Modelling with Ordinary Differential Equations

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CRC

CRC Press
Boca Raton Ann Arbor London Tokyo

Library of Congress Cataloging-in-Publication Data

Dreyer, T. P.
Modelling with ordinary differential equations / T. P. Dreyer.
p. cm.
Includes bibliographical references and index.
ISBN 0-8493-8636-5
1. Mathematical models. 2. Differential equations. I. Title.
QA401.D74 1993
511'.—dc20 93-9636
CIP

Direct all inquiries to CRC Press, Inc., 2000 Corporate Blvd., N.W., Boca Raton, Florida 33431.

International Standard Book Number 0-8493-8636-5

Library of Congress card number 93-9636

Printed in the United States 1 2 3 4 5 6 7 8 9 0
Printed on acid-free paper

PREFACE

This is a book about mathematical modelling. The aim of the book is threefold:

1. To teach the basic skills of modelling
2. To introduce a basic tool for modelling, namely, ordinary differential equations
3. To show the wide scope of applications which can be modelled with ordinary differential equations

The book is not a reference manual and does not pretend to cover completely either one of the fields of modelling or ordinary differential equations. It is primarily meant as an elementary text to be used in a lecture room. It can, however, also be read as a self-study program since each model is developed from first principles, examples are provided in the text to clarify new definitions, and carefully chosen exercises are given in each section to strengthen the understanding of the material and the skill to construct models. Supplementary references are also given at the end of each section to stimulate further reading.

The only prerequisite for this book is a fair background in differential and integral calculus of one variable (including Taylor series) and the basic notions of a partial derivative and a vector in two or three dimensions.

The first version of this book in the form of classroom notes was written in 1972. As the material was tested in the classroom over two decades, some material was discarded, new models were added, and the selection of exercises and projects was refined. This book is the end result of a marriage between the academic goals set by the teacher and the feedback through questionnaires and examinations of the students. The favorable comments of ex-students working as applied mathematicians in industry indicate that the choice of material and the level of sophistication in the book are satisfactory.

Some mathematicians think that modelling is a rather obvious exercise, and hence a textbook should concentrate on mathematical theory and leave the rest to common sense. Anyone who has done some mathematical work for industry knows that this viewpoint is false and that the mathematical solution of some mathematical equations is only a part (many times the easiest part) of the modelling process. In this book, the theory of differential equations and the modelling process are interwoven purposefully, not only to convey the importance of both, but also to highlight the essential interaction between them.

There are a number of interesting models in the subsequent pages, and it is hoped that the reader shall enjoy reading about them, but to learn something about modelling the exercises must be done. It is nice to watch a good game of tennis, but as long as you are sitting in the stand, your game will not inprove! You can check your answers at the back of the book.

Apart from the exercises, there is a section on projects at the end of each chapter. These are mainly intended for the computer-literate reader, and can also be used in tutorials.

The proofs of theorems appear in a section at the end of each chapter to facilitate the flow of the argument in the model-building process. It does not mean that the proofs are unimportant or that they should be left out.

The book consists of seven chapters. In the first chapter some basic notions are introduced. The second chapter is a diverse collection of real-life situations where a first order differential equation pops up in the mathematical modelling. Each problem was chosen with a specific purpose in mind. In general it is shown that there is much more to these mathematical models than merely solving differential equations; on the contrary, in some cases it is not even necessary to solve the differential equation.

In the third chapter some elementary numerical methods are discussed, with the accent on ways to develop more accurate methods rather than presenting fairly sophisticated numerical methods. In a real-life situation the applied mathematician will probably use an efficient standard numerical program which is part of the software of his computer.

Laplace transforms are introduced in the fourth chapter as a very necessary tool for solving systems of linear differential equations. In the next chapter we look at mathematical models in which systems of differential equations appear. Again each problem is chosen to highlight different aspects of mathematical models and/or the theory of differential equations.

In the sixth chapter the standard applications of second order differential equations in mechanical vibrations and electrical networks are discussed, but it also includes a mathematical model on the ignition of a car which teaches us to think before we spend a lot of time solving differential equations. In the final chapter two situations are modelled, namely, the pendulum and competing species, to illustrate qualitative solutions.

Through the years many people have contibuted to the creation of this book. The discussions with my colleagues were very helpful, in particular the ideas of Gerhard Geldenhuys and Philip Fourie. The suggestion to write this book came from Alan Jeffrey of the University of Newcastle on Tyne. His interest and comments are deeply appreciated. The encouragement of Navin Sullivan in London carried the project through. Finally, without the help of Hester Uys and Jan van Vuuren on the computer, and Christelle Goldie with the sketches, there would not be a book at all.

Although this book may look like a jumble of disconnected applications, be warned - there is method in the madness, and you may learn more than you thought you would. So start right away and enjoy the book.

TP Dreyer

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Introduction

1.1 Mathematical Modelling

Although mathematics had already been applied to real-life problems by the Egyptians and other ancient civilizations, the term "mathematical model" is a fairly recent addition to the mathematical vocabulary. The term signifies an attempt to describe the interplay between the physical world on the one hand and abstract mathematics on the other hand. It is customary to refer to a collection of equations, inequalities, and assumptions as the "model"; but the term "mathematical modelling" means more than that: it is an orderly structured manner in which a real-life problem might be tackled.

The process of mathematical modelling can be broken up into seven different stages, as is shown in the flow chart in Figure 1.1.1.

(1) **Identification:** The questions to be answered must be clarified. The underlying mechanism at work in the physical situation must be identified as accurately as possible. Formulate the problem in words, and document the relevant data.

(2) **Assumptions:** The problem must be analysed to decide which factors are important and which factors are to be ignored so that realistic assumptions can be made.

(3) **Construction:** This is the translation of the problem into mathematical language which normally results in a collection of equations and/or inequalities after the variables had been identified. The "word" problem is transformed into an abstract mathematical problem.

(4) **Analysis:** The mathematical problem is solved so that the unknown variables are expressed in terms of known quantities, and/or it is analysed to obtain information about parameters.

(5) **Interpretation:** The solution to the abstract mathematical problem must be compared to the original "word" problem to see if it makes sense in the real-world situation. If not, go back to formulate more realistic assumptions, and construct a new model.

(6) **Validation:** Check whether the solution agrees with the data of the real-world problem. If the correlation is unsatisfactory, return to the "word" problem for a re-appraisal of the data and the assumptions. Modify or add assumptions and construct a new model.

(7) **Implementation:** If the solution agrees with the data, then the model can be used to predict what will happen in the future, or conclusions can be drawn to help in future planning, etc. In the case of predictions care should be taken to determine the time interval in which the predictions are valid.

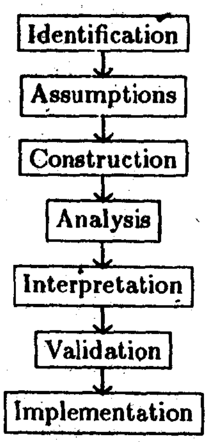

Figure 1.1.1: **Flow chart of the modelling process**

In a specific problem we may not use all seven stages, or some stages may be trivial. However, even though we will not state each stage explicitly in every model in this book, it is always the way in which we think about each problem.

Obviously a single book cannot cover all the different types of mathematical models. To narrow the field one could differentiate models either by the type of physical problem or by the type of mathematics needed to solve the problem. For example, in the first case only problems related to epidemics could be discussed, or in the second case the models can be restricted to those using systems of linear equations. In this book the second option was chosen with ordinary differential equations as the unifying theme of the different mathematical models.

- Do: Exercise 1 in §1.5.

1.2 Boundary Value Problems

Ordinary differential equations originate naturally in most of the branches of science. In this book we shall look at examples in diverse fields like physics, engineering, biology, economics, and medicine. A thorough knowledge of differential equations is a powerful tool in the hands of an applied mathematician with which to tackle these problems. We shall try to equip you with this tool in the pages that follow. But first a few general definitions and background material.

Definition 1.1

An ordinary differential equation *is an equation in which an unknown function $y(x)$ and the derivatives of $y(x)$ with respect to x appear. If the n-th derivative of y is the highest derivative in the equation, we say that the differential equation is of the* n-th order. *The* domain *of the differential equation is the set of values of x on which $y(x)$ and its derivatives are defined.*

Usually the domain is an interval I on the real line where I could be a finite interval (a, b), the positive real numbers $(0, \infty)$, the non-negative real numbers $[0, \infty)$, or even the whole real line $(-\infty, \infty)$. We shall use this notation: a square bracket denotes that the endpoint is included and a round bracket that the endpoint is not included in the interval.

Definition 1.2

If the differential equation is linear in the dependent variable y and all its derivatives, we refer to it as a linear differential equation. *If not, we shall call it a* nonlinear differential equation. *The general form of a linear differential equation of the n-th order is*

$$a_n(x)\frac{d^n y}{dx^n} + \ldots + a_1(x)\frac{dy}{dx} + a_0(x)y = b(x) \quad (1.2.1)$$

The main purpose of this book is to teach you how to construct *mathematical models* of selected real-life problems. These problems were chosen to include an ordinary differential equation in the model. To obtain a meaningful answer to the given problem, the differential equation must be solved. The solution must then be interpreted in the light of the assumptions that were made in the construction of the model, as we have seen in §1.1.

Apart from the differential equation, the mathematical model will typically also include prescribed values of y and/or the derivatives of y at isolated points in the interval I, usually one or both of the endpoints.

Definition 1.3

The prescribed values of y and/or the derivatives of y at the endpoints of the interval are called boundary values, *and the problem of finding $y(x)$ where y appears in a differential equation as well as in prescribed boundary values is called a* boundary value problem. *If the independent variable x represents time and all the boundary values are specified at the left endpoint of the interval, the boundary value problem is called an* initial value problem.

When a boundary value problem is encountered, there are four questions which must be answered:

(1) What is meant by a "solution"?

(2) Does a solution exist?

(3) Is there more than one solution?

(4) Is the solution a continuous function of the boundary values?

Let us briefly discuss each of these important questions.

There are different ways in which "solution" can be understood. For example, do we require that the function $y(x)$ should satisfy the differential equation everywhere in I, or could there be a few exceptional points where the highest derivative does not exist. Let $i(I)$ denote the interior of I which means the largest open interval in I. If I is itself an open interval, then of course $i(I) = I$; otherwise one or both the endpoints will be excluded.

In this book, unless specifically stated otherwise, we shall use the word "solution" in the following sense:

Definition 1.4

A solution *of a given boundary value problem on the interval I is a function which is continuous on I, satisfies the differential equation at every point $x \in i(I)$, and agrees with the prescribed boundary values.*

Note that the definition implies that the derivatives which appear in the equation must exist everywhere in i(I). A function which is differentiable on an open interval is also continuous there (see [16] p. 166), but the converse is not true – for example $f(x) = |x|$ is continuous on $[-1; 1]$, but the derivative does not exist at $x = 0$, even though the left hand derivative and the right hand derivative do exist. (Remember that a function is continuous at a point $x = a$ if the left and right hand limits exist and are both equal to $f(a)$; and similarly for a derivative, both left and right hand limits must exist and be equal to each other.)

However, the necessity of a more general definition of the term "solution" will be shown in §2.8. There we shall need a slightly weaker condition than continuity, namely piecewise continuity. We shall also use piecewise continuous functions in §4.2.

Definition 1.5

A function f is said to be piecewise continuous *on a finite interval $[a, b]$ if the interval can be subdivided into a finite number of intervals with f continuous on each of these intervals and if the jump in the value of f at each of the endpoints of these intervals is finite.*

Note that the jump in the value of f is the difference between the left and right hand limits, which must both exist if the jump is to be finite. We shall continue the discussion of piecewise continuity in §2.8 and §4.2.

- Do: Exercise 2 in §1.5.

The *existence* of a solution is usually settled by finding the solution explicitly with the aid of mathematical techniques. However, there are many differential equations whose solutions cannot be expressed in terms of elementary functions (For example, the pendulum equation $\theta'' + \omega^2 \sin\theta = 0$ – see §7.2.) Then numerical techniques must be utilized to calculate the value of the solution approximately at selected points in the interval I. These calculations only make sense if one knows beforehand that a solution does indeed exist. In these cases the existence of a solution to the boundary value problem must be proven indirectly, without knowing what the solution really looks like. The construction of proofs for the so-called existence theorems is an important research area in mathematics.

Once the question of the existence of a solution is settled, the next logical question is whether the known solution is the only possible solution. If this is the case, then we say the solution is *unique*. Clearly the uniqueness of the solution is very important for the interpretation of the results of the model – in fact, if the solution is not unique, then all the possible solutions must be found before any meaningful conclusions can be drawn. Sometimes the uniqueness follows immediately by the manner in which the solution was found; in other cases it is more complicated and one usually relies on special theorems which were proved beforehand. Both these approaches will be encountered in this book.

Finally, it is also very important to know how sensitive the solution is to changes in the boundary values. The main reason for this is that they are subject to errors. The crucial question is whether a small error in the boundary values will cause a small error in the solution. If this is the case the solution must be a continuous function of the boundary values.

- Do: Exercises 3, 4, 5 in §1.5.

1.3 Direction Fields

Consider the initial value problem

$$\frac{dy}{dx} = F(x,y), \qquad y(0) = \alpha \tag{1.3.1}$$

where α is a prescribed initial value and F is a given function. If a solution $y = f(x)$ exists, then the graph of this solution is a curve in the (x,y)–plane passing through the point $(0,\alpha)$ on the y– axis. For different values of α, different solution curves are obtained so that one can think of the (x,y)–plane as being filled with solution curves in the sense that through every point in the plane passes a solution curve corresponding to some value of α (provided, of course, that F is defined and smooth enough). The slope of the tangent to each solution curve at any point $(a, f(a))$ on the curve is $F(a, f(a))$ by (1.3.1). Consider, for example, the boundary value problem

$$\frac{dy}{dx} = \frac{2x\left(x^2+1\right)}{y} \tag{1.3.2}$$

$$y(0) = 1 \tag{1.3.3}$$

We shall see in Chapter 2 that the solution is $y = x^2 + 1$ (see Exercise 32 in §2.12).

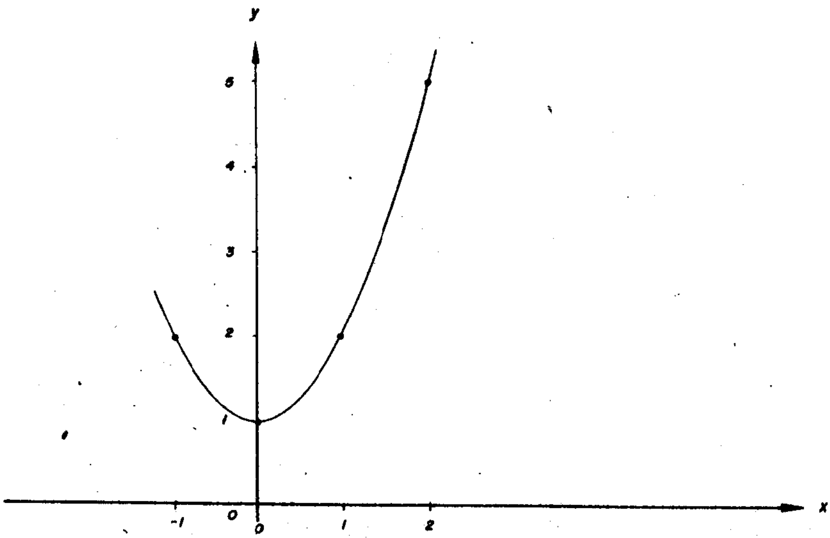

Figure 1.3.1: Solution of (1.3.2) and (1.3.3)

By differentiation it follows that the slope of this solution curve is $2x$. On the other hand, by (1.3.2) we have

$$\frac{dy}{dx} = \frac{2x\,(x^2+1)}{x^2+1} = 2x.$$

The function $F(x, y)$ in (1.3.1) prescribes at each point (x, y) in the plane where F is defined, a slope (or direction) for the solution curve which passes through (x, y). We say briefly that the differential equation prescribes a ***direction field***. If we choose a suitable rectangular mesh in the plane, a useful picture of the direction field can be obtained. At each point (a, b) of the mesh in the (x, y)–plane, a small line segment (also called a lineal element) is drawn from (a, b) with slope $F(a, b)$. Since each of these line segments is a tangent to a solution curve, the picture of all these line segments gives an indication of the shape of the solution curves. Obviously, the finer the rectangular mesh, the better this indication will be – provided that the mesh is not finer than the thickness of the line segments!

In this way an idea of the behaviour of the solution curves can be obtained, even though the solution of (1.3.1) may not be known.

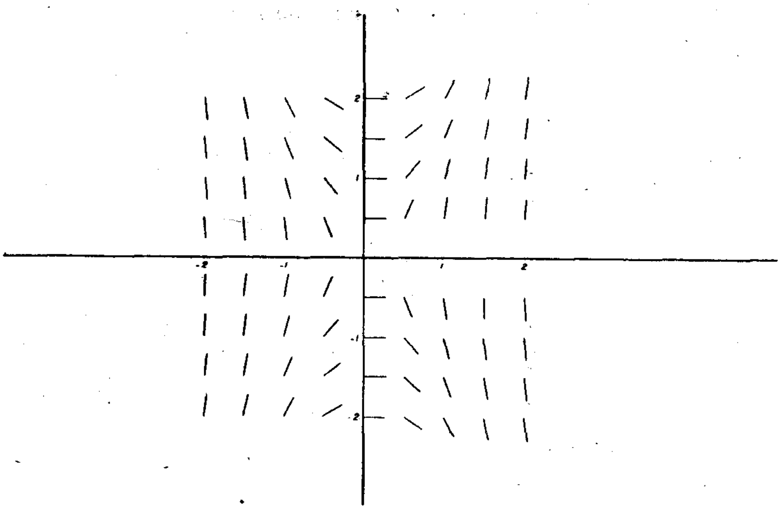

Figure 1.3.2: **Direction field of (1.3.2)**

Let us draw the direction field for the differential equation in (1.3.2) on the subset $R = \{(x, y) : \; -2 \leq x \leq 2, \; -2 \leq y \leq 2\}$ with the rectangular mesh points at ± 2, ± 1.5, ± 1.0, ± 0.5, and 0 for both x and y. Then we need the slope at 81 mesh points. At the origin the slope is