Elliptic Functions and Elliptic Curves

PATRICK DU VAL

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PATRICK DU VAL

Ordinarius Professor of Geometry,
University of Istanbul

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Preface

The lectures on which the following notes are based were given in various forms in University College, London, from about 1964 to 1969. Generally they were an optional undergraduate course, containing the substance of Chapters 1-6, and part of Chapter 8. Once or twice they were given to graduate students in geometry, and then included also the bulk of Chapters 9-13. Chapter 7, with the part of Chapter 11 which depends on this, and the cubic transformations in Chapter 8, never figured in the course, but it seemed to me very desirable to add them to the published notes. There is of course much more that I would have liked to include (such as transformations at least of order 5, some study of the connexion between modular relations and the subgroups of finite index in the modular group, a general examination of rectification problems, and the parametrisation of confocal quadrics and of the tetrahedroid and wave surfaces); but a limit of length is laid down for this series of publications, which I fear I have already strained to the utmost.

In my treatment of elliptic functions I have tried above all to present a unified view of the subject as a whole, developing naturally out of the Weierstrass function; and to give the essential rudiments of every aspect of the subject, while unable to enter in very great detail into any one of these. In particular I have been concerned to emphasize the dependence of the properties of the functions on the shape of the lattice; it is for this reason that the modular function is introduced at such an early stage, and that equal prominence is given throughout (except in the context of the Jacobi functions) to the rhombic and the rectangular lattices.

The treatment of the theta functions will be seen to be rather slight. They are in themselves a large subject, of which our study is in a considerable measure independent, since our approach (based on Neville's) to the Jacobi functions obviates any need for the theta functions as a preliminary, except for the expression of invariants such as k, K, J in terms of τ or q, i. e. in terms of the lattice shape.

I have kept the analytic apparatus required to a minimum, largely because I am no expert analyst myself; all that I assume ought, I think, to be familiar to any graduate or third-year honours student, and is to be found in any such general textbook as Whittaker and Watson [43] or Copson [5]. For the study of elliptic curves I have of course had to assume some knowledge of algebraic geometry. The general theory sketched in Section 85 can be read up in detail in such works as van der Waerden [38] or Hodge and Pedoe [21]; and the properties of the genus used in Section 89 in any book on algebraic curves, such as Walker [40] or Semple and Kneebone [35]. For any assumed properties of the plane cubic and twisted quartic, probably the best sources are still the two classics of Salmon [32, 33], now available in modern reprints; and for the finite groups $\underline{V}$, $\underline{T}$, $\underline{O}$ etc. perhaps the easiest reference is my own monograph [9].

In conclusion, I would like to express my gratitude to the London Mathematical Society for making this publication possible; to the general editor of the series, Professor G. C. Shephard, for his patience; to Dr D. G. Larman for assistance with the bibliography; and particularly to my wife for her help in reading the proofs.

Istanbul, 1971 Patrick Du Val

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1.

For any complex number $z = x + iy$ (x, y real, $i^2 = -1$) we define $\operatorname{Re}(z) = x$, $\operatorname{Im}(z) = y$, $|z| = (x^2 + y^2)^{\frac{1}{2}}$, $\bar{z} = x - iy$. If $y = 0$ (i. e. if $\bar{z} = z$), z is real, if $y \neq 0$, z is imaginary, if $x = 0$, z is pure imaginary (note that 0 is pure imaginary without being imaginary) and if $|z| = 1$, z is unimodular. The real and pure imaginary axes in the Argand plane are horizontal and vertical respectively.

Lattices. A lattice Ω of complex numbers is an aggregate of complex numbers with the two properties: (i) Ω is a group with respect to addition; (ii) the absolute magnitudes of the non-zero elements are bounded below, i. e. there is a real number $k > 0$ such that $|\omega| \geq k$ for all $\omega \neq 0$ in Ω. Every lattice is either (i) trivial, consisting of 0 only; (ii) simple, consisting of all integer multiples of a single generating element, which is unique except for sign; or (iii) double, consisting of all linear combinations with integer coefficients of two generating elements ω_1, ω_2, whose ratio is imaginary. These are not unique; if ω_1, ω_2 generate Ω, so do

$$\omega_1' = p\omega_1 + q\omega_2\,, \qquad \omega_2' = r\omega_1 + s\omega_2\,,$$

where p, q, r, s are any integers satisfying $ps - qr = \pm 1$. It is usual however to require ω_1, ω_2 to be so ordered that $\operatorname{Im}(\omega_2/\omega_1)$ shall be positive; and if ω_1', ω_2' are to be similarly ordered, this requires $ps - qr = +1$.

2. Lattice shapes

If Ω is any lattice, and m any non zero complex number, $m\Omega$ denotes the aggregate of complex numbers $m\omega$ for all ω in Ω. This is also a lattice, which is said to be similar to Ω; similarity is an equivalence relation between lattices, an equivalence class being a lattice shape. All simple lattices are similar, i. e. constitute one lattice shape. The lattice points (i. e. elements of the lattice, represented as points in the Argand plane) are (for a simple lattice) at equal intervals along one line through the origin, but in general (for a double lattice) are the vertices of a pattern of parallelograms filling the whole plane, whose sides can be taken to be any pair of generators. The lattice point patterns for similar lattices are similar in the elementary sense.

$\overline{\Omega}$ denotes the aggregate of complex numbers $\overline{\omega}$ for all ω in Ω; $\overline{\Omega}$ is also a lattice. If $\overline{\Omega} = \Omega$, Ω is called real. This is the case if and only if either: (i) Ω is simple, its generator (and hence all its elements) being either real or pure imaginary; (ii) generators can be so chosen that ω_1 is real and ω_2 pure imaginary, in which case Ω is called rectangular, the lattice points being the vertices of a pattern of rectangles, whose sides are horizontal and vertical, i. e. parallel to the real and imaginary axes; or (iii) generators can be chosen which are conjugate complex, in which case Ω is called rhombic, the lattice points being the vertices of a pattern of rhombi, whose diagonals are horizontal and vertical. Any lattice similar to a rectangular or rhombic lattice is also rectangular or rhombic, but is only real if the sides of the rectangles (diagonals of the rhombi) are horizontal and vertical. The real rectangular or rhombic lattice will be called horizontal or vertical, according as the longer sides of the rectangles (longer diagonals of the rhombi) are horizontal or vertical.

Besides the simple lattice, there are two special lattice shapes: (i) square (ordinary squared paper pattern); this is both rectangular and rhombic, and may be said to be in the rectangular or rhombic position if the sides or diagonals respectively of the squares are horizontal and vertical (it is real in both cases); (ii) triangular (pattern of equilateral triangles filling the plane); this is rhombic in three ways, a rhombus (with diagonals in the ratio $\sqrt{3}:1$) consisting of any two triangles with a common side. Every lattice satisfies $\Omega = -\Omega$; the only cases in which $\Omega = k\Omega$, with $k \neq \pm 1$, are the square lattice ($\Omega = i\Omega$) and the triangular lattice ($\Omega = \varepsilon\,\Omega$, where ε is a primitive cube root of unity; we shall throughout denote these cube roots by ε, ε^2 instead of the more usual ω, ω^2, to avoid confusion with the use of ω for an element of a lattice).

3. Residue classes

If z is any value of a complex variable, $z + \Omega$ denotes the aggregate of values $z + \omega$ for all ω in the lattice Ω. This aggregate is called a residue class (mod Ω). The residue classes (mod Ω) form a continuous group under addition, defined in the obvious way, namely $(z + \Omega) + (w + \Omega) = (z + w) + \Omega$. Ω itself is a residue class (mod Ω), the zero element of the group.

By a fundamental region of Ω we mean a simply connected region of the Argand plane which contains exactly one member of each residue class (mod Ω). If Ω is the trivial lattice, each residue class consists only of a single value of z, and the only fundamental region is the whole plane. If Ω is the simple lattice generated by ω, a fundamental region is an infinite strip, bounded by two parallel lines, one of which is the locus of $z + \omega$ for all z on the other; these bounding lines need not be perpendicular to ω, nor straight, though it is usually convenient to take them so; but they must not intersect. One of the two lines is included in the fundamental region, and the other is not, i. e. the strip is closed on one side and open on the other. If Ω is a double lattice, a fundamental region can be chosen in many ways; the simplest, and usually the most convenient, is what is called a unit cell, i. e. a parallelogram with sides ω_1, ω_2 (any pair of generators), including one of each pair of parallel sides, and one vertex, but excluding the rest of the boundary.

We obtain a topological model of the residue class group by identifying the points congruent (mod Ω) on the boundary of the fundamental region, i. e. joining up the open edges to the corresponding closed edges. For the simple lattice, identifying the points z, $z + \omega$ throughout the bounding lines of the strip, we obtain an infinite cylinder, with generators perpendicular to ω. This is topologically equivalent to a sphere with two pinholes, corresponding to the open ends of the cylinder (compare the Mercator map of the sphere, rolled up thus into a cylinder, on which every point of the sphere is mapped uniquely, except the two poles). For the double lattice, identifying points z, $z + \omega_2$ on the sides of the unit cell parallel to ω_1 we obtain a finite cylinder of length $|\omega_1|$ with ends perpendicular to its generators; to identify corresponding points on these two ends, the cylinder must be bent round (and also twisted, unless the sides of the unit cell are perpendicular, i. e. unless Ω is rectangular) to form a ring surface or torus.

In particular, the torus

$$x = (a + b\cos\phi)\cos\theta, \quad y = (a + b\cos\phi)\sin\theta, \quad z = b\sin\phi,$$
$$(x^2 + y^2 + z^2 + a^2 - b^2)^2 = 4a^2(x^2 + y^2)$$

$(a^2 > b^2 > 0)$, obtained by rotating the circle

$$(x - a)^2 + z^2 = b^2, \quad y = 0$$

about the z axis, is not only a topological model of the residue class group, but a conformal model of the fundamental region, for a rectangular lattice whose generators satisfy

$$\omega_2/\omega_1 = ib/\sqrt{(a^2 - b^2)} .$$

This means that the angle between the transverse common tangents of the two circles $(x \pm a)^2 + z^2 = b^2$, which are the section of the torus by the meridian plane $y = 0$, is equal to that between the diagonals of the rectangular unit cell of Ω.

Proof. The element of arc on the surface is given by

$$ds^2 = (a + b\cos\phi)^2 d\theta^2 + b^2 d\phi^2 = (a + b\cos\phi)^2(d\xi^2 + d\eta^2) ,$$

where $\xi = \theta$ and

$$\eta = \int \frac{b\,d\phi}{a + b\cos\phi} = \frac{2b}{\sqrt{(a^2 - b^2)}} \tan^{-1}\left(\sqrt{\left(\frac{a-b}{a+b}\right)} \tan\tfrac{1}{2}\phi\right) ,$$

so that the mapping of the point (θ, ϕ) of the torus on the point with cartesian coordinates (ξ, η) thus defined in a plane is conformal; and the torus, cut open along the meridian $\theta = \pm\pi$ and the parallel $\phi = \pm\pi$, is mapped (1, 1) on the rectangle between the lines

$$\xi = \pm\pi, \quad \eta = \pm b\pi/\sqrt{(a^2 - b^2)} ,$$

the fundamental region in question. //

No surface is known in three-dimensional Euclidean space, on which the residue class group modulo a non-rectangular lattice can be mapped in this way, so as to give at the same time a conformal map of the fundamental region. (Such a surface exists in eight-dimensional Euclidean space, but this is beyond our scope.)

4. Summation over a lattice

If Ω is any lattice and $f(z)$ any function of a complex variable, we shall denote by $\sum_{\Omega} f(\omega)$ the sum of $f(\omega)$ over all elements ω of Ω, and by $\sum_{\Omega}' f(\omega)$ the sum over all non-zero elements, i.e. the same sum with the term for $\omega = 0$ omitted.

Theorem 1.1. <u>For any lattice</u> Ω <u>and any integer</u> $n > 2$, $S_n(\Omega) = \sum_{\Omega}' \omega^{-n}$ <u>converges absolutely</u>.

Proof. It is well known that for $n > 1$, $\sum_{r=1}^{\infty} r^{-n}$ converges absolutely; denote this sum by s_n (it is in fact the Riemann zeta function $\zeta(n)$; but the use of the letter ζ here is unacceptable, since in the context of elliptic functions this letter has a quite different but equally well established meaning, to which we shall come later). If Ω is simple with generator ω, for even n, $S_n(\Omega) = 2\omega^{-n}s_n$, and for odd n, $S_n(\Omega) = 0$, as the terms $(r\omega)^{-n}$, $(-r\omega)^{-n}$ cancel. If Ω is a double lattice, the lattice points can be distributed into sets lying on the perimeters of a sequence of concentric parallelograms, similar to the unit cell, those on the r^{th} perimeter being of the form $p\omega_1 + q\omega_2$, where $|p|$, $|q|$ both $\leq r$, and at least one of them $= r$. Denote by $\sum_{r} f(\omega)$ the sum of terms with ω on the r^{th} perimeter; then $\sum_{\Omega}' f(\omega) = \sum_{r=1}^{\infty} \sum_{r} f(\omega)$. Now if h is the lesser diameter of the unit cell perpendicular to an edge, every ω on the r^{th} perimeter satisfies $|\omega| \geq rh$, the inequality being strict for most of them; and they are 8r in number. Thus $\sum_{r} |\omega|^{-n} < 8r(rh)^{-n}$, so that the series $\sum_{r=1}^{\infty} \sum_{r} |\omega|^{-n}$ is majorised by the absolutely convergent series $8h^{-n} \sum_{r=1}^{\infty} r^{1-n}$, and is thus itself absolutely convergent. //

The quantities $S_n(\Omega)$ thus defined clearly satisfy the homogeneity property $S_n(k\Omega) = k^{-n} S_n(\Omega)$, for all complex numbers $k \neq 0$ and all integers $n > 2$, since every term in the series on the left is k^{-n} times the corresponding term in that on the right. It follows that if n is odd, $S_n(\Omega) = 0$, for every lattice Ω, since $\Omega = -\Omega$, $S_n(\Omega) = S_n(-\Omega) = -S_n(\Omega)$. Similarly, if Ω is square, as $\Omega = i\Omega$, $S_n(\Omega) = 0$ for all n not divisible by 4, and if Ω is triangular, as $\Omega = \varepsilon\,\Omega$, $S_n(\Omega) = 0$ for all n not divisible by 6. If Ω is real, $S_n(\Omega)$ is real for all n, conjugate complex elements of the lattice giving rise to conjugate complex terms in the sum, and real elements to real terms; and in general $S_n(\overline{\Omega}) = \overline{S_n(\Omega)}$.

The simple lattice generated by ω can be regarded as the limit of a double lattice, of which one generator $\omega_1 = \omega$ remains constant, and the other ω_2 varies continuously in such a way that $\text{Im}(\omega_2/\omega_1)$ tends to infinity, as all the lattice points except the integer multiples of ω recede to infinity, leaving the plane empty of lattice points except those of the simple lattice. The simple lattice will therefore be called a degenerate double lattice.

Theorem 1.2. When a double lattice Ω, varying continuously, tends to the degenerate limit, with generator ω, $S_n(\Omega)$ tends, uniformly in $\text{Re}(\omega_2/\omega_1)$, to the limit $2\omega^{-n} s_n$, its value for the simple lattice.

Proof. Denote ω_2/ω_1 by τ; on account of the homogeneity, it is sufficient to prove the theorem for the lattice Ω_τ, generated by 1, τ. Now for any even n, pairing off the equal terms for ω, $-\omega$, we can write

$$S_n(\Omega_\tau) = 2s_n + 2 \sum_{q=1}^{\infty} \sum_{p=-\infty}^{\infty} (p + q\tau)^{-n}.$$

Now let k be any integer. For each value of q, we can divide the values of p into sets of kq consecutive integers, according as $\mathrm{Re}(p + q\tau)$ lies between consecutive multiples of kq. If $\mathrm{Im}(\tau) > k$, whatever $\mathrm{Re}(\tau)$ may be, for the two such sets of values of p defined by

$$rkq \le \mathrm{Re}(p + q\tau)kq, < (r + 1)kq, \quad -(r + 1)kq \le \mathrm{Re}(p + q\tau) < -rkq$$

we have $|p + q\tau| > kq\sqrt{(1 + r^2)}$, so that for each value of q,

$$\left| \sum_{p=-\infty}^{\infty} (p + q\tau)^{-n} \right| < 2(kq)^{1-n} \sum_{r=0}^{\infty} (1 + r^2)^{-\frac{1}{2}n}$$

$$< 2(kq)^{1-n}(1 + 2^{-\frac{1}{2}n} + s_n) ,$$

replacing $(1 + r^2)^{-\frac{1}{2}n}$ by $(r - 1)^{-n}$, in all but the first two terms, since $(r - 1)^2 < 1 + r^2$. Hence

$$\left| 2 \sum_{q=1}^{\infty} \sum_{p=-\infty}^{\infty} (p + q\tau)^{-n} \right| < 4k^{1-n}s_{n-1}(1 + 2^{-\frac{1}{2}n} + s_n) ,$$

irrespective of the value of $\mathrm{Re}(\tau)$. Thus by taking $\mathrm{Im}(\tau)$ greater than a sufficiently large integer k, we can make

$$|s_n(\Omega_\tau) - 2s_n|$$

as small as we like, uniformly in $\mathrm{Re}(\tau)$; the theorem is thus proved for Ω_τ, and follows immediately for any $\Omega = \omega_1\Omega_\tau$. //

5. Functions and periods

We recall that a function f(u) of a complex variable u is analytic at $u = a$ if it has an expansion as a power series $f(u) = \sum_{r=0}^{\infty} c_r(u - a)^r$, with constant coefficients $c_0, c_1, \ldots,$ converging absolutely and uniformly in some circle $|u - a| < k$, where $k > 0$. f(u) is meromorphic at $u = a$ if for some integer n, $(u - a)^n f(u)$ is analytic at $u = a$; if $n > 0$ is the least integer for which this holds, f(u) has an expansion

$$f(u) = \sum_{r=1}^{n} b_r(u - a)^{-r} + \sum_{r=0}^{\infty} c_r(u - a)^r ,$$

with $b_n \neq 0$; in this case $u = a$ is a pole of $f(u)$, of order n; the terms $\sum_{r=1}^{n} b_r(u-a)^{-r}$ are called the infinite part of the function $f(u)$, b_n its leading coefficient, and b_1 its residue, at $u = a$. (This well established use of the word residue has of course nothing to do with residue classes, to which unfortunately we occasionally have to refer in the same contexts.) Similarly $u = a$ is a zero of order n of $f(u)$ if $f(u)$ is analytic at $u = a$, $f(a) = 0$, and n is the greatest integer such that $(u-a)^{-n}f(u)$ is analytic at $u = a$, i.e. c_n is the first non-zero coefficient in the expansion of $f(u)$ at $u = a$, which is accordingly of the form $f(u) = \sum_{r=n}^{\infty} c_r(u-a)^r$.

A function is said to be analytic or meromorphic in a given region, or in the whole plane, if it is so at every point of the region or of the plane. If $f(u)$ is analytic and non-zero at any point, in any region, or in the whole plane, so is $\frac{1}{f(u)}$; if $f(u)$ is meromorphic, so is $\frac{1}{f(u)}$, the poles of each being the zeros of the other, and of the same order. The poles of a function meromorphic in any region are a discrete set, i.e. for each pole, the distances of other poles from it are bounded below; and if $f(u)$ is meromorphic in any finite region, including its boundary, $f(u)$ can only have a finite number of poles in the region. As $\frac{1}{f(u)}$ is also meromorphic, $f(u)$ can only have a finite number of zeros in the region; and as $f(u) - c$ is meromorphic (for any constant c) $f(u)$ can only assume a given value c in a finite set of points in the region.

A period ω of a function $f(u)$ is a constant such that $f(u + \omega) = f(u)$ for all u. The sum of two periods is also trivially a period, and if ω is a period, so is $-\omega$. Thus the periods of any function form a group with respect to addition. On the other hand, unless the absolute magnitude of non-zero periods is bounded below, the function must be constant in any region in which it is differentiable, since $\frac{f(u+h) - f(u)}{h} = 0$ for some arbitrarily small but non-zero values of h. Thus the periods of a non-constant meromorphic function must be a lattice. Zero is of course a period of every function; if it is the only one, the lattice of periods is the trivial lattice, and the function is called non-periodic. If a function has a simple or double lattice of periods, it is called simply or doubly periodic. Familiar examples of simply periodic functions are $\sin u$, $\tan u$, e^u, with simple lattices of periods generated by 2π, π, $2i\pi$ respectively.

6. Definition

An elliptic function is a function of a complex variable, which is meromorphic in the whole plane, and doubly periodic. Since it has the same value in all points of any residue class (mod Ω), where Ω is its lattice of periods, it can be thought of as a function of the residue class, rather than of the individual value of u, i. e. a function of position on the torus model of the residue class group rather than of position in the plane. Before proving (by construction) the existence of some functions with these properties, it is convenient to prove some elementary consequences of the definition, assuming that such functions exist.

7. Liouville's theorem

This states that any function which is analytic and bounded in the whole plane is a constant. Also, a function which is analytic in any finite region (including its boundary) is bounded in that region. Hence, an elliptic function which has no residue classes of poles is bounded in the fundamental region, and so in the whole plane, and is accordingly a constant. This principle is applied in two main ways to elliptic functions:

Theorem 1. 3. If two elliptic functions have the same lattice of periods, the same residue classes of poles, and the same residue classes of zeros, of the same order in each case, the ratio of the two functions is a non-zero constant.

Proof. If $f(u)$, $g(u)$ have either zeros or poles of the same order at $u = a$, $\frac{f(u)}{g(u)}$ is analytic and non-zero at $u = a$. //

Theorem 1. 4. If two elliptic functions have the same lattice of periods, and the same residue classes of poles, with the same infinite part in each pole, the functions differ by a constant.

Proof. If $f(u)$, $g(u)$ have the same infinite part at $u = a$, $f(u) - g(u)$ is analytic there, i. e. has no pole. //

8. Contour integration theorems

For any function meromorphic in a simply connected region R bounded by a closed contour C, we recall the three classical theorems on integration round the contour C: Let $f(u)$ be meromorphic in R, with zeros of orders $m_1, \ldots, m_h$ at $u = a_1, \ldots, a_h$ and poles of order $n_1, \ldots, n_k$ at $u = b_1, \ldots, b_k$, with residues $r_1, \ldots, r_k$ respectively, all these zeros and poles being in R but none on C. Then

$$\text{I.} \quad \int_C f(u)du = 2\pi i \sum_{j=1}^{k} r_j .$$

$$\text{II.} \quad \int_C \frac{f'(u)du}{f(u)} = 2\pi i\Big(\sum_{j=1}^{h} m_j - \sum_{j=1}^{k} n_j\Big) .$$

$$\text{III.} \quad \int_C \frac{uf'(u)du}{f(u)} = 2\pi i\Big(\sum_{j=1}^{h} m_j a_j - \sum_{j=1}^{k} n_j b_j\Big) .$$

From this we deduce

Theorem 1.5. Let $f(u)$ be an elliptic function with the lattice Ω of periods, zeros of order $m_1, \ldots, m_h$ in the residue classes $a_1+\Omega, \ldots, a_h+\Omega$, and poles of order $n_1, \ldots, n_k$ in the residue classes $b_1+\Omega, \ldots, b_k+\Omega$, with residues $r_1, \ldots, r_k$ respectively. Then

$$\text{I.} \quad \sum_{j=1}^{k} r_j = 0 ;$$

$$\text{II.} \quad \sum_{j=1}^{h} m_j = \sum_{j=1}^{k} n_j ;$$

$$\text{III.} \quad \sum_{j=1}^{h} m_j a_j \equiv \sum_{j=1}^{k} n_j b_j \pmod{\Omega} .$$

Proof. Take the contour C to be the boundary of a unit cell, starting from a chosen point $u = c$, and travelling along straight lines to $u = c + \omega_1$, $c + \omega_1 + \omega_2$, $c + \omega_2$, and back to $u = c$ in turn, c being chosen so that the path does not pass through any zero or pole. If $\phi(u)$ is any function of u

$$\int_C \phi(u)du = \int_c^{c+\omega_1}(\phi(u) - \phi(u+\omega_2))du + \int_c^{c+\omega_2}(\phi(u+\omega_1) - \phi(u))du . \tag{8.1}$$

If $f(u)$ is an elliptic function with period lattice Ω, generated by ω_1, ω_2, so is $\frac{f'(u)}{f(u)}$, and in both the integrals I, II, the integrand in both terms on the right in (8.1) is identically zero, which gives the results I, II of the theorem. As for integral III, $\frac{uf'(u)}{f(u)}$ is not of course an elliptic function; but as in this case $\phi(u) - \phi(u + \omega_2) = -\omega_2 f'(u)/f(u)$, the first term on the right in (8.1) becomes

$$-\omega_2 \int_c^{c+\omega_1} d \log f(u) = \omega_2(\log f(c) - \log f(c + \omega_1)) ;$$

and as $f(c + \omega_1) = f(c)$, the difference between their logarithms as obtained from the integral must be an integer multiple of $2\pi i$, say $-2q\pi i$; thus the first term in (8.1) reduces to $2\pi i . q\omega_2$; and similarly the other term reduces to $2\pi i . p\omega_1$. Thus the integral III is equal to $2\pi i$ times an element $p\omega_1 + q\omega_2$ of Ω, which proves the result III of the theorem. //

9. Order of an elliptic function

Just as an s-ple zero of a polynomial $f(x)$ is commonly and conveniently regarded as being s coincident zeros of $f(x)$, or roots of the equation $f(x) = 0$, and this convention enables us to say that an equation of degree n has exactly n roots, when we make due allowance for coincidences; so an s-ple zero or pole of any meromorphic function is conventionally to be regarded as s coincident zeros or poles; and in the case of elliptic functions, with period lattice Ω, if $u = a$ is an s-ple zero or pole, so is every member of the residue class $a + \Omega$, which is regarded as s coincident residue classes of zeros or poles. With this convention the results II, III of Theorem 1.5 can be restated as

Theorem 1.6. An elliptic function $f(u)$ assumes any value c in a number n of residue classes which is independent of c and characteristic of $f(u)$, making due allowance for coincidences among these n residue classes for some values of c; moreover, the sum of these n residue classes is independent of c.