# NUMBER THEORY AND COMBINATORICS JAPAN 1984

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World Scientific

*Published by*

**World Scientific Publishing Co Pte Ltd.**
**P. O. Box 128, Farrer Road, Singapore 9128**
**242, Cherry Street, Philadelphia PA 19106-1906, USA**

**Library of Congress Cataloging-in-Publication Data**

Number Theory and Combinatorics
Japan 1984

1. Numbers, Theory of — Congresses. 2. Combinatorial analysis — Congresses. I. Akiyama, J. II. Title.
QA241.C6885 1984 512'.7 85-22698
ISBN 9971-978-77-6

# FOREWORD

Professor Paul Erdős of the Hungarian Academy of Sciences visited Japan for about two weeks beginning on January 23, 1984. Although his name has been well-known in Japan and there are quite a few Japanese mathematicians who have met him before in Europe, in U. S. and elsewhere, he had never visited Japan up until then, and accordingly, many of us were very happy that his long awaited visit to Japan had finally become possible. In order to commemorate this happy occasion a number of conferences were organized; one was held in Tokyo on January 27-28 devoted to analytic number theory and related areas, another in Okayama on January 30-31 dealing with the connection between number theory and analysis, and the third in Kyoto on February 1 concentrating on combinatorics.

In these proceedings we put together the invited lectures delivered by a number of mathematicians at the above mentioned conferences, and papers contributed by other mathematicians attending the conferences. Some of these works are concerned with problems in analytic and elementary number theory, while others treat problems in combinatorics related with number theoretic questions; needless to say, the development of many of these questions was influenced one way or the other by Professor Erdős. We should mention that all the papers included in this volume had been sent to appropriate referees, and that they received the approval of the referees before they were accepted for publication by the editorial committee.

For the editors,
J. Akiyama
Y. Ito
I. Shiokawa

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# ON THE CONVERGENCE OF $\sum_{n=1}^{\infty} n^{-d}$ EXP $(2\pi i n^{\beta}\theta)$

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0. In [1] it is proved that $\sum_{n=1}^{\infty} n^{-\alpha}\sin(n^{\beta}\theta)$ is convergent for all $\theta>0$ if $1<\beta<2\alpha$. Now in this paper we improve it to prove the following theorems.

We note their connection with [4].

<u>Theorem 1</u> If $1 > \alpha > \xi = \max(\frac{\beta}{2} - \rho(\beta-1),\ \beta-1)$, $2 > \beta > 1$, and if $\nu=\beta/(\beta-1)$ is not an integer, then

$$(1) \qquad \sum_{n=1}^{\infty} n^{-\alpha}\exp(2\pi i n^{\beta}\theta) \qquad (\ i^2 = -1\ )$$

converges for all $\theta>0$, where $\rho = (600[2+\nu]^2\log[2+\nu])^{-1}$.

<u>Theorem 2</u> If $1 > \alpha > \xi = \max(\frac{\beta}{2} - \rho'(\beta-1),\ \beta-1)$, $2 > \beta > 1$, $\nu=\beta/(\beta-1)$ is an integer and $\theta^{\nu-1}$ is irrational, then (1) converges, where

$$\rho' = \left(\frac{\tau}{3(\nu-1)^2\log(12(\nu-1)\nu/\tau)}\right)\left(1+\frac{1}{30(\nu-1)}\right) \quad \text{if } \nu \geqq 12,$$

$$\rho' = \left(\frac{\tau}{3\cdot 11^2\log(12\cdot 11\cdot 12/\tau)}\right)\left(1+\frac{1}{30\cdot 11}\right) \quad \text{if } 3 \leqq \nu \leqq 11.$$

Here $\tau$ is defined in Lemma 4.

<u>Theorem 3</u> If $0 < \alpha < 1$, $2\alpha \leqq \beta < 2$, $\beta > 1$ and $\nu=\beta/(\beta-1)$ is an integer, then (1) diverges for some $\theta>0$, where $\theta^{\nu-1}$ is rational: for example, if $\theta=K_0^{-(\beta-1)}(\beta-1)^{\beta-1}\beta^{-\beta}$ and $K_0$ is a positive integer, or if $\theta=$

$(a_0/p^\mu)^{-(\beta-1)}(\beta-1)^{\beta-1}\beta^{-\beta}$, p is a prime, $a_0$ and $\mu$ are positive integers, $\mu>1$, $\mu\leqq\nu$, $(a_0,p)=(\nu,p)=1$.

Theorem 4 If $0<\alpha<1$, $2>\beta>-2\alpha+2$, $\nu=\beta/(\beta-1)$ is an integer and $\theta^{\nu-1}$ is rational, then (1) converges for $\theta=(a_0/p)^{-(\beta-1)}(\beta-1)^{\beta-1}\beta^{-\beta}$, where p is a prime, $a_0$ is a positive integer and $(a_0,p)=(\nu,p-1)=1$.

1. Lemmas.

Lemma 1 (cf.[5]) We put

$$I(X) = \int_a^b \exp(iXp(t))dt. \tag{2}$$

In (2), a, b and the function p(t) are real and independent of the positive parameter X, a being finite and b(>a) finite or infinite. The function p(t) is required to satisfy the properties (i)∼(iii) below:

(i) In (a,b) p'(t) is continuous and positive.

(ii) p(t) can be expanded in powers of t-a with a nonzero radius of convergence. p'(a)=0. Then

$$p(t) = p(a) + \sum_{s=0}^{\infty} p_s(t-a)^{s+2}, \text{ where } p_0 \neq 0. \tag{3}$$

The above expansion is the Taylor series.

(iii) $p(b) = \lim_{t\to b-} p(t)$ is finite and $P_0(t) = 1/p'(t)$ tends to a finite limit as $t \to b-$.

Otherwise, $p(b) = +\infty$, $\lim_{t\to b-} P_0(t) = 0$ and (2) converges at $t = b$ uniformly for all sufficiently large X.

Then we have

$$I(X) = \exp(iXp(a))\exp(\tfrac{\pi}{4}i)\Gamma(\tfrac{1}{2})\frac{1}{2\sqrt{p_0X}} + \delta(X) + O(X^{-1}) \text{ if } p(b)<+\infty,$$

$$I(X) = \exp(iXp(a))\exp(\tfrac{\pi}{4}i)\Gamma(\tfrac{1}{2})\frac{1}{2\sqrt{p_0X}} + \delta(X) \quad \text{if} \quad p(b)=+\infty.$$

$$|\delta(X)| \leq [|Q_{1,1}(a)| + |Q_{1,1}(b)| + V_{a,b}\{Q_{1,1}(t)\}]X^{-1},$$

where $Q_{1,1}(t) = P_0(t) - 1/(2\sqrt{p_0}\sqrt{p(t)-p(a)})$, and $V_{a,b}$ denotes the total variation of the function $Q_{1,1}(t)$.

Lemma 2  Assume the conditions and notations of Lemma 1. Let $p'(t)$ be negative in $(a,b)$, and when $b$ is finite, $p'(b)=0$ and let

$$p(t) = p(b) + \sum_{s=0}^{\infty} r_s(b-t)^{s+2}, \text{ where } r_0 \neq 0. \tag{4}$$

Then we have

$$I(X) = \exp(iXp(b))\exp(\tfrac{\pi}{4}i)\Gamma(\tfrac{1}{2})\frac{1}{2\sqrt{r_0X}} + \delta(X) + O(X^{-1}).$$

Proof  In the proof of Lemma 1, put $v=p(t)-p(b)$ ((3.1) of [5]), and we can prove this lemma similarly.

Lemma 3 (A variant of Theorem 5 in Chapter IV of [2])  Let $h$ and $Q$ be integers ($h\geq 4$, $Q\geq 2$), and let $g(x)$ be real-valued and have continuous derivatives up to the $(h+1)$th order in $[P+1,P+Q]$.

Suppose there exists a $\lambda \in (0,1)$ such that for all $x \in [P+1,P+Q]$

$$\lambda \leq \frac{g^{(h+1)}(x)}{(h+1)!} \leq 2\lambda, \tag{5}$$

or the same for $-g^{(h+1)}(x)$ in place of $g^{(h+1)}(x)$, where

$$\lambda^{-1/3} \leq Q \leq \lambda^{-1}. \tag{6}$$

Then

$$(7) \qquad \left| \sum_{n=P+1}^{P+Q} e^{2\pi i g(n)} \right| \leq K\exp(ch(\log^2 h))Q^{1-\rho},$$

where $\rho = (600h^2\log h)^{-1}$, and c, K are absolute constants.

The proof can be easily obtained if we choose

$$\eta = (9h^2\log h)^{-1},$$

$$q = [\lambda^{-(1-\eta)/(h+1)}] + 1$$

in Theorem 4 in Chapter IV of [2].

First, in page 104, line 10, we replace

$$(2\lambda(k+1)Q+1)\left(\frac{2q}{k+1}+1\right) \leq (2k+3)\left(\frac{2q}{k+1}+1\right) \leq 3k\left(\frac{q}{4}+q\right) < 4kq,$$

since $Q \leq \lambda^{-1}$, $q \geq 1$, $k \geq 7$

by

$$(2\lambda(h+1)Q+1)\left(\frac{2q}{h+1}+1\right) \leq (2h+3)\left(\frac{2q}{h+1}+1\right) \leq 3h\left(\frac{2}{5}q+1\right) < 3hq,$$

since $Q \leq \lambda^{-1}$, $q \geq 2$, $h \geq 4$.

Next, in page 104, line 1 below, we replace

$$|C| \leq 4Q^{1-1/(21)}\{4k\lambda^{-k}q^{3/2-k(k+1)}e^{c1k(\log^2 k)}\}^{1/(21)}+8\pi k\lambda q^{k+1}Q + q$$

by

$$|C| \leq 4Q^{1-1/(21)}\{3h\lambda^{-h}q^{3/2-h(h+1)}e^{c1h(\log^2 h)}\}^{1/(21)}+ 8\pi h\lambda q^{h+1}Q + q.$$

Thirdly, in page 105, lines 2～4, we replace

$$q \leqq Q^{4/(k+1)}, \quad \lambda q^{k+1} \leqq \lambda^{\eta} \leqq Q^{-\eta},$$

$$\lambda q^{k+1} \geqq 2^{-(k+1)}\lambda^{\eta} \geqq 2^{-(k+1)}Q^{-4\eta}$$

by

$$q \leqq 2Q^{3/(h+1)}, \quad \lambda q^{h+1} \leqq 2^{h+1}\lambda^{\eta} \leqq 2^{h+1}Q^{-\eta},$$

$$\lambda q^{h+1} \geqq \lambda^{\eta} \geqq Q^{-3\eta}.$$

Fourthly, in page 105, line 6, we replace

$$|C| \leqq c_2 \exp(c_1 k(\log^2 k))Q^{1-1/(21)}Q^{2\eta k/1}Q^{3/((k+1)1)} + 8\pi k Q^{1-\eta} + Q^{4/(k+1)}$$

by

$$|C| \leqq c_2 \exp(c_1 h(\log^2 h))Q^{1-1/(21)}Q^{3\eta h/(21)}Q^{9/(41(h+1))} + 2^{h+4}\pi h Q^{1-\eta} + 2Q^{3/(h+1)}.$$

Finally, in page 105, line 8, we replace

$$\frac{1}{2} - 3(k+1)^{-1} - 2\eta k \geqq \frac{1}{2} - \frac{3}{8} - \frac{1}{21} \geqq \frac{1}{14}$$

by

$$\frac{1}{2} - \frac{9}{4(h+1)} - \frac{3}{2}\eta h \geqq \frac{1}{2} - \frac{9}{20} - \frac{1}{24} = \frac{1}{120}.$$

<u>Lemma 4</u> (cf.[7] Chapter VI Theorem I) Let $Q$, $n(\geqq 11)$ be positive integers. Let

$$f(x) = a_{n+1}x^{n+1} + \ldots.. + a_1 x,$$

where $a_{n+1}(\neq 0),\ldots,a_1$ are real, and let

$$S_1 = \sum_{x=1}^{Q} \exp(2\pi i f(x)).$$

Let r be one of the numbers n+1,...,2, and suppose that

$$a_r = \frac{a}{q} + \frac{\omega}{q^2} \quad ((a,q)=1,\ |\omega| \leqq 1 \text{ and } q>0).$$

Then

$$S_1 \ll Q^{1-\rho'},$$

where $$\rho' = \left(\frac{\tau}{3n^2 \log(12n(n+1)/\tau)}\right)\left(1 + \frac{1}{30n}\right).$$

Here $\tau$ is defined in terms of q and Q by

| | | |
|---|---|---|
| (i) | $q = c_1 Q^{\tau}$ | for $1 < q \leqq c_1 Q$, |
| (ii) | $\tau = 1$ | for $c_1 Q \leqq q \leqq c_2 Q^{n-1}$, |
| (iii) | $q = c_2 Q^{n-\tau}$ | for $c_2 Q^{n-1} \leqq q < c_2 Q^n$, |

where $c_1$, $c_2$ are any selected positive constants (e.g. $c_1 = c_2 = 1$).

Thus $0 < \tau \leqq 1$ always.

Lemma 5 (cf.[3] Theorem 185) If $\xi$ is irrational, then there are infinitely many fractions p/q which satisfy

$$|p/q - \xi| < q^{-2} \quad ((p,q)=1).$$

Lemma 6 (cf.[7] Chapter II Lemma 4) Let $\mu$, a and $n(\geqq 3)$ be integers. Let p be a prime, and suppose that $1<\mu\leqq n$ and $(a,p)=(n,p)=1$. Then

$$S(a,p^{\mu}) = \sum_{x=0}^{p^{\mu}-1} \exp\left(2\pi i \frac{a}{p^{\mu}} x^n\right) = p^{\mu-1}.$$

Lemma 7 (cf.[7] Chapter II Lemma 5) Let $\mu$ be an integer and $\mu > n(\geqq 3)$. Then

$$S(a,p^{\mu}) = p^{n-1} S(a,p^{\mu-n}).$$

Lemma 8 (cf.[7] Chapter II Lemma 3) Assume the conditions and notations of Lemma 6. Then

$$|S(a,p)| \leq (\delta-1)p^{1/2}, \text{ where } \delta=(n,p-1).$$

Lemma 9 (cf.[7] Chapter II Lemma 1) For any integers $q_1,\dots,q_k$ which are relatively prime in pairs, we have

$$S(a_1,q_1)\dots S(a_k,q_k) = S(a_1Q_1+\dots+a_kQ_k,q_1\dots q_k),$$

where $Q_s = q_1\dots q_k q_s^{-1}$ for $s = 1,2,\dots,k$.

2. Proof of Theorems.

Proof of Theorem 1 Put

$$S(N) = \sum_{n=1}^{N} \exp(2\pi i f(n)), \tag{8}$$

where $f(x) = \theta x^{\beta}$ $(1/2 \leqq x \leqq N)$ and $1<\beta<2$.

Since $f'(x)$ is strictly increasing, there exists an $\alpha_r \in [1/2,N]$ such that $f'(\alpha_r) = r-1/2$, where $r$ is a positive integer. All $\alpha_r$ belong to the set $\{ \alpha_1>0,\dots,\alpha_M,\alpha_{M+1} \}$, where $M = [\beta\theta N^{\beta-1}] \geqq 1$.

Then we have

$$S(N) = \sum_{n\leqq\alpha_1} + \sum_{\alpha_1<n\leqq\alpha_2} +\dots+ \sum_{\alpha_M<n\leqq\alpha_{M+1}} + \sum_{\alpha_{M+1}<n\leqq N}, \tag{9}$$

where any empty sum is zero.

Since $f''(x) > 0$ and $f'(\alpha_1) = \frac{1}{2}$, we have

$$|f'(x)| \leqq \frac{1}{2} \quad \text{for } \frac{1}{2} \leqq x \leqq \alpha_1.$$

By van der Corput's lemma (cf.[6] Lemma 4.8 and Lemma 4.2), we have

$$\sum_{n \leq \alpha_1} \exp(2\pi i f(n)) = \int_{1/2}^{\alpha_1} \exp(2\pi i f(x))dx + O(1)$$

$$= O\left( \max_{1/2 \leq x \leq N} \frac{1}{\sqrt{f''(x)}} \right) + O(1).$$

Similarly, since $\exp(2\pi i f(n)) = \exp(2\pi i(f(n)-(M+1)n))$, we have

$$\sum_{\alpha_{M+1} < n \leq N} \exp(2\pi i f(n)) = O\left( \max_{1/2 \leq x \leq N} \frac{1}{\sqrt{f''(x)}} \right) + O(1).$$

Therefore it follows from (9) that

$$S(N) = \sum_{r=1}^{M} S_r + O\left( \max_{1/2 \leq x \leq N} \frac{1}{\sqrt{f''(x)}} \right) + O(1), \tag{10}$$

where $S_r = \sum_{\alpha_r < n \leq \alpha_{r+1}} \exp(2\pi i f(n))$.

Now we have

$$\max_{1/2 \leq x \leq N} \frac{1}{\sqrt{f''(x)}} \ll \theta^{-1/2} N^{1-\beta/2},$$

since $f''(x) = \beta(\beta-1)\theta x^{\beta-2}$. Let us write

$$S_r = \left( S_r - \int_{\alpha_r}^{\alpha_{r+1}} \exp(2\pi i(f(x)-rx))dx \right) + \int_{\alpha_r}^{\alpha_{r+1}} \exp(2\pi i(f(x)-rx))dx.$$

Then we have, by van der Corput's lemma (cf.[6] Lemma 4.8),

$$S_r = \sum_{\alpha_r < n \leq \alpha_{r+1}} \exp(2\pi i(f(n)-rn))$$

$$= \int_{\alpha_r}^{\alpha_{r+1}} \exp(2\pi i(f(x)-rx))dx + O(1)$$

(11) $$= I_r + O(1), \text{ say.}$$

We now consider the integral $I_r$, and put $F(x)=2\pi(f(x)-rx)$. Then

$$F'(x)= 2\pi(f'(x)-r) = 2\pi(\beta\theta x^{\beta-1}-r),$$

$$F''(x)= 2\pi\beta(\beta-1)\theta x^{\beta-2} \geqq 0 \qquad \text{for } 0 \leqq x \leqq \alpha_r,$$

and $$F'(x) \leqq F'(\alpha_r) = -\pi < 0 \qquad \text{for } 0 \leqq x \leqq \alpha_r.$$

Hence it follows from van der Corput's lemma (cf.[6] Lemma 4.2) that

(12) $$\left| \int_0^{\alpha_r} \exp(2\pi i(f(x)-rx))dx \right| \leqq \frac{4}{\pi}.$$

Similarly

(13) $$\left| \int_{\alpha_{r+1}}^{\infty} \exp(2\pi i(f(x)-rx))dx \right| \leqq \frac{4}{\pi}.$$

We put

$$J_r = \int_0^{\infty} \exp(2\pi i(f(x)-rx))dx,$$

and then

$$I_r = J_r - \int_0^{\alpha_r} - \int_{\alpha_{r+1}}^{\infty}.$$

Therefore it follows from (12),(13) that

(14) $$I_r = J_r + O(1).$$

In the integral $J_r$ we put

$$x = \frac{X}{r}t, \text{ where } X = \theta^{-1/(\beta-1)} r^{\beta/(\beta-1)}.$$

Then

$$(15)\qquad \frac{r}{X}J_r = \int_0^{\infty} \exp(2\pi i X p(t))dt,$$

where $\quad p(t) = t^{\beta} - t.$

We divide the range of the above integral at the stationary point $t=t_0=\beta^{-1/(\beta-1)}$, that is, $p'(t_0)=0$.

Hence we have

$$(16)\qquad \int_0^{\infty} \exp(2\pi i X p(t))dt = \int_0^{t_0} + \int_{t_0}^{\infty}$$

$$= J_{(1)} + J_{(2)}, \text{ say.}$$

By Lemmas 1 and 2, we have

$$(17)\qquad J_{(1)} = \exp(\tfrac{\pi}{4}i)\exp(2\pi i X p(t_0))\sqrt{\frac{\pi}{X}}\frac{1}{2\sqrt{p_0}} + O(X^{-1}),$$

$$(18)\qquad J_{(2)} = \exp(\tfrac{\pi}{4}i)\exp(2\pi i X p(t_0))\sqrt{\frac{\pi}{X}}\frac{1}{2\sqrt{r_0}} + O(X^{-1}),$$

where $\quad p_0 = r_0 = \frac{1}{2}(\beta-1)\beta^{1/(\beta-1)}.$

Therefore it follows from (15),(16),(17),(18) that

$$J_r = \frac{X}{r}(J_{(1)} + J_{(2)})$$

$$= \frac{X}{r}\exp(\tfrac{\pi}{4}i)\exp(2\pi i X p(t_0))\sqrt{\frac{\pi}{X}}\frac{1}{\sqrt{p_0}} + O(r^{-1})$$

$$= \sqrt{2\pi}\exp(\tfrac{\pi}{4}i)(\beta-1)^{-1/2}\beta^{-1/(2\beta-2)}\theta^{-1/(2\beta-2)}r^{(2-\beta)/(2\beta-2)}\exp\{-2\pi i(\beta-1)$$

$$\beta^{-\beta/(\beta-1)}\theta^{-1/(\beta-1)}r^{\beta/(\beta-1)}\} + O(r^{-1}).$$

(19) $$= A\theta^{-1/(2\beta-2)} r^{(2-\beta)/(2\beta-2)} \exp(2\pi i K r^{\beta/(\beta-1)}) + O(r^{-1}),$$

where we have put

$$A = \sqrt{2\pi}\, \exp(\tfrac{\pi}{4} i)(\beta-1)^{-1/2} \beta^{-1/(2\beta-2)} ,$$

$$B = -(\beta-1)\beta^{-\beta/(\beta-1)},$$

$$K = B\theta^{-1/(\beta-1)}.$$

Then we have to estimate

(20) $$T_M = \sum_{r=1}^{M} r^{(2-\beta)/(2\beta-2)} \exp(2\pi i K r^{\beta/(\beta-1)}).$$

Put

(21) $$U_M = \sum_{r=1}^{M} \exp(2\pi i K r^{\beta/(\beta-1)}),$$

then we have, by partial summation,

(22) $$T_M \ll M^{(2-\beta)/(2\beta-2)} \max_{1 \leqq v \leqq M+1} |U_v| .$$

Next we shall estimate $U_M$.

When $\nu=\beta/(\beta-1)$ is not an integer, we put $g(x)=Kx^{\beta/(\beta-1)}$. Then there exists a positive integer h such that

(23) $$\frac{g^{(h+1)}(x)}{(h+1)!} = \frac{C}{x^{3-\gamma}} ,$$

where $C$ is a constant and $0 < \gamma < 1$.