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陳省身数学論文選集


Chern at 3, with Grandmother


Hamburg, 1935

Curriculum Vitae of Shiing-shen Chern

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National Medal of Science 1975.

Alexander von Humboldt Award, Germany 1982.

Steele Prize, American Mathematical Society 1983.

Wolf Prize, Israel 1983–84.

A Summary of My Scientific Life and Works*

By Shiing-shen Chern

I was born on October 26, 1911 in Kashing, Chekiang Province, China. My high school mathematics texts were the then popular books *Algebra* and *Higher Algebra* by Hall and Knight, and *Geometry* and *Trigonometry* by Wentworth and Smith, all in English. Training was strict and I did a large number of the exercises in the books. In 1926 I enrolled as a freshman in Nankai University, Tientsin, China. It was clear that I should study science, but my disinclination with experiments dictated that I should major in mathematics. The Mathematics Department at Nankai was a one-man department whose Professor, Dr. Li-Fu Chiang, received his Ph.D. from Harvard with Julian Coolidge. Mathematics was at a primitive state in China in the late 1920s. Although there were universities in the modern sense, few offered a course on complex function theory and linear algebra was virtually unknown. I was fortunate to be in a strong class of students and such courses were made available to me, as well as courses on non-Euclidean geometry and circle and sphere geometry, using books by Coolidge.

The period around 1930, when I graduated from Nankai University, saw great progress in Chinese science. Many students of science returned from studies abroad. At the center of this development was Tsing Hua University of Peking (then called Peiping), founded through the return of the Boxer's Indemnity by the U.S. I was an assistant at Tsing Hua in 1930–1931 and was a graduate student from 1931–1934. My teacher was Professor Dan Sun, a former student of E.P. Lane at Chicago. Therefore, I began my mathematical career by writing papers on projective differential geometry.

In 1934 I was awarded a fellowship to study abroad. I went to Hamburg, Germany, because Professor W. Blaschke lectured in Peking in 1933 on the geometry of webs and I was attracted by the subject. I arrived at Hamburg in the fall of 1934 when Kähler's book *Einführung in die Theorie der Systeme von Differentialgleichungen* was published and he gave a seminar based on it. In a less than two-year stay in Hamburg I worked in more depth on the Cartan–Kähler theory than any other topic. I received my D.Sc. in February 1936.

The completion of the degree fulfilled my obligation to the fellowship. It was natural to look forward to a carefree postdoctoral year in Paris with the master himself, Elie Cartan. It turned out to be a year of hard work. In 1936–1937 in Paris I learned moving frames, the method of equivalence, more Cartan–Kähler theory, and, most importantly, the mathematical language and the way of thinking of Cartan. Even now I frequently find Cartan easier to follow than some of his expositors.

* Originally written 1978; updated and revised 1988.

I returned to China in the summer of 1937 to become Professor of Mathematics at Tsing Hua University. I crossed the Atlantic on the S.S. Queen Elizabeth and, after a month long tour of the United States, I crossed the Pacific on the S.S. Empress of Canada. The Sino-Japanese war broke out while I was on board and I never reached Peking.

During the war Tsing Hua University moved to Kunming in Southwest China and became a part of Southwest Associated University. Mathematically it was a period of isolation. I taught courses on advanced topics (such as conformal differential geometry, Lie groups, etc.) and had good students.

In 1943 I became a member of the Institute for Advanced Study; both Veblen and Weyl were aware of my work. During the period 1943–1945 I learned algebraic topology and fiber bundles and did my work on characteristic classes, among other things. The war ended in 1945 and I decided to return to China. Postwar transportation difficulties delayed my trip so that I did not arrive in Shanghai until March 1946. I was called to organize a new institute of Mathematics of the Academic Sinica in Nanking. The work lasted only for about two years. On December 31, 1948 I left Shanghai for the United States, again on an invitation of the Institute for Advanced Study. (See Weil's article in Volume I. Before leaving China I was offered a position at the Tata Institute in Bombay, then at a planning stage, which I was not able to accept. The offer must have come on the initiative of D.D. Kosambi, the first professor of mathematics at Tata, who knew well my work on path geometry.) I spent the winter term of 1949 at the Institute. During 1949–1960 I was a Professor at the University of Chicago.

In 1960 I moved to Berkeley where I became Professor Emeritus in 1979. Together with C.C. Moore and I.M. Singer I submitted a proposal to the National Science Foundation for a Mathematical Institute in Berkeley. It was granted and I became the Director of the Mathematical Sciences Research Institute in 1981–84. After my retirement I started a mathematical institute at my alma mater, Nankai University, Tianjin, China. I am hoping that my last retirement will come soon.

In the following I will try to give a summary of my mathematical works.

1. Projective Differential Geometry

Einstein's general relativity provided the great impetus to the study of Riemannian geometry and its generalizations. Before that, geometry was dominated by Felix Klein's Erlangen Program announced in 1871, which assigns to a space a group of transformations which is to play the fundamental role. Thus the Euclidean space has the group of rigid motions and the projective space has the group of projective collineations, etc. Along the lines to classical curve and surface theory in the tradition of Serret–Frenet, Euler, Monge, and Gauss, projective differential geometry was founded by E.J. Wilczynski and G. Fubini and E. Cech. Its main problem is to find a complete system of local invariants of a submanifold under the projective group and interpret them geometrically through osculation by simpler geometrical figures. The main difficulty lies in that the projective group is relatively large and invariants can only be reached through a high order of osculation. Moreover, the group of isotropy is non-compact, a fact which excludes many beautiful geometrical properties.

In my first papers [1], [2] I avoided the first difficulty by studying more complicated figures. The papers are nothing more than exercises, but the philosophy behind them found an echo in the recent works of P.A. Griffiths on webs, Abel's theorem, and their applications to algebraic geometry. For example, instead of studying an algebraic curve of degree d in the plane, one can study the configuration consisting of d points on each line of the plane, its points of intersection with the curve. One gets in this way d arcs in correspondence. Paper [1] studies two arcs in correspondence.

My next paper [3] concerns projective line geometry, now a forgotten subject. A line complex is, in modern terminology, a hypersurface in the Plüker-Grassmann manifold of all lines in the three-dimensional projective space. While the consideration of tangent spheres of a surface leads to the fundamental notions of lines of curvature and principal curvatures and that of the tangent quadrics of a projective surface leads to the quadrics of Darboux and Lie, the use of quadratic line complexes in the study of general line complexes was initiated in this paper.

Several years later I returned to projective differential geometry by introducing new invariants of contact of a pair of curves in a projective space of n dimensions, and also of surfaces [17], [19]. They include as a special case, the invariant of Mehmke–Smith, which plays a role in some questions on singularities in several complex variables. Generally speaking, the study of diffeomorphism invariants of a jet at a singularity has recently attracted wide attention (H. Whitney, R. Thom). The projective invariants, studied extensively by Italian differential geometers, should enter into the more refined questions.

The Laplace transforms of a conjugate net was a favorite topic in the theory of transformations of surfaces. It is a beautiful geometric construction which leads to a transformation of linear homogeneous hyperbolic partial differential equations of the second order in two variables. In [24], [35] a generalization was given to a class of submanifolds of any dimension. This generalization could be related to the recent search of high-dimensional solitons and their Bäcklund transformations.

From projective spaces it is natural to pass to spaces with paths where the straight lines are replaced by the integral curves of a system of ordinary differential equations of the second order, an idea which could be traced back to Hermann Weyl. Such spaces are said to be projectively connected or to have a projective connection. Projective relativity (O. Veblen, J.A. Schouten) aims at singling out the projectively connected spaces whose paths are to be identified with the trajectories of a free particle in a unified field theory. They are defined by a system of "field equations." A new system of field equations was proposed in [111].

From the mathematical viewpoint, projectively connected spaces are of intrinsic interest. Relating projective spaces and general projectively connected spaces is the imbedding problem. Given a submanifold M in a projective space, an induced projective connection can be defined on M by taking a field of linear subspaces transversal to the tangent spaces of M and projecting neighboring tangent spaces from them. In [7] I proved an analogue of the Schläfli–Janet–Cartan imbedding theorem for Riemannian spaces of which the following is a special case: A real analytic normal (in the sense of Cartan) projective connection on a space of dimension n can be locally induced by an imbedding in a projective space of dimension $n(n + 1)/2 + [n/2]$. The dimension needed is thus generally higher than in Schläfi's case.

The fundamental theorem on projective connections is the theorem associating a unique normal projective connection to a system of paths. I announced in [23] that the same is true when there is in a space of dimension n a family of k-dimensional submanifolds depending on $(k + 1)(n - k)$ parameters and satisfying a completely integrable system of differential equations. The case $k = 1$ is classical and the case $k = n - 1$ was the main conclusion of M. Hachtroudi's Paris thesis. My derivation was long and was never published. A geometrical treatment was later given by C.T. Yen (*Annali di Matematica* 1953).

In the Princeton approach to non-Riemannian geometry led by Veblen and T.Y. Thomas, a main tool is the use of normal coordinates relative to which the normal extensions of tensors are defined. Normal coordinates in the projective geometry of paths can be given different definitions; their existence is generally not easy to establish. In [8] I showed that Thomas's normal coordinates are in general different from the normal coordinates defined naturally from Cartan's concept of a projective connection.

In my recent joint works with Griffiths on webs [112] we came across a theorem characterizing a flat normal projective connection as one with ∞^2 totally geodesic hypersurfaces suitably distributed; the classical theorem needs ∞^n totally geodesic hypersurfaces, n being the dimension of the space.

In concluding this section, I wish to say that I believe that projective differential geometry will be of increasing importance. In several complex variables and in the transcendental theory of algebraic varieties the importance of the Kähler metric cannot be over-emphasized. On the other hand, projective properties are in the holomorphic category. They will appear when the problems involve, directly or indirectly, the linear subspaces or their generalizations.

2. Euclidean Differential Geometry

Before the nineteen-forties, a mathematical student was usually introduced to differential geometry through a course on curves and surfaces in Euclidean space, known in European universities as "applications of the infinitesimal calculus to geometry". I was particularly fascinated by Blaschke's book for its emphasis on global problems. I was, however, able to do some work only after I began to treat surface theory by moving frames. In [29] I observed that Hilbert's proof of the rigidity of the sphere gives the more general theorem that a closed strictly convex surface in E^3 (= three-dimensional Euclidean space) is a sphere if one principal curvature is a monotone decreasing function of the other.

More generally, a natural area of investigation in Euclidean differential geometry is concerned with the W-hypersurfaces, where there is a functional relation between the principal curvatures. If a hypersurface is closed and strictly convex, its Gauss map into the unit hypersphere is one-to-one and we can identify functions on the hypersurface with those on the unit hypersphere. Let σ_r, $1 \le r \le n$, be the rth elementary symmetric function of the reciprocals of the principal curvatures of a convex hypersurface in

E^{n+1}. In [68] I proved that if, for a certain r, the σ_r functions of two closed strictly convex hypersurfaces Σ, Σ^* in E^{n+1} agree as functions on the unit hypersphere, then Σ and Σ^* differ by a translation. The condition means geometrically that σ_r are the same at points of Σ, Σ^* at which the normals are parallel. In [69], Hano, Hsiung, and I proved a similar uniqueness theorem by replacing the conditions by $\sigma_r \leq \sigma_r^*$, $\sigma_{r+1} \geq \sigma_{r+1}^*$ for a certain r. The proofs depend on the establishment of some integral formulas.

In [81] I considered hypersurfaces with boundary in the Euclidean space and found upper bounds on their size if certain curvature conditions are satisfied. This generalized some work of E. Heinz and S. Bernstein for surfaces in E^3.

Again using integral formulas, Hsiung and I proved in [77] that a volume-preserving diffeomorphism of two k-dimensional compact submanifolds in E^n is an isometry if a certain additional condition is satisfied.

In [62] and [66] Lashof and I studied the total curvature of a compact immersed submanifold in E^n. The total curvature is defined as the measure of the image of the unit normal bundle on the unit hypersphere of E^n under the Gauss map. (Observe that independent of the dimension of a submanifold the unit normal bundle has dimension $n-1$, which is the dimension of the unit hypersphere of E^n.) The total curvature was considered by J. Milnor following his work on that of a knot. Generalizing the classical theorems of Fenchel for the total curvature of a closed space curve, Lashof and I proved that the total curvature of a compact immersed submanifold in E^n, when properly normalized, has a universal lower bound and that it is reached when and only when the submanifold is a convex hypersurface. As a corollary, it is proved that a closed surface of non-negative Gaussian curvature in E^3 is convex, generalizing a classical theorem of Hadamard. In this work, a lemma on the local behavior of a hypersurface with degenerate second fundamental form plays a fundamental role. Total curvature and tight immersion have received many interesting developments in recent years (Kuiper, Banchoff, Pohl, and Chen).

Among these is Banchoff's introduction of the notion of a taut immersion, which means that the distance function of a point of the submanifold from any point in space has the smallest number of critical points. This is a stronger property than tight immersion. In [143] Tom Cecil and I proved that tautness is invariant under the Lie group of sphere transformations (=group formed by all contact transformations carrying spheres to spheres). We also introduced some basic notions of the differential geometry in Lie sphere geometry, such as the Legendre map and the Dupin submanifold.

It was Bonnet who studied isometric deformations of surfaces in E^3 preserving the mean curvature. The problem leads to a complicated over-determined system of partial differential equations which has been studied by many authors. In [133] I showed that these are either surfaces of constant mean curvature or form an exceptional family, depending on 6 constants, which consists of W-surfaces. In the analytical treatment the connection form of the unit tangent bundle is heavily used.

Given an oriented (two-dimensional) surface in E^4, its Gauss map has as image the Grassmann manifold of all oriented planes through a point. The latter is homeomorphic to $S^2 \times S^2$. As a result the map defines a pair of integers. Spanier and I [47] proved that if the surface is imbedded, these two integers are equal when the spaces are properly oriented.

In [50] Kuiper and I introduced two integers to an immersed manifold in E^n: the indices of nullity and of relative nullity. Inequalities are established between them and the dimension and codimension of a compact submanifold in E^n. The origin of this work was a theorem of Tompkins that there is no closed surface in E^3 whose Gaussian curvature is identically zero.

The smoothness requirements of various theorems in surface theory have been thoroughly investigated by P. Hartman and A. Wintner in a long series of papers. In [55] we studied the critical case for the isothermic coordinates, namely, the minimum conditions so that the metric in the isothermic coordinates has the same smoothness.

Finally I wish to mention a result on complex space-forms. In his thesis, Brian Smyth determined the complete Einstein hypersurfaces in a Kählerian manifold of constant holomorphic sectional curvature by using the classification of symmetric Hermitian spaces. The result turns out to be a local one. The problem leads to an over-determined differential system and I showed in [87] that the theorem follows from a careful study of the integrability conditions. The hypersurfaces in question are either totally geodesic or are hyperspheres.

Euclidean differential geometry is comparable to elementary number theory in its beauty of simplicity. Unlike the latter more remains to be discovered.

3. Geometrical Structures and Their Intrinsic Connections

A Riemannian structure is governed by its Levi-Civita connection, and a path structure by its normal projective connection. A fundamental problem of local differential geometry is to associate to a structure a connection which describes all the properties. An effective way of doing this is by Elie Cartan's method of equivalence. In the years 1937–1943 when I was isolated in the interior of China I carried out the program in many cases:

The geometry of the equation of the second order

$$y'' = F(x, y, y'), \qquad y' = dy/dx, \qquad y'' = d^2y/dx^2$$

in the (x, y)-plane was studied by A. Tresse. Tresse's results were formulated in terms of the Lie theory; it would be more geometrical to say that a normal projective connection can be defined in the space of line elements (x, y, y'). I studied the equation of the third order

$$y''' = F(x, y, y', y'')$$

under the group of contact transformations in the plane and showed that in an important case a conformal connection can be defined intrinsically [6], [13]. I also defined affine connections from structures arising from webs [9] (cf. §8).

Local differential geometrical structures are defined either by differential systems or by metrics, the two typical cases being projective geometry and Euclidean geometry. When the paths are the integral curves of a system of ordinary differential equations, the allowable parameter change has an important bearing on the resulting geometry. D.D. Kosambi considered a system of differential equations of the second order with an allowable affine transformation of parameters and attached to the structure an affine connection. I proved in [10] the result by the method of equivalence and went on in [11] to solve the corresponding problem when the paths are defined by a system of differential equations of higher order.

Geometrically it is more natural that a family of submanifolds is given with unrestricted parametrization, i.e., the parameters are allowed arbitrary (smooth) changes. Generalizing Tresses's problem to n dimensions, the given data should be $\infty^{2(n-1)}$ curves satisfying a differential system such that through any point and tangent to any direction at the point there is exactly one such curve. With these curves taking the place of the straight lines, a generalized projective geometry, i.e., a normal projective connection, can be defined. As mentioned in §1, I extended this result to the case when there is given $\infty^{(k+1)(n-k)}$ k-dimensional submanifolds satisfying a differential system. In the same vein I defined in [20] a Weyl connection, giving ∞^2 surfaces in $\mathbb{R}^3$ as "isotropic surfaces." This was extended to n dimensions in [21], but the details of the n-dimensional case were never published.

In [22], [42] I studied the connections to be attached to a Finsler metric and showed that there is more than one natural choice.

In 1972 Moser found a local normal form of a non-degenerate real hypersurface in C_2 and asked me to identify his invariants with those of Elie Cartan. Years before I had extended Cartan's work to a real hypersurface in C_{n+1}. I have not published the results, partly because a paper of Tanaka on the same subject appeared in the meantime, although Tanaka made an assumption on the hypersurface (which he removed in a later paper). In [105] Moser and I gave both the normal form of a non-degenerate real hypersurface in C_{n+1} and its intrinsic connection as a CR-manifold and identified the two sets of invariants. When the hypersurface is real analytic, I defined in [107] a projective connection. The latter does not give all the invariants, but has the advantage that its invariants are in the holomorphic category.

All these are special cases of a G-structure. Some G-structures, such as the complex structures, admit an infinite pseudo-group of transformations. In [54] I gave an introduction to G-structures, including the notion of a torsion form and an exposition of Cartan's theory of infinite continuous pseudo-groups. A more complete account of G-structures was given in [83].

In [61] I observed that the Hodge harmonic theory is valid for a torsionless G-structure, with $G \subset O(n)$; the Hodge decomposition can then be generalized to the decomposition of a harmonic form into irreducible summands under the action of G. This viewpoint also gives a better understanding of Hodge's results.

Among mathematical disciplines the area of geometry is not so well defined. Perhaps the notion of a G-structure is of sufficient scope to fulfill the current requirements for the mainstream of geometry.

4. Integral Geometry

I went to Hamburg in 1934 when Blaschke, in his usual style, started a series of papers entitled "Integral Geometry". Although I have a keen interest in the subject, my works on it have been scattered.

I observed that integral geometry in the tradition of Crofton deals with two homogeneous spaces with the same group. Call the group G. If the homogeneous spaces are realized as coset spaces G/H and G/K, H and K being subgroups of G, two cosets aH and bK, $a, b \in G$, are called incident if they have an element in common. With this notion of incidence, Crofton's formula was established in a very general context [14], [16], [18]. This notion of incidence was appreciated by Weil and found useful in later works of Helgason and Tits.

My other work on integral geometry concerns the kinematic density of Poincaré. With Chih-Ta Yen I gave a proof of the fundamental kinematic formula in E^n [15], [48].

In his formula for the volume of a tube, Weyl introduced a number of scalar invariants of an imbedded manifold in E^n, half of which depend only on the induced metric. If M^p and M^q are closed imbedded manifolds of E^n, with M^p fixed and M^q moving, I proved in [84] a simple formula expressing the integral of an invariant of the intersection $M^p \cap M^q$ over the kinematic measure. This complements the fundamental kinematic formula, which deals with hypersurfaces.

5. Characteristic Classes

My introduction to characteristic classes was through the Gauss–Bonnet formula, known to every student of surface theory. Long before 1943, when I gave an intrinsic proof of the n-dimensional Gauss–Bonnet formula [25, 30], I knew, by using orthonormal frames in surface theory, that the classical Gauss–Bonnet is but a global consequence of the Gauss formula which expresses the "theorema egregium." The algebraic aspect of the proof in [25] is the first instance of a construction later known as transgression, which is destined to play a fundamental role in the homology theory of fiber bundles, and in other problems.

The Gauss–Bonnet formula is concerned with the Euler–Poincaré characteristic. It was natural to look at corresponding results for the general Stiefel–Whitney characteristic classes, then newly introduced. I soon realized that the latter are essentially defined only mod two and relating them with curvature forms would be artificial. Technically its cause lies in the complicated homology structure of the orthogonal group, such as the presence of torsion. The Grassmann manifold and the Stiefel manifold over the complex numbers have no torsion, and the same is true of the unitary group. In [33] I introduced the characteristic classes of complex vector bundles and related them via the de Rham theorem, with the curvature forms of an Hermitian structure in the bundle. Actually this paper contains, through the explicit construction of differential forms, the essence of the homology structure of a principal bundle with the unitary group as structure group: transgression, characteristic classes, universal bundle, etc. These characteristic classes are defined for algebraic manifolds, but their definition, whether via an Hermitian structure or via the universal bundle, is not algebraic. In [51] I showed that by considering an associated bundle with the flag manifold as fibers the characteristic classes can be defined in terms of those of line-bundles. As a consequence the dual homology class of a characteristic class of an algebraic manifold contains a representative algebraic cycle.

The study of the homology structure of a fiber bundle through the use of a connection merges local properties into global properties and combines differential geometry with differential topology. The general case of a principal bundle with an arbitrary Lie group as structure group, of which my work above is a special case concerning the unitary group, was carried out by Weil in 1949 in an unpublished manuscript. Part of Weil's results was presented in [I, 1]. The main conclusion is the so-called Weil homomorphism which identifies the characteristic classes (through the curvature forms) with the invariant polynomials under the action of the adjoint group. This identification, whose importance should be immediately recognizable, has recently been found crucial in the heat equation proof of the Atiyah–Singer index theorem and in Bott's theorem on foliations.

Actually, the characteristic forms themselves, which represent the characteristic classes via the de Rham theorem, contain more information. The vanishing of the characteristic forms, not just their classes, (which only means that the forms are exact), leads to the secondary characteristic classes. These were studied with James Simons in [98, 103]. The secondary characteristic classes depend on the choice of the connection, but enjoy strong invariance properties under a change of the connection. They have been found to play a role in various problems, such as conformal immersions and the η-invariant defined by the spectrum of a compact Riemannian manifold. A duality theorem for characteristic forms was given in a joint paper with White [108].

In [39] I determined the mod two cohomology ring of the real Grassmann manifold. As a consequence it follows that the Stiefel–Whitney classes generate the mod two characteristic ring of a sphere bundle. The result plays a role in the estimation of the number of closed geodesics on a compact Riemannian manifold.

When the base manifold has a complex structure, its ring of complex-valued exterior differential forms has also a more refined structure. Forms have a bidegree and there are two exterior differentiations, one with respect to the complex structure and the other to its conjugate complex structure, denoted usually by ∂, $\bar{\partial}$ respectively. In [80, 92] Bott and I studied the forms of a holomorphic Hermitian vector bundle relative to the operator $i\partial\bar{\partial}$. This has applications to complex geometry, and in particular to the study of the zeroes of holomorphic sections, which contains as a particular case the classical theory of value distributions of meromorphic functions.

In [101] I gave an elementary proof (without sheaf cohomology) of Bott's theorem on characteristic numbers and the residues of a meromorphic vector field on a compact complex manifold. The proof is in the spirit of a transgression.

On a manifold it is necessary to use covariant differentiation; curvature measures its non-commutativity. Its combination as a characteristic form measures the non-triviality of the underlying bundle. This train of ideas is so simple and natural that its importance can hardly be exaggerated.

6. Holomorphic Mappings

The simplest case of a holomorphic mapping is $\mathbb{C} \to P_1$, where $\mathbb{C}$ is the complex line and P_1 is the complex projective line. In usual terminology $\mathbb{C}$ is called the Gaussian plane and P_1 the Riemann sphere; the mapping is known as a meromorphic function. The geometrical basis of the classical value distribution theory consists of two theorems, known as the first and second main theorems, which are but the Gauss–Bonnet theorem applied to the Hopf bundle and the canonical bundle of P_1 respectively. From these the Nevanlinna defect relation follows by calculus-type inequalities.

In [70] these viewpoints were made precise by the study of holomorphic mappings of a non-compact Riemann surface into a compact one. As a differential geometer I have naturally been interested in the theory of a family of meromorphic functions interpreted as a holomorphic curve in P_n, the complex projective space of n dimensions, as developed by Henri Cartan, H. and J. Weyl, and Ahlfors. A geometrical treatment was given in [99] for P_2; the corresponding results for P_n were worked out by H. Yamaguchi in an unpublished manuscript. An essential ingredient for the good distributional behavior of a non-compact holomorphic curve lies in the validity of Frenet-type formulas. Cowan, Vitter, and I considered holomorphic curves in any complex manifold M and showed that Frenet formulas will be valid only when M has very special properties, which are close to being of constant holomorphic sectional curvature [104].

It is natural to consider holomorphic mappings in higher dimensions, a broad subject of which much remains to be understood. In [75] I gave some general observations. Following some work of H. Levine, done with my supervision, I studied in [71] a holomorphic mapping $f: C_n \to P_n$ and proved that under some growth conditions the set $P_n - f(C_n)$ is of measure zero.

In [80] Bott and I reformulated the value distribution problem as one on the distribution of zeroes of the holomorphics sections of a holomorphic vector bundle. A preparatory algebraic problem consists of the study of complex transgression, i.e., transgression relative to the operation $i\partial\bar{\partial}$ [80], [92]. Characteristic classes are defined in a refined sense, which is of importance in applications to problems pertaining to the holomorphic category.

In [88] I proved a Schwarz lemma in high dimensions as a volume-decreasing property. With S.I. Goldberg [106] an analogous theorem was proved for a class of harmonic mappings of Riemannian manifolds.

In [90] I introduced with H. Levine and L. Nirenberg intrinsic pseudo-norms in the real cohomology vector spaces of a complex manifold M. The definition utilizes the pluri-subharmonic functions. The pseudo-norm becomes a norm when there are enough pluri-subharmonic functions in M. Under a holomorphic mapping the pseudo-norm is a non-increasing function.

Geometry occupies an important position in complex function theory. Its role in several complex variables will be even greater in the future.

7. Minimal Submanifolds

The Grassmann manifold $\tilde{G}_{2,n}$ of all the oriented planes through a point in E^n has a complex structure invariant under the action of $SO(n)$. On the other hand, an oriented surface in E^n has a complex structure through its induced Riemannian metric. The surface is minimal if and only if the Gauss map is anti-holomorphic [79]. This theorem was proved by Pinl for $n = 4$ and is clearly the starting point in relating minimal surfaces with complex function theory. One of the fundamental theorems on minimal