t_0, then it is clear that $\mathbf{x}_0$ is also an equilibrium point of (1) at all times $t_1 \geq t_0$. Furthermore, if (1) is autonomous, then $\mathbf{x}_0 \in R^n$ is an equilibrium point of (1) at *some* time if and only if it is an equilibrium point of (1) at *all* times. Therefore we may speak of an equilibrium point of an autonomous system without specifying the time.

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[2]Hereafter referred to as [1.1(17)].

The physical significance of an equilibrium point is as follows: Suppose $\mathbf{x}_0 \in R^n$ is an equilibrium point of (1) at time t_0. Then, whenever $t_1 \geq t_0$, the equation

7 $$\dot{\mathbf{x}}(t) = \mathbf{f}[t, \mathbf{x}(t)], \quad t \geq t_1; \qquad \mathbf{x}(t_1) = \mathbf{x}_0$$

has the unique solution

8 $$\mathbf{x}(t) = \mathbf{x}_0, \qquad \forall\, t \geq t_1$$

Conversely, if an element $\mathbf{x}_0 \in R^n$ has the property that the unique solution of (7) is given by (8) whenever $t_1 \geq t_0$, then it follows by simple differentiation that $\mathbf{x}_0$ satisfies (6), i.e., that $\mathbf{x}_0$ is an equilibrium point of (1) at time t_0. Thus, in other words, $\mathbf{x}_0$ is an equilibrium point of (1) at time t_0 if, should any solution $\mathbf{x}(\cdot)$ of (1) assume the value $\mathbf{x}_0$ at some time $t_1 \geq t_0$, it then remains at that value $\mathbf{x}_0$ for all $t \geq t_1$. The terms *stationary point* and *singular point* are also used in place of *equilibrium point*.

9 **Example.** Consider the motion of a frictionless simple pendulum, and let θ denote the angle of the pendulum from the vertical. Then the motion of the pendulum is described by

10 $$\ddot{\theta}(t) + \frac{g}{l} \sin \theta(t) = 0$$

where g is the acceleration due to gravity and l is the length of the pendulum. If we define

11 $$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \triangleq \begin{bmatrix} \theta(t) \\ \dot{\theta}(t) \end{bmatrix}$$

then the dynamics of the system are described by the state variable equations

12 $$\begin{bmatrix} \dot{x}_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ -(g/l) \sin x_1(t) \end{bmatrix}$$

Notice first of all that the system is autonomous. Next, we have that $x_0 = [x_{10} \quad x_{20}]'$ is an equilibrium point of (12)[3] if and only if

13 $$x_{20} = 0$$

14 $$\sin x_{10} = 0$$

i.e., the set of equilibrium points of (12) is the set of points in R^2 of the form

15 $$(n\pi, 0), \qquad \text{where } n = 0, \pm 1, \pm 2, \ldots$$

---

[3]Notice that we need not specify the time because the system is autonomous.

Because we commonly identify two values of θ that differ by a multiple of 2π, this system has basically two equilibrium points, namely (0, 0) and $(0, \pi)$. Of these, the first equilibrium point corresponds to the pendulum hanging straight down, while the second equilibrium point corresponds to the pendulum being at rest pointing straight up. Of course, we do not ever expect to find a real pendulum at rest pointing straight up, because the slightest perturbation (such as wind drafts present in the room) would knock the pendulum out of this equilibrium position. This is intimately connected with the question of the stability of an equilibrium point, which is studied in Chap. 5.

16 **Example.** Consider the one-dimensional motion of a particle in a potential field. Let r denote the position of the particle, m the mass of the particle, and $p(r)$ the potential energy at r. We assume that $p(r)$ is a continuously differentiable function of r. The motion of the particle is described by

17 $$\ddot{r}(t) = \frac{1}{m}\frac{dp(\xi)}{d\xi}\bigg|_{\xi=r(t)} = \frac{f[r(t)]}{m}$$

where $f(r) = dp(r)/dr$ denotes the force at r. To obtain a state variable description, define

18 $$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \triangleq \begin{bmatrix} r(t) \\ \dot{r}(t) \end{bmatrix}$$

Then the state equations are

19 $$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ f[x_1(t)] \end{bmatrix}$$

From (19), we see that the set of equilibrium points of this (autonomous) system consists of all points of the form $(r_0, 0)$, where $f(r_0) = 0$. Therefore this system is in an equilibrium state if the particle has zero velocity and is at a position where the force is zero, i.e., if the potential energy is stationary.

20 *[definition]* An equilibrium point $\mathbf{x}_0$ at time t_0 of (1) is said to be *isolated* if there exists a neighborhood N of $\mathbf{x}_0$ in R^n such that N contains no equilibrium points at time t_0 of (1) other than $\mathbf{x}_0$.

21 **Example.** Both the equilibrium points of the system in Example (9) are isolated. In the system of Example (16), an equilibrium point $(r_0, 0)$ is isolated if and only if r_0 is an isolated zero of the function $f(\cdot)$, i.e., if there exists a $\delta > 0$ such that $f(r) \neq 0$ whenever $0 < |r - r_0| < \delta$.

22 **Example.** Consider the linear vector differential equation

23 $$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t), \qquad t \geq 0$$

Clearly $\mathbf{0}$ is an equilibrium point of (23) at all times $t_0 \geq 0$. Suppose now that $\mathbf{A}(t_0)$ is nonsingular for some t_0. This means that $\mathbf{A}(t_0)\mathbf{x} = \mathbf{0}$ implies $\mathbf{x} = \mathbf{0}$. In this case, $\mathbf{0}$ is the only equilibrium point at time t_0 of (23) and is hence isolated.

24 **fact** Consider the system (1), and suppose $\mathbf{x}_0$ is an equilibrium point at time t_0 of (1); i.e., suppose (6) holds. Suppose further that $\mathbf{f}(t_0, \cdot)$ is continuously differentiable, and define

25 $$\mathbf{A}(t_0) = \left.\frac{\partial \mathbf{f}(t_0, \mathbf{x})}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{x}_0}$$

If $\mathbf{A}(t_0)$ is nonsingular, then $\mathbf{x}_0$ is an isolated equilibrium point at time t_0 of (1).

proof For each $\mathbf{x} = [x_1 \ldots x_n]'$ in R^n, define

26 $$\|\mathbf{x}\|_2 = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}$$

The real number $\|\mathbf{x}\|_2$ is known as the Euclidean norm of the vector $\mathbf{x}$.[4] If $\mathbf{A}(t_0)$ is nonsingular, then there exists a *positive* constant c such that

27 $$\|\mathbf{A}(t_0)\mathbf{x}\|_2 \geq c\,\|\mathbf{x}\|_2, \qquad \forall\ \mathbf{x} \in R^n$$

Because $\mathbf{f}(t_0, \cdot)$ is continuously differentiable, we can expand $\mathbf{f}(t_0, \mathbf{x})$ in the form

28 $$\mathbf{f}(t_0, \mathbf{x}) = \mathbf{f}(t_0, \mathbf{x}_0) + \mathbf{A}(t_0)(\mathbf{x} - \mathbf{x}_0) + \mathbf{r}(t_0, \mathbf{x})$$

where the "remainder" term $\mathbf{r}(t_0, \cdot)$ satisfies the condition

29 $$\lim_{\|\mathbf{x}-\mathbf{x}_0\|_2 \to 0} \frac{\|\mathbf{r}(t_0, \mathbf{x})\|_2}{\|\mathbf{x} - \mathbf{x}_0\|_2} = 0$$

However, because $\mathbf{x}_0$ is an equilibrium point at time t_0 of (1), we have $\mathbf{f}(t_0, \mathbf{x}_0) = \mathbf{0}$; therefore,

30 $$\mathbf{f}(t_0, \mathbf{x}) = \mathbf{A}(t_0)(\mathbf{x} - \mathbf{x}_0) + \mathbf{r}(t_0, \mathbf{x})$$

Now, pick a number $d > 0$ such that

31 $$\frac{\|\mathbf{r}(t_0, \mathbf{x})\|_2}{\|\mathbf{x} - \mathbf{x}_0\|_2} \leq \frac{c}{2} \text{ whenever } \|\mathbf{x} - \mathbf{x}_0\|_2 < d$$

Such a choice for d is always possible in view of the limit condition (29). Let N be the neighborhood of $\mathbf{x}_0$ defined by

32 $$N = \{\mathbf{x} \in R^n : \|\mathbf{x} - \mathbf{x}_0\|_2 < d\}$$

---

[4] A detailed discussion of norms, including the explanation for the subscript 2, is found in Chap. 3. For the present, it is enough to note that $\|\mathbf{x}\|_2 > 0$ whenever $\mathbf{x} \neq \mathbf{0}$.

We shall show that N contains no equilibrium points at time t_0 of (1) other than $\mathbf{x}_0$. By definition (20), this is enough to show that $\mathbf{x}_0$ is isolated.

Accordingly, suppose $\mathbf{x} \in N$ and $\mathbf{x} \neq \mathbf{x}_0$; we shall show that $\mathbf{f}(t_0, \mathbf{x}) \neq \mathbf{0}$. We have, whenever $\|\mathbf{x} - \mathbf{x}_0\|_2 < d$, that

33
$$\begin{aligned}
\|\mathbf{f}(t_0, \mathbf{x})\|_2 &= \|\mathbf{A}(t_0)(\mathbf{x} - \mathbf{x}_0) + \mathbf{r}(t_0, \mathbf{x})\|_2 \\
&\geq \|\mathbf{A}(t_0)(\mathbf{x} - \mathbf{x}_0)\|_2 - \|\mathbf{r}(t_0, \mathbf{x})\|_2 \\
&\geq c\|\mathbf{x} - \mathbf{x}_0\|_2 - \frac{c}{2}\|\mathbf{x} - \mathbf{x}\|_2 \\
&= \frac{c}{2}\|\mathbf{x} - \mathbf{x}_0\|_2 \\
&> 0 \text{ whenever } \mathbf{x} \neq \mathbf{x}_0
\end{aligned}$$

Hence, whenever $\mathbf{x} \in N$ and $\mathbf{x} \neq \mathbf{x}_0$, we have $\|\mathbf{f}(t_0, \mathbf{x})\|_2 > 0$, i.e., $\mathbf{f}(t_0, \mathbf{x}) \neq \mathbf{0}$. Thus N contains no equilibrium points at time t_0 of (1) other than $\mathbf{x}_0$, and therefore $\mathbf{x}_0$ is isolated. ∎

Problem 1.2. For the Volterra predator-prey equations

$$\dot{x}_1 = a_1 x_1 + b_1 x_1 x_2$$
$$\dot{x}_2 = a_2 x_2 + b_2 x_1 x_2$$

(a) Show that (0, 0) is an equilibrium point.

(b) Show that (0, 0) is an isolated equilibrium point if and only if both a_1 and a_2 are nonzero.

Problem 1.3. Consider the tunnel-diode circuit of Figure 1.1, where

$$i_d = v_d - 2v_d^2 + v_d^3 \triangleq f(v_d)$$

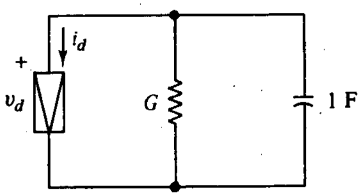

FIG. 1.1

(a) Show that the voltage v_d is governed by the equation

$$\frac{dv_d}{dt} = -Gv_d - f(v_d)$$

(b) Find all the equilibrium points of this system, when (i) $G = 0$, (ii) $G = 0.1$, (iii) $G = 1$.