LASER HANDBOOK

VOLUME 1

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Preface

The laser field is fascinating in many ways. Much of the fascination, no doubt, stems from the rapid pace of progress that this field has witnessed ever since its beginnings. Many factors have aided the rapid growth. There was – and still is – the prospect that one may make significant scientific and technical contributions. There was – and still is – the realization that laser work offers favorable returns on comparatively modest investments. There was – and still is – an appreciable number of highly competent research groups whose combined efforts have brought about progress at an astounding rate. Last but not least, much of the enthusiasm for laser work may have to do with the fact that this was – and still is – a young field.

Indeed, the laser field is young on almost any scale. It is young compared to the two or three productive decades a scientist may hope to experience. It is young compared to other fields of similar scientific and technological impact, such as that of semiconducting silicon and germanium. Even if one considers the usual development cycle of several years that is needed to transfer novel phenomena from the laboratory environment to routine industrial engineering, the laser field is young and hence promises ample opportunity for further growth.

Historically, the laser field came into being through an extension of stimulated amplification techniques from the microwave to the optical region of the electromagnetic spectrum. Stimulated emission of radiation as a way to obtain amplification and oscillation at microwave frequencies had independently been proposed by Townes (1951) [1], Weber (1953) [2], and Basov and Prokhorov (1954) [3]. Bloembergen (1956) [4] proposed a successful and versatile maser pumping scheme. The demonstration of *m*icrowave *a*mplification by *s*timulated *e*mission of *r*adiation, or *maser* action for short, by Gordon, Zeiger and Townes (1954) [5], and Scovil, Feher and Seidel (1957) [6], and subsequent research and development efforts have verified the potential of this principle towards the realization of highly monochromatic oscillators and amplifiers with very low noise. The first proposal to extend the maser principle to optical frequencies was made in 1958 by Schawlow and Townes [7]. Their paper assembled and extrapolated knowledge from the fields of microwave masers and optical spectroscopy. It suggested that open resonators such as the Fabry–Perot interferometer would selectively provide a few radiation modes with a high Q, leaving the enormous number of remaining modes at low Q. It then went on to predict

* Numbers in square brackets correspond to references given on page xii.

the properties of laser light such as coherence, directionality, linewidth, and noise.

Laser [8] action was first observed by Maiman [9] in 1960. It was obtained from ruby at room temperature, optically pumped by a flashlamp similar to those used in photography. This discovery triggered an avalanche of laser work in laboratories everywhere and soon led to the demonstration of laser action in other systems, in other solids, in gases, in semiconductors, and in liquids.

We do not intend at this point to continue retelling the history of subsequent laser and related research. Some of it will briefly be mentioned in chapters of this Handbook dealing with one or the other specialty of the laser field. Here it may suffice to point out that, through the concerted effort of numerous research groups, the laser field has reached a remarkable degree of maturity despite its youth. While this statement may not apply to the entire field, it does apply to a number of important specialties which were selected fairly early and vigorously pursued in the ensuing years.

A few examples may be mentioned to illustrate the state of affairs.

(*i*) In solid-state ionic lasers, all potentially important laser-active ions are probably already known since some time. The present performance in terms of c.w. power, pulse peak power, and pulse energy density comes close to fundamental limits imposed by the destruction of laser materials and optical components so that an appreciable further increase seems out of the question. It is conceded, however, that one may find further suitable laser host materials and that continuing engineering advances may be expected to result in easier handling and maintenance of these devices.

(*ii*) In gas lasers, it is likely that all relevant atomic excitation mechanisms have been identified. On the other hand, the number of gases, metal vapors, and gas mixtures capable of laser operation has steadily increased over the years and is likely to continue to do so. The same is true for the number of laser transitions. Today's performance in c.w. or pulse output power, minimum linewidth, and frequency stability has reached impressive levels. While theoretical reasoning indicates that considerable improvement should be possible, it is reasonable to expect that any order-of-magnitude improvement in one of these parameters will require increasingly greater technological efforts. On the other hand, recent advances in gas lasers through novel means of excitation, such as transverse discharges, gas dynamics, or chemical reaction show there is still room for alternative types of technology.

(*iii*) In semiconductor lasers, the progress of the last few years has been characterized by greatly increased sophistication in the junction structure of GaAs devices. Barring an unforeseen breakthrough, progress will continue to require considerable technological investments for a slow advance in the achievable performance. On the other hand, the potential of materials other than GaAs is largely unexplored.

(*iv*) Liquid lasers, by comparison, are younger and less developed. Much progress may be anticipated. However, it is hard to imagine that they could outperform solid-state lasers in some respect by a large margin, with the exception of tunability.

(*v*) Laser theory, within the boundaries of certain approximations, has reached a high degree of perfection. Laser models of various degrees of complexity have been analyzed to various degrees of sophistication. Most significant perhaps, those aspects of laser light which reflect the quantum nature of radiation, such as fluctuations and photon statistics, are adequately described by theory. This leads one to suspect that probably the most important work in laser theory has already been done.

Among the approximations alluded to, one implies that the laser active atoms are statistically independent in some sense, permitting the density matrix to be factored. Another allows the so-called adiabatic elimination of atomic variables. It is assumed that the photon lifetime, i.e. the time constant of the light field in the laser cavity, is much longer than the response time of atomic polarization. Hence the atomic polarization is always in equilibrium with the instantaneous field. Both approximations lead to a considerably simplified description of laser dynamics. The underlying physical assumptions appear to be well justified in existing lasers. It is not clear, however, whether or not there may be lasers for which one or the other approximation is not valid. For those lasers, an alternative type of theory would have to be developed.

As one looks for natural extensions of laser theory, these will probably apply to systems other than lasers. One views laser theory as the general solution to a specific nonlinear problem in quantum-statistical mechanics. However one may hope that similar techniques can be developed for more complicated multimode cooperative quantum systems capable of showing instabilities, for example in the form of second order phase transitions. This aspect [10] will briefly be discussed in chapter A2.

(*vi*) In nonlinear optics, where lasers are used as primary light sources, much of the groundwork has been done. The symmetry properties of appropriate material parameters are understood, as are the limitations which atomic structure imposes on the magnitude of the parameters. However, many material parameters remain to be measured. Thus the exploratory research work approaches completion while much remains to be done in device-oriented work.

(*vii*) In scientific laboratories, notably although not exclusively those concerned with solid-state physics research, lasers have been used in experiments of increasing complexity and sophistication. Many highly successful experimental techniques have been devised and much valuable knowledge has been gained. Certainly this trend will continue or likely even accelerate.

(*viii*) In industrial applications, laser usage has been slower to gain acceptance than had been anticipated by many. To date the most popular laser applications are rather mundane ones where the radiation is just a convenient raw form of energy for drilling and welding, where a scale of wavelengths serves to measure distances, or where the beam defines a reference direction in construction work. It seems safe to assume, however, that other more exacting laser applications will gain wider acceptance in due time.

To repeat, we view the laser field as one which has become reasonably mature despite its youth. The maturity is most evident in the well researched fundamental processes and basic techniques. The youthfulness is most clearly visible in the fields of scientific and technical laser applications.

In this situation we thought it timely to collect the present knowledge in the laser field in terms of an encyclopedic review. Such a book can offer a review of lasting value in the more mature areas of laser research, and also provide a useful contemporary survey of the more youthful areas of laser applications. In early discussions between the editors, some of their friends, and the publisher, it became apparent that the Laser Handbook, as we came to call it, would fill a need that was felt by many. There is need for a book that presents the available knowledge and experience in the more established aspects of the laser field where little substantial change may be anticipated. With respect to those areas, the Handbook should offer self-contained treatments of these various subjects. Tutorially written to make for easy reading, they should nevertheless lead close to the frontiers of research. As a rule it should be possible to study original research papers after consultation of the appropriate Handbook chapter. With regard to the more volatile areas of laser research and applications, the purpose of the Handbook would be somewhat different. There it should concentrate on the established fundamentals. To a lesser extent it should review the achieved results or point out the direction of likely future developments. While Handbook chapters in this category would tend to have less permanent value, they should nevertheless offer a convenient entry into a particular specialty, its problems, its terminology, and literature.

These aims distinguish the Laser Handbook from the many introductory text books which were published during the past few years [11]. However, we feel it advisable that the laser novice first become acquainted with the field through one of these texts before he can use the Laser Handbook to full advantage. Likewise, the Laser Handbook cannot and does not intend to take the place of the exhaustive specialized monograph.

In our early discussions it became apparent that the Laser Handbook will find its foremost usage within scientific and technical teams. We mention a few typical situations.

(*i*) There are industrial development groups who work towards perfection of advanced manufacturing processes. To them the laser is just a tool, albeit a sophisticated and versatile one. The Handbook offers to these workers an adequate background without recourse to the original literature. The material presented may suggest to them more effective alternatives or ramifications of their original approach.

(*ii*) There are scientific groups who study a range of phenomena for which the laser, again, is merely a diagnostic tool or a means of exciting other interesting processes. Here, too, the background information provided by the Handbook may prove invaluable in designing modified experiments, for example to achieve greater accuracy, speed, sensitivity, or for work in another region of the spectrum.

(*iii*) There are other research groups whose activity is centered around one of the special branches of the laser field itself. To such a group, the Handbook will give a solid introduction to its own specialty, an opportunity probably most appreciated by new members of the group. On the other hand, active work in one of these special fields clearly requires intimate familiarity with current research publications, so that the knowledge provided just by a Handbook article would not be adequate for the research specialist. The function of the Handbook to such a group will more likely be to provide a good background of other, perhaps related specialized laser topics. This background, difficult as it were to assemble from the primary literature, may stimulate progress in directions which were not obvious from the outset.

(*iv*) There are laser theory groups whose primary function is theoretical analysis and perhaps numerical computation of laser phenomena and related topics of quantum-statistical mechanics. To such a group the Handbook will present descriptions of a range of phenomena which otherwise might escape the attention of the theorist, and it would acquaint the group with problems of concern to the applied scientist or device engineer. There is hope that this confrontation may lead to projects of mutual interest to theorists and engineers alike and, failing that, it would at least help to counteract an ivory tower mentality. For the theoretical novice, the brief chapters on laser theory in the Handbook may also prove a convenient point of entry to the field.

In our early discussions on the scope of the Handbook, it also became clear that a significant number of contributing authors were needed. In fact, the number envisagd grew somewhat during the incubation time of the project. We now count a total of 40 chapters which were written by 53 authors. We consider ourselves fortunate that our solicitation was met with pleasant cooperation by most of those we invited. We did not attempt to make the list of authors as international as possible nor representative in any sense. In fact our list of authors lacks members of several very productive laser laboratories and nations. Nevertheless we are happy to count among the authors of Handbook chapters members of many important research organizations almost all over the world. Needless to say, the cooperation on this project has strengthened old friendships and established new ones.

Let us give just one example to justify the large number of chapters we feel are necessary for an adequate presentation of the field. Take the case of solid-state ionic lasers. A treatment of this subject naturally breaks down into several more restricted subject areas. Laser theory (chapters A2 and A3) is clearly involved, as is the theory of optical resonators (chapter A4) although both not in a very specific way. An important topic is the solid-state spectroscopy of transition and rare-earth metal ions which make up the active laser materials (chapter B2). Beyond the choice of an active material, laser design involves a suitable pumping arrangement with cooling and other provisions (chapter C1). Many special techniques were developed for solid-state lasers, such as *Q*-switching and mode-locking (chapter C2) as well as high-power pulse amplification (chapter C3). Finally solid-state ionic lasers find use in too many applications to mention the respective chapters at this point. It is clear that few if any authors are experts in all of these various specialized topics at the same time. We therefore decided to have these topics treated separately by authors (or teams of authors as the case may be) who are among the leading experts in these respective specialties.

For convenience the Handbook is produced in two volumes. The first one is primarily concerned with lasers by themselves. Section A is devoted to theory, including that of coherence, of laser operation and fluctuations, of optical resonators, and of photon-counting statistics, the latter topic with a discussion of experimental techniques and results. Section B describes the principles of the various laser types, namely gas lasers, solid-state ionic lasers, dye lasers, and semiconductor lasers. Section C deals with a number of important laboratory techniques which are needed for practical work with lasers. This includes optical pumping arrangements, Q-switching and mode-locking, and high-power amplification as used with solid-state lasers; gas-laser design and techniques for manipulating the output characteristics; modulation of laser beams and photodetection; tunable parametric oscillators; and finally optical thin-film techniques as used in the design of laser circuit components. Section D discusses solid-state materials used in the study of nonlinear interactions of light with light, or with electric, magnetic, and sonic fields.

The second volume is devoted to laser applications in a broad sense of the word. Section E assembles a number of fruitful applications which lasers have found in the physics laboratory, most of them in solid-state research. This includes accounts of spontaneous and stimulated scattering processes such as Raman, Brillouin, and Rayleigh; optical second-harmonic generation and two-photon spectroscopy; self-focusing in Kerr-liquids and similar effects where laser beams modify their propagation characteristics through induced changes in the optical medium; coherent resonant propagation effects such as the so-called π- and 2π-pulses; use of lasers to generate and study high-temperature plasmas. Section F similarly brings together a number of technical laser applications which have shown practical importance or at least promise to do so. This includes holography, pattern recognition, optical information processing and optical implementations of computer logic and memory; the measurement of length or distance in high-precision metrology and optical radar range-finding and guidance; laser machining techniques such as drilling, welding, and trimming; high-speed photography; communications; and finally laser applications in biology and medicine. A subject index concludes this volume.

This outline and the general considerations above indicate that the main emphasis in the Handbook is on the presently available devices and techniques in the visible and near-infrared range of the spectrum. We feel that the choice of subjects was appropriate and in some sense even optimum at the time the Handbook project was started. Unfortunately, owing to the long lead time required in book production, it was not possible to include some more recent developments. There have been outstanding achievements in the field of tunable visible and infrared sources, and in the field of very powerful continuous or pulsed infrared lasers (transversely excited CO_2-laser, gasdynamic laser). A similar accomplishment is the demonstration of lasers for very short ultraviolet wavelengths. Laser control of chemical reactions [12] has become a research area of considerable significance and it is quite possible that it will develop into a standard chemical laboratory technique. We regret that the Handbook does not deal with these topics.

It is even less feasible, of course, to write about subjects where no work is in progress as yet although one may expect this to be true some time in the future. We feel that the current thrust towards short wavelengths which already has produced u.v. and vacuum u.v. lasers, will in due time also lead to lasers in the nanometer (x-ray) wavelength range – let us call them 'nanomasers' for short. If and when this breakthrough comes, it will trigger related work in systems and components. For example, light guides, focusers, resonators, filters, modulators, beam deflectors, detectors would all have to be developed for this novel spectral range and according to unconventional design criteria. Furthermore, the domain of potential applications would increase appreciably. Just remember that it took eight years to identify the molecular structure of vitamin B_{12} [13] from x-ray diffraction data (where phase information is lost in the recording process). Now imagine one may record a hologram (where phase information is faithfully recorded) with coherent nanometer radiation and reconstruct the image with visible light. This would make the molecular structure directly visible and hence reduce the labor of the analyst by several orders of magnitude.

These last considerations make it desirable that the contents of the present Handbook from time to time be complemented by publishing up-to-date appendices. These presumably can be short since the pertinent tutorial background should already be contained in both original volumes, and hence they may be published with less delay. The editors [14] invite proposals from the readers on topics which should be treated in this fashion. In addition, they welcome any comments the readers may have concerning the present two volumes.

Finally, it is a pleasure to thank all those who helped in the production of this Handbook. We are grateful to many friends and colleagues who freely gave their advice although we regret that we cannot mention them individually. The Subject Index was kindly prepared by Dr. A. Vendramini. The IBM Zurich Laboratory generously supported this project by granting secretarial and other assistance as required. We are indebted to the publishing company for the excellent cooperation throughout the production period, in particular to Dr. W.H. Wimmers, managing editor, and Dr. J.C. van Eijbergen, desk editor. The project of a 'Laser Handbook' was first proposed by Prof. H. Haken in discussions with Dr. W.H. Wimmers.

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A1 Photon Statistics

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Abstract

This chapter deals with the quantum theory of electromagnetic fields, in particular with the theory of coherence and photon correlations. It provides the required analytical tools such as coherent states and the P-representation.

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1 Introduction

Statistical experiments on photon fields have only recently become a practical reality. To give some examples of what we mean by photon statistics let us think of some simple experiments which can be performed with a light source and a photon counter. We can imagine a shutter of some sort, either mechanical or electrical, to be placed between the source and the counter. We may open the shutter for a specified time and then close it again, observing how many photons are recorded by the counter in the interval. That number is clearly a random variable. Repetitions of the experiment will lead to a statistical distribution for the number of photons counted. A satisfactory theory should furnish us with a prediction of that distribution, its moments and other properties.

There are other kinds of experiments which can be performed with essentially the same apparatus. If the shutter is opened during two different time intervals, for example, we may ask for the joint distribution, or the degree of correlation of the random numbers of photons counted during the pairs of intervals. If we have a second photon counter available we may begin to inquire about spatial as well as temporal correlations. If the two counters are arranged to register in delayed coincidence with one another then we may measure the combined space–time correlations of their counting rates. The prediction of all of these statistical quantities should be the task of a well-formulated theory. But more important perhaps, the theory should be capable of dealing with a much broader variety of light sources than have been familiar in the past.

The light source in the experiment need not be of the usual gas discharge or incandescent type. It might be any of numerous varieties of laser oscillators or it might be one of the new types of amplifiers operating at optical frequencies. We might alternatively have a compound source which emits scattered light, the light from a laser, say, scattered by the density fluctuations of a fluid. All of these sources can emit fields which are rather different in character from one another. The best way of making these differences evident is through experiments of the kind we have noted, that is to say experiments which reveal a good deal more about the statistical properties of a field than the usual static measurements of its intensity and its spectrum.

It is only within roughly the last ten years that measurements of the sort we are discussing have actually been carried out. The first such experiment, which was performed by Hanbury Brown and Twiss (1956), used a gas discharge tube as a source and revealed a distinct tendency for two photon detectors, placed in equivalent positions in the field, to register photons simultaneously. Many of the early experiments on photon statistics were in effect repetitions of this experiment with one or another improvement of method. More recently, as the technique of making statistical measurements upon fields has become refined and as the new coherent light sources have been developed, we have seen an increase in the number of such measurements and in their variety as well.

Since the applications of such experiments are likely to increase further it is worth developing a systematic way of discussing them. Let us begin therefore by noting that there are two rather different kinds of statistical uncertainty which are probed by each of the types of experiments we have mentioned. One kind of uncertainty concerns the state of the radiation source. The source is a macroscopic system which can rarely if ever be prepared in a pure quantum state. The best we can hope to do in general is to describe its uncertain state, or the uncertain state of the field to which it leads, by the methods of statistical mechanics and in that way to predict the associated fluctuations.

The second source of statistical uncertainty is inherent in the photodetection process. The photon counter is an intrinsically quantum mechanical instrument. Even if an incident field is fully predetermined in its behavior (a condition which can only be approached in the classical or strong field limit), the response of the photon counter to the field is never fully predictable. The random integers it records are numbers which are meaningless in the context of classical electromagnetic theory. The only language which can be used with logical consistency in discussing counting experiments is the language of quantum mechanics (Glauber 1963a, b, c, 1965, 1966). It will be evident nontheless that many features of the quantum mechanical discussion exhibit simple correspondences with classical theory. In a sense what we propose to do is in fact to extend the classical theory of noise in electromagnetic fields down into the quantum domain and in so doing to take account also of the noise which is an unavoidable part of the detection process. It may provide some useful background therefore if we begin by saying a few words about classical noise theory.

2 *Classical theory*

In order to be able to describe the electromagnetic field in terms of a discrete set of variables let us consider the field inside a volume of finite dimensions. A set of homogeneous boundary conditions which we need not specify for present purposes is assumed to constrain the fields at the surface; they could be periodic boundary conditions for example, or the conditions appropriate to reflecting (i.e. perfectly conducting) walls. For an appropriate set of frequencies ω_k we can find a sequence of vector mode functions $\boldsymbol{u}_k(\boldsymbol{r})$ which satisfy the Helmholtz equation

$$(\nabla^2 + \omega_k^2/c^2)\,\boldsymbol{u}_k(\boldsymbol{r}) = 0, \tag{1}$$