FOUNDATIONS OF
MODERN ANALYSIS

Enlarged and Corrected Printing

J. DIEUDONNE

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PREFACE TO THE ENLARGED AND CORRECTED PRINTING

This book is the first volume of a treatise which will eventually consist of four volumes. It is also an enlarged and corrected printing, essentially without changes, of my "Foundations of Modern Analysis," published in 1960. Many readers, colleagues, and friends have urged me to write a sequel to that book, and in the end I became convinced that there was a place for a survey of modern analysis, somewhere between the "minimum tool kit" of an elementary nature which I had intended to write, and specialist monographs leading to the frontiers of research. My experience of teaching has also persuaded me that the mathematical apprentice, after taking the first step of "Foundations," needs further guidance and a kind of general bird's eye-view of his subject before he is launched onto the ocean of mathematical literature or set on the narrow path of his own topic of research.

Thus I have finally been led to attempt to write an equivalent, for the mathematicians of 1970, of what the "Cours d'Analyse" of Jordan, Picard, and Goursat were for mathematical students between 1880 and 1920.

It is manifestly out of the question to attempt encyclopedic coverage, and certainly superfluous to rewrite the works of N. Bourbaki. I have therefore been obliged to cut ruthlessly in order to keep within limits comparable to those of the classical treatises. I have opted for breadth rather than depth, in the opinion that it is better to show the reader rudiments of many branches of modern analysis rather than to provide him with a complete and detailed exposition of a small number of topics.

Experience seems to show that the student usually finds a new theory difficult to grasp at a first reading. He needs to return to it several times before he becomes really familiar with it and can distinguish for himself which are the essential ideas and which results are of minor importance, and only then will he be able to apply it intelligently. The chapters of this treatise are therefore samples rather than complete theories: indeed, I have systematically tried not to be exhaustive. The works quoted in the bibliography will always enable the reader to go deeper into any particular theory.

However, I have refused to distort the main ideas of analysis by presenting them in too specialized a form, and thereby obscuring their power and generality. It gives a false impression, for example, if differential geometry is restricted to two or three dimensions, or if integration is restricted to Lebesgue measure, on the pretext of making these subjects more accessible or "intuitive."

On the other hand I do not believe that the essential content of the ideas involved is lost, in a first study, by restricting attention to separable metrizable topological spaces. The mathematicians of my own generation were certainly right to banish hypotheses of countability wherever they were not needed: this was the only way to get a clear understanding. But now the situation is well understood: the most central parts of analysis (let us say those which turn on the notion of a finite-dimensional manifold) involve only separable metrizable spaces, in the great majority of important applications. Moreover, there exists a general technique, which is effective and usually easy to apply, for passing from a proof based on hypotheses of countability to a general proof. Broadly speaking, the recipe is to "replace sequences by filters." Often, it should be said, the result is simply to make the original proof more elegant. At the risk of being reviled as a reactionary I have therefore taken as my motto "only the countable exists at infinity": I believe that the beginner will do better to concentrate his attention on the real difficulties involved in concepts such as differential manifolds and integration, without having at the same time to worry about secondary topological problems which he will meet rather seldom in practice.†

In this text, the whole structure of analysis is built up from the foundations. The only things assumed at the outset are the rules of logic and the usual properties of the natural numbers, and with these two exceptions all the proofs in the text rest on the axioms and theorems proved earlier.‡ Nevertheless this treatise (including the first volume) is not suitable for students who have not yet covered the first two years of an undergraduate honours course in mathematics.

† In the same spirit I have abstained (sometimes at the cost of greater length) from the use of transfinite induction in separable metrizable spaces: not in the name of philosophical scruples which are no longer relevant, but because it seems to me to be unethical to ban the uncountable with one hand whilst letting it in surreptitiously with the other.

‡ This logical order is not followed so rigorously in the problems and in some of the examples, which contain definitions and results that have not up to that point appeared in the text, or will not appear at all.

A striking characteristic of the elementary parts of analysis is the small amount of algebra required. Effectively all that is needed is some elementary linear algebra (which is included in an appendix at the end of the first volume, for the reader's convenience). However, the role played by algebra increases in the subsequent volumes, and we shall finally leave the reader at the point where this role becomes preponderant, notably with the appearance of advanced commutative algebra and homological algebra. As reference books in algebra we have taken R. Godement's "Abstract Algebra,"§ and S. A. Lang's "Algebra"¶ which we shall possibly augment in certain directions by means of appendices.

As with the first volume, I have benefited greatly during the preparation of this work from access to numerous unpublished manuscripts of N. Bourbaki and his collaborators. To them alone is due any originality in the presentation of certain topics.

Nice, France
April, 1969

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§ Godement, R., "Abstract Algebra." Houghton-Mifflin, New York, 1968. (Original French edition published by Hermann, Paris, 1963.)

¶ Lang, S. A., "Algebra." Addison-Wesley, Reading, Massachusetts, 1965.

PREFACE

This volume is an outgrowth of a course intended for first year graduate students or exceptionally advanced undergraduates in their junior or senior year. The purpose of the course (taught at Northwestern University in 1956–1957) was twofold: (a) to provide the necessary elementary background for all branches of modern mathematics involving "analysis" (which in fact means everywhere, with the possible exception of logic and pure algebra); (b) to train the student in the use of the most fundamental mathematical tool of our time—the axiomatic method (with which he will have had very little contact, if any at all, during his undergraduate years).

It will be very apparent to the reader that we have everywhere emphasized the *conceptual* aspect of every notion, rather than its *computational* aspect, which was the main concern of classical analysis; this is true not only of the text, but also of most of the problems. We have included a rather large number of problems in order to supplement the text and to indicate further interesting developments. The problems will at the same time afford the student an opportunity of testing his grasp of the material presented.

Although this volume includes considerable material generally treated in more elementary courses (including what is usually called "advanced calculus") the point of view from which this material is considered is completely different from the treatment it usually receives in these courses. The fundamental concepts of function theory and of calculus have been presented within the framework of a theory which is sufficiently general to reveal the scope, the power, and the true nature of these concepts far better than it is possible under the usual restrictions of "classical analysis." It is not necessary to emphasize the well-known "economy of thought" which results from such a general treatment; but it may be pointed out that there is a corresponding "economy of notation," which does away with hordes of indices, much in the same way as "vector algebra" simplifies classical analytical geometry. This has also as a consequence the necessity of a strict adherence to axiomatic methods, with no appeal whatsoever to "geometric intuition," at least in the formal proofs: a necessity which we have emphasized by deliberately abstaining from introducing any diagram in the book. My opinion is that the graduate student of today must, as soon as possible, get a thorough training in this abstract and axiomatic way of thinking, if he is ever to understand what is currently going on in mathematical research. This volume aims to help the student to build up this "intuition of the abstract" which is so essential in the mind of a modern mathematician.

It is clear that students must have a good working knowledge of classical analysis before approaching this course. From the strictly logical point of view, however, the exposition is not based on *any* previous knowledge, with the exception of:

1. The first rules of mathematical logic, mathematical induction, and the fundamental properties of (positive and negative) integers.

2. Elementary linear algebra (over a field) for which the reader may consult Halmos [11], Jacobson [13], or Bourbaki [4]; these books, however, contain much more material than we will actually need (for instance we shall not use the theory of duality and the reader will know enough if he is familiar with the notions of vector subspace, hyperplane, direct sum, linear mapping, linear form, dimension, and codimension).

In the proof of each statement, we rely exclusively on the axioms and on theorems already proved in the text, with the two exceptions just mentioned. This rigorous sequence of logical steps is somewhat relaxed in the examples and problems, where we will often apply definitions or results which have not yet been (or ever will never be) proved in the text.

There is certainly room for a wide divergence of opinion as to what parts of analysis a student should learn during his first graduate year. Since we wanted to keep the contents of this book within the limits of what can materially be taught during a single academic year, some topics had to be eliminated. Certain topics were not included because they are too specialized, others because they may require more mathematical maturity than can usually be expected of a first-year graduate student or because the material has undoubtedly been covered in advanced calculus courses. If we were to propose a general program of graduate study for mathematicians we would recommend that *every* graduate student should be expected to be familiar with the contents of this book, whatever his future field of specialization may be.

I would like to express my gratitude to the mathematicians who have helped me in preparing these lectures, especially to H. Cartan and N. Bourbaki, who allowed me access to unpublished lecture notes and manuscripts, which greatly influenced the final form of this book. My best thanks also go to my colleagues in the Mathematics Department of Northwestern University, who made it possible for me to teach this course along the lines I had planned and greatly encouraged me with their constructive criticism.

April, 1960 J. DIEUDONNÉ

NOTATIONS

In the following definitions the first digit refers to the number of the chapter in which the notation occurs and the second to the section within the chapter.

$=$	equals: 1.1
$\neq$	is different from: 1.1
$\in$	is an element of, belongs to: 1.1
$\notin$	is not an element of: 1.1
$\subset$	is a subset of, is contained in: 1.1
$\supset$	contains: 1.1
$\not\subset$	is not contained in: 1.1
$\{x \in X \mid P(x)\}$	the set of elements of X having property P: 1.1
$\emptyset$	the empty set: 1.1
$\{a\}$	the set having *a* as unique element: 1.1
$\mathfrak{P}(X)$	the set of subsets of X: 1.1
$X - Y$, $\complement_X Y$, $\complement Y$	complement of Y in X: 1.2
$\cup$	union: 1.2
$\cap$	intersection: 1.2
(a,b)	ordered pair: 1.3
$pr_1 c$, $pr_2 c$	first and second projection: 1.3
$G(x)$, $G^{-1}(y)$	cross sections of $G \subset X \times Y$: 1.3
$X \times Y$	product of two sets: 1.3
$X_1 \times X_2 \times \ldots \times X_n$	product of *n* sets: 1.3
$pr_i z$	*i*th projection: 1.3
$pr_{i_1 i_2 \ldots i_k}(z)$	partial projection: 1.3
X^n	product of *n* sets equal to X: 1.3
$F(x)$	value of the mapping F at *x*: 1.4

Y^X, $\mathscr{F}(X,Y)$	set of mappings of X into Y: 1.4
1_X	identity mapping of X: 1.4
$x \to T(x)$	mapping: 1.4
$F(A)$	direct image: 1.5
$F^{-1}(A)$	inverse image: 1.5
$F^{-1}(y)$	inverse image of a one element set $\{y\}$: 1.5
$F(.\,,y)$, $F(x,.)$	partial mappings of a mapping F of $A \subset X \times Y$ into Z: 1.5
j_A	natural injection: 1.6
F^{-1}	inverse mapping of a bijective mapping: 1.6
$G \circ F$	composed mapping: 1.7
$(x_\lambda)_{\lambda \in L}$	family: 1.8
N	set of natural integers: 1.8
$\{x_1, \ldots, x_n\}$	set of elements of a finite sequence: 1.8
$\bigcup_{\lambda \in L} A_\lambda$, $\bigcup_{\lambda} A_\lambda$	union of a family of sets: 1.8
$\bigcap_{\lambda \in L} A_\lambda$, $\bigcap_{\lambda} A_\lambda$	intersection of a family of sets: 1.8
X/R	quotient set of a set X by an equivalence relation R: 1.8
$\prod_{\lambda \in L} X_\lambda$	product of a family of sets: 1.8
pr_J	projection on a partial product: 1.8
(u_λ)	mapping into a product of sets: 1.8
R	set of real numbers: 2.1
$x + y$	sum of real numbers: 2.1
xy	product of real numbers: 2.1
0	element of **R**: 2.1
$-x$	opposite of a real number: 2.1
1	element of **R**: 2.1
x^{-1}, $1/x$	inverse in **R**: 2.1
$x \leqslant y$, $y \geqslant x$	order relation in **R**: 2.1
$x < y$, $y > x$	relation in **R**: 2.1
$]a,b[$, $[a,b]$, $[a,b[$, $]a,b]$	intervals in **R**: 2.1
$\mathbf{R}_+$, $\mathbf{R}_+^*$	set of real numbers $\geqslant 0$ (resp. > 0): 2.2
$\lvert x \rvert$, x^+, x^-	absolute value, positive and negative part of a real number: 2.2
Q	set of rational numbers: 2.2
Z	set of positive or negative integers: 2.2
l.u.b. X, sup X	least upper bound of a set: 2.3
g.l.b. X, inf X	greatest lower bound of a set: 2.3
$\sup_{x \in A} f(x)$, $\inf_{x \in A} f(x)$	supremum and infimum of f in A: 2.3
$\overline{\mathbf{R}}$	extended real line: 3.3

$+\infty$, $-\infty$	points at infinity in $\overline{\mathbf{R}}$: 3.3
$x \leqslant y$, $y \geqslant x$	order relation in $\overline{\mathbf{R}}$: 3.3
$d(\mathrm{A},\mathrm{B})$	distance of two sets: 3.4
$\mathrm{B}(a;r)$, $\mathrm{B}'(a;r)$, $\mathrm{S}(a;r)$	open ball, closed ball, sphere of center a and radius r: 3.4
$\delta(\mathrm{A})$	diameter: 3.4
$\mathring{\mathrm{A}}$	interior: 3.7
$\bar{\mathrm{A}}$	closure: 3.8
$\mathrm{fr}(\mathrm{A})$	frontier: 3.8
$\lim\limits_{x \to a,\, x \in \mathrm{A}} f(x)$	limit of a function: 3.13
$\lim\limits_{n \to \infty} x_n$	limit of a sequence: 3.13
$\Omega(a;f)$	oscillation of a function: 3.14
$\log_a x$	logarithm of a real number: 4.3
a^x	exponential of base a (x real): 4.3
$\mathbf{C}$	set of complex numbers: 4.4
$z + z'$, zz'	sum, product of complex numbers: 4.4
0, 1, i	elements of $\mathbf{C}$: 4.4
$\mathscr{R}z$, $\mathscr{I}z$	real and imaginary parts: 4.4
$\bar{z}$	conjugate of a complex number: 4.4
$\lvert z \rvert$	absolute value of a complex number: 4.4
$x + y$, λx, x	sum and product by a scalar in a vector space: 5.1
0	element of a vector space: 5.1
$\lVert x \rVert$	norm: 5.1
$\sum\limits_{n=0}^{\infty} x_n$	sum of a series, series: 5.2
$\sum\limits_{\alpha \in \mathrm{A}} x_\alpha$	sum of an absolutely summable family: 5.3
(c_0)	space of sequences tending to 0: 5.3, prob. 5
$\mathscr{L}(\mathrm{E};\mathrm{F})$	space of linear continuous mappings: 5.7
$\lVert u \rVert$	norm of a linear continuous mapping: 5.7
$\mathscr{L}(\mathrm{E}_1, \ldots, \mathrm{E}_n; \mathrm{F})$	space of multilinear continuous mappings: 5.7
l^1	space of absolutely convergent series: 5.7, prob. 1
l^∞	space of bounded sequences: 5.7, prob. 1
$(x \mid y)$	scalar product: 6.2
P_{F}	orthogonal projection: 6.3
l^2, $l^2_{\mathbf{R}}$, $l^2_{\mathbf{C}}$	Hilbert spaces of sequences: 6.5
$\mathscr{B}_{\mathrm{F}}(\mathrm{A})$, $\mathscr{B}_{\mathbf{R}}(\mathrm{A})$, $\mathscr{B}_{\mathbf{C}}(\mathrm{A})$	spaces of bounded mappings: 7.1
$\mathscr{C}_{\mathrm{F}}(\mathrm{E})$	space of continuous mappings: 7.2
$\mathscr{C}^\infty_{\mathrm{F}}(\mathrm{E})$	space of bounded continuous mappings: 7.2
$f(x+)$, $f(x-)$	limits to the right, to the left: 7.6
$f'(x_0)$, $\mathrm{D}f(x_0)$	(total) derivative at x_0: 8.1

f', $\mathrm{D}f$	derivative (as a function): 8.1
$f'(\alpha)$, $\mathrm{D}_+f(\alpha)$	derivative on the right: 8.4
$f'_g(\beta)$, $\mathrm{D}_-f(\beta)$	derivative on the left: 8.4
$\int_\alpha^\beta f(\xi)\,d\xi$	integral: 8.7
e, $\exp(x)$, $\log x$	(x real): 8.8
$\mathrm{D}_1 f(a_1, a_2)$, $\mathrm{D}_2 f(a_1, a_2)$	partial derivatives: 8.9
$f'_{\xi_i}(\xi_1, \ldots, \xi_n)$, $\frac{\partial}{\partial \xi_i} f(\xi_1, \ldots, \xi_n)$	partial derivatives: 8.10
$\frac{\mathrm{D}(f_1, \ldots, f_n)}{\mathrm{D}(\xi_1, \ldots, \xi_n)}$, $\frac{\partial(f_1, \ldots, f_n)}{\partial(\xi_1, \ldots, \xi_n)}$	jacobian: 8.10
$f''(x_0)$, $\mathrm{D}^2 f(x_0)$, $f^{(p)}(x_0)$, $\mathrm{D}^p f(x_0)$	higher derivatives: 8.12
$f * \rho$	regularization: 8.12, prob. 2
$\mathscr{E}_{\mathrm{F}}^{(p)}(\mathrm{A})$	space of p times continuously differentiable **mappings**: 8.13.
$\|\alpha\|$, M_α, D^α, $\mathrm{D}_{\mathrm{M}_\alpha}$	(α composite index): 8.13
e^z, $\exp(z)$	(z complex): 9.5
$\sin z$, $\cos z$	sine and cosine: 9.5
π	9.5
$\log z$, $\mathrm{am}(z)$, $\binom{t}{n}$, $(1+z)^t$	(z, t complex numbers): 9.5, prob. 8
γ^0	opposite path: 9.6
$\gamma_1 \vee \gamma_2$	juxtaposition of paths: 9.6
$\int_\gamma f(z)\,dz$	integral along a road: 9.6
$j(a; \gamma)$	index with respect to a circuit: 9.8
$\mathrm{E}(z, p)$	primary factor: 9.12, prob. 1
$\Gamma(z)$	gamma function: 9.12, prob. 2
γ	Euler's constant: 9.12, prob. 2
$\int_\gamma f(z)\,dz$	integral along an endless road: 9.12, prob. 3
$\omega(a; f)$, $\omega(a)$	order of a function at a point: 9.15
$\mathscr{L}(\mathrm{E})$	algebra of operators: 11.1
uv	composed operator: 11.1
$\mathrm{sp}(u)$	spectrum: 11.1
$\mathrm{E}(\zeta)$, $\mathrm{E}(\zeta; u)$	eigenspace: 11.1
$\tilde{u}$	continuous extension: 11.2
$\mathrm{N}(\lambda)$, $\mathrm{N}(\lambda; u)$, $\mathrm{F}(\lambda)$, $\mathrm{F}(\lambda; u)$	subspaces attached to an eigenvalue of a compact operator: 11.4
$k(\lambda)$, $k(\lambda; u)$	order of an eigenvalue: 11.4
u^*	adjoint operator: 11.5

CONTENTS

CHAPTER I

ELEMENTS OF THE THEORY OF SETS

We do not try in this chapter to put set theory on an axiomatic basis; this can however be done, and we refer the interested reader to Kelley [15] and Bourbaki [3] for a complete axiomatic description. Statements appearing in this chapter and which are not accompanied by a proof or a definition may be considered as axioms connecting undefined terms.

The chapter starts with some elementary definitions and formulas about sets, subsets and product sets (Sections 1.1 to 1.3); the bulk of the chapter is devoted to the fundamental notion of *mapping*, which is the modern extension of the classical concept of a (numerical) *function* of one or several numerical "variables" Two points related to this concept deserve some comment:

1. The all-important (and characteristic) property of a mapping is that it associates to any "value" of the variable a *single* element; in other words, there is no such thing as a "multiple-valued" function, despite many books to the contrary. It is of course perfectly legitimate to define a mapping whose values are *subsets* of a given set, which may have more than one element; but such definitions are in practice useless (at least in elementary analysis), because it is impossible to define in a sensible way *algebraic operations* on the "values" of such functions. We return to this question in Chapter IX.

2. The student should as soon as possible become familiar with the idea that a function f is a single object, which may itself "vary" and is in general to be thought of as a "point" in a large "functional space"; indeed, it may be said that one of the main differences between the classical and the modern concepts of analysis is that, in classical mathematics, when one writes $f(x)$, f is visualized as "fixed" and x as "variable," whereas nowadays *both* f

and x are considered as "variables" (and sometimes it is x which is fixed, and f which becomes the "varying" object).

Section 1.9 gives the most elementary properties of denumerable sets; this is the beginning of the vast theory of "cardinal numbers" developed by Cantor and his followers, and for which the interested reader may consult Bourbaki ([3], Chapter III) or (for more details) Bachmann [2]. It turns out, however, that, with the exception of the negative result that the real numbers do not form a denumerable set (see (2.2.17)), one very seldom needs more than these elementary properties in the applications of set theory to analysis.

1. ELEMENTS AND SETS

We are dealing with objects, some of which are called *sets*. Objects are susceptible of having *properties*, or *relations* with one another. Objects are denoted by *symbols* (chiefly letters), properties or relations by combinations of the symbols of the objects which are involved in them, and of some other symbols, characteristic of the property or relation under consideration. The relation $x = y$ means that the objects denoted by the symbols x and y are the same; its negation is written $x \neq y$.

If X is a set, the relation $x \in X$ means that x is an *element* of the set X, or *belongs* to X; the negation of that relation is written $x \notin X$.

If X and Y are two sets, the relation $X \subset Y$ means that every element of X is an element of Y (in other words, it is equivalent to the relation $(\forall x)(x \in X \Rightarrow x \in Y)$); we have $X \subset X$, and the relation ($X \subset Y$ and $Y \subset Z$) implies $X \subset Z$. If $X \subset Y$ and $Y \subset X$, then $X = Y$, in other words, two sets are equal if and only if they have the same elements. If $X \subset Y$, one says that X is *contained* in Y, or that Y *contains* X, or that X is a *subset* of Y; one also writes $Y \supset X$. The negation of $X \subset Y$ is written $X \not\subset Y$.

Given a set X, and a property P, there is a unique subset of X whose elements are all elements $x \in X$ for which $P(x)$ is true; that subset is written $\{x \in X \mid P(x)\}$. The relation $\{x \in X \mid P(x)\} \subset \{x \in X \mid Q(x)\}$ is equivalent to $(\forall x \in X)(P(x) \Rightarrow Q(x))$; the relation $\{x \in X \mid P(x)\} = \{x \in X \mid Q(x)\}$ is equivalent to $(\forall x \in X)(P(x) \Leftrightarrow Q(x))$. We have, for instance, $X = \{x \in X \mid x = x\}$ and $X = \{x \in X \mid x \in X\}$. The set $\varnothing_X = \{x \in X \mid x \neq x\}$ is called the *empty set* of X; it contains no element. If P is any property, the relation $x \in \varnothing_X \Rightarrow P(x)$ is true for every x, since the negation of $x \in \varnothing_X$ is true for every x (remember that $Q \Rightarrow P$ means "not Q or P"). Therefore, if X and Y are sets, $x \in \varnothing_X$ implies $x \in \varnothing_Y$, in other words $\varnothing_X \subset \varnothing_Y$, and similarly $\varnothing_Y \subset \varnothing_X$, hence $\varnothing_X = \varnothing_Y$, all empty sets are equal, hence noted $\varnothing$.

If a is an object, the set having a as unique element is written $\{a\}$.

If X is a set, there is a (unique) set the elements of which are all subsets of X; it written $\mathfrak{P}(X)$. We have $\varnothing \in \mathfrak{P}(X)$, $X \in \mathfrak{P}(X)$; the relations $x \in X$, $\{x\} \in \mathfrak{P}(X)$ are equivalent; the relations $Y \subset X$, $Y \in \mathfrak{P}(X)$ are equivalent.

PROBLEM

Show that the set of all subsets of a finite set having n elements ($n \geqslant 0$) is a finite set having 2^n elements.

2. BOOLEAN ALGEBRA

If X, Y are two sets such that $Y \subset X$, the set $\{x \in X \mid x \notin Y)$ is a subset of X called the *difference* of X and Y or the *complement* of Y with respect to X, and written $X - Y$ or $\complement_X Y$ (or $\complement Y$ when there is no possible confusion).

Given two sets X, Y, there is a set whose elements are those which belong to both X and Y, namely $\{x \in X \mid x \in Y\}$; it is called the *intersection* of X and Y and written $X \cap Y$. There is also a set whose elements are those which belong to one at least of the two sets X, Y; it is called the *union* of X and Y and written $X \cup Y$.

The following propositions follow at once from the definitions:

(1.2.1) $X - X = \varnothing, \qquad X - \varnothing = X.$

(1.2.2) $X \cup X = X, \qquad X \cap X = X.$

(1.2.3) $X \cup Y = Y \cup X, \qquad X \cap Y = Y \cap X.$

(1.2.4) The relations $X \subset Y$, $X \cup Y = Y$, $X \cap Y = X$ are equivalent.

(1.2.5) $X \subset X \cup Y, \qquad X \cap Y \subset X.$

(1.2.6) The relation "$X \subset Z$ and $Y \subset Z$" is equivalent to $X \cup Y \subset Z$; the relation "$Z \subset X$ and $Z \subset Y$" is equivalent to $Z \subset X \cap Y$.

(1.2.7) $X \cup (Y \cup Z) = (X \cup Y) \cup Z$, written $X \cup Y \cup Z$.

$X \cap (Y \cap Z) = (X \cap Y) \cap Z$, written $X \cap Y \cap Z$.

(1.2.8) $X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$

$X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$ (distributivity).