Richard J. Trudeau

The Non-Euclidean Revolution

With an Introduction by H.S.M. Coxeter

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Preface

epistemology n. The study of the nature and origin of knowledge.

When I was in my early teens I felt adults were lying to me much of the time. They made dogmatic statements they couldn't defend. "God will punish you if you take His name in vain," they said. "A sentence must never end with a preposition.[1]" At the time what confidence I had was derived mostly from my ability to reason, so I reacted intellectually. I became obsessed with discovering what, if anything, is *true*, and in what sense.

About this time two things happened. I studied plane geometry, which I found fascinating, and I noticed that the phrase "mathematically proven" was a folk-wisdom synonym for "absolutely certain." I concluded that if absolute truth is to be found anywhere it must lie in mathematics.

In college I studied mathematics and philosophy. I learned to formulate my epistemological knot more precisely: to what extent is mathematics the truth? But I made little progress untangling it.

After college, what with graduate school, adjusting to work, and learning to like myself and trust my emotions, the knot was pushed to the back of my mind.

I became a college mathematics teacher. In the spring of 1971 my chairman asked me to develop a course for the fall term in non-Euclidean geometry. I had heard of non-Euclidean geometry but had never studied it. That summer I did study it, along with the 2100-year-old controversy that had culminated in its invention, and suddenly I was able to untangle my long-neglected epistemological knot in a most satisfactory manner. I felt that at last I understood the extent to which mathematics is true, more importantly the extent to which it is not, and by inference the extent to which any general statement in science or philosophy can claim to be true. It was a heady experience; I felt as if I had been transported to a vantage from which I could see—actually see—the limits of reason.

What I learned that summer was that a struggle with the notion of *mathematics as truth* similar to my own had unfolded in mathematical and philosophical circles from about 400 B.C. into the 19th century; that it had climaxed in the first half of the 19th century with the invention of non-Euclidean geometry; and that as a result over the second half of that century mathematicians and scientists changed the way they viewed their subjects. An entire scientific revolution had taken place that I had never heard of!

Moreover, prescinding from my special interest in the matter, I felt that this intellectual adventure I had stumbled upon made a terrific story. Thus this book; for, being a teacher, whenever I hear a good story I immediately want to retell it, in my own way, to someone else.

I presuppose that you studied plane geometry in high school. However I do not expect that you have done so recently, or that you did particularly well in the course, or that you remember much about it. And while I occasionally draw upon high school algebra to illustrate a point, if you've never studied that subject you won't be at any real disadvantage.

The book proceeds on three levels. On one it's just a geometry book with extra material on history and philosophy. For a while we will talk about Euclidean geometry—the "plane geometry" of high school—then switch to "hyperbolic geometry," another plane geometry invented around 1820. We will compare the two and reflect on what we have done.

On another level this book is about a scientific revolution, every bit as significant as the Copernican revolution in astronomy, the Darwinian revolution in biology, or the Newtonian or 20th-century revolutions in physics, but which is largely unsung because its effects have been more subtle—a revolution brought about by the invention of an alternative to traditional Euclidean geometry. Hyperbolic geometry is as logically consistent as Euclid's, has as much claim to being "true" as Euclid's, and yet extensively *contradicts* Euclid's. In Euclidean geometry the angles of a triangle add up to 180°; in hyperbolic geometry they add up to less, and the sum varies from triangle to triangle. In Euclidean geometry the Theorem of Pythagoras[2] holds; in hyperbolic geometry it does not. The effect of this paradoxical situation on 19th-century mathematicians and scientists was profound. Mathematicians embarked on an agonizing reappraisal of their subject that would last for decades; and scientists found themselves asking whether science wasn't in fact a very different thing than they had always thought.

On the third and most speculative level this book is about the possibility of significant, absolutely certain knowledge about the world. It offers striking evidence—though of course it cannot prove—that such knowledge is impossible.

I said that I assume you have studied Euclidean geometry. If in addition—and I think this is likely—you have *not* studied non-Euclidean geometry, and your epistemology of mathematics is as nebulous as mine used to be, then the story I retell in this book will provide you a rare opportunity to actually *experience* the intellectual and intuitive disorientation scientific revolutions cause. In fact the opportunity may be unique. If you are an average educated person it would probably be difficult for you, reading an account of one of the

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other scientific revolutions I mentioned, to feel the confusion (and excitement!) that originally surrounded the event, because *you already believe* the once-revolutionary theory to be substantially correct. You have been brought up to believe the earth moves around the sun and is held to its path by gravity (so much for Copernicus and Newton); you may have doubts about the specific mechanism Darwin proposed to *explain* evolution, but you probably consider it a fact that evolution has occurred, which is what the fuss was really about; and while you may not know much about 20th-century physics, the spectacle of a nuclear explosion is terrifying proof that there is something to it. With regard to geometry, however, you are almost certainly a committed Euclidean, and consider the possibility of a logical, "truthful" geometry contradicting Euclid's to be absurd. You are like a 16th-century astronomer hearing of Copernicanism for the first time.

The coming of non-Euclidean geometry was basically a mathematical event, so learning about it involves reading mathematics. Mathematics is more demanding than light fiction, so take your time. Don't try to push on when you're tired. In parts of this book you may not want to read more than two or three pages at a sitting.

On the other hand, this book is supposed to be fun. Feel free to skip parts too technical for your taste. You will be able to pick up the thread again, after the technicalities subside. Feel free, especially, to skip proofs (in Chapters 2, 4, and 6). They are the bookkeeping, included to show that matters stand as I say they do, but skippable if you'd as soon take my word for it.

I have included some exercises, in case you'd like to try your hand. If not, they too can be skipped without loss of continuity.

Richard J. Trudeau

Notes

[1] *preposition.* The record for largest number of consecutive terminal prepositions (five) seems to be held by the child's question at the end of the following anecdote, whose author I have unfortunately been unable to trace.

A child is in bed with a cold. "Mommy, can you come up and read me a story?" he calls down to his mother. As she reaches the top of the stairs, he recognizes the book in her hand as one he doesn't care for. He asks, "What did you bring the book that I don't like to be read to out of up for?"

[2] *Theorem of Pythagoras.* In every right triangle, $c^2 = a^2 + b^2$ where c denotes the length of the longest side and a and b the lengths of the other two.

Introduction

Felix Klein described non-Euclidean geometry as "one of the few parts of mathematics which is talked about in wide circles, so that any teacher may be asked about it at any moment." This old observation is now reinforced by our knowledge that astronomical space is only approximately Euclidean. Trudeau's book provides the reader with a non-technical description of the progress of thought from Plato and Euclid to Kant, Lobachevsky, and Hilbert. There is a pleasantly discursive treatment of Pontius Pilate's unanswered question "What is truth?" The text is enlivened by abundant quotations and amusing cartoons. The final chapter includes a clear account of experiments which seem to indicate that the world as seen by our two eyes is not even approximately Euclidean but hyperbolic!

H. S. M. Coxeter

Contents

CHAPTER 1

First Things

The Origin of Deductive Geometry

Once upon a time—around 600 B.C.—there was a man named Thales, who invented what we call "science."

Before Thales, thinkers did not think abstractly. Instead of looking for principles behind the curious events with which nature confronted them, they looked for personalities. Their findings—myths—were stories populated with an assortment of gods and goddesses whose interactions with one another and with human beings produced natural phenomena like spring, thunder, eclipses, etc.

Nowadays it is common to sneer at ancient myths, but they were created by men and women of genius. The myths provided a comprehensive explanation of natural phenomena and a link between humanity and nature that made the universe less frightening. In fact science has only one advantage over myth, by also predicting natural phenomena to a degree myth never could.

(By the way, this story I'm telling about pre-Euclidean geometry is itself a sort of myth. It credits a few legendary characters with subtle intellectual developments that must actually have involved numerous people over considerable time. The story has evolved, in the virtual absence of hard data, from a few legends, mathematicians' longing to know the origin of their subject, and their sense, as mathematicians, of what the milestones probably were.)

Thales (c.625–c.547) believed nature operated not by whimsy, or by the gods' romantic entanglements, but by principles intelligible to human beings. Thales introduced abstraction into the contemplation of nature.

In particular, Thales introduced abstraction into geometry. Before Thales, "geometry" had meant "surveying" (the Greek *geometrein* means "to measure land"), and geometric figures had been particular objects like corrals and fields. Instead Thales conceived of geometric figures as abstract shapes. This enabled him, when he examined in this light the hodgepodge of geometric recipes, rules-of-thumb, and empirical formulae that had been transmitted from Babylonia and Egypt,[1] to detect an order. He noticed that some geometric facts were deducible from others. And he made the extraordinary suggestion that geometry should become, as much as possible, a purely mental activity.

Greek Civilization in 550 B.C.

Thales was Greek, of course, and lived in a city that was then at the center of Greek culture: Miletos, on the western coast of Asia Minor (currently Turkey). Just a few miles from Miletos there is an island called Samos, where Pythagoras was born when Thales was in his fifties.

When Pythagoras (c.570–c.495) grew up he learned of Thales' scientific ideas. He was particularly captivated by Thales' proposal for geometry.

Pythagoras left Asia Minor, and for a while studied in Egypt. Eventually he settled in Kroton, a Greek city in southern Italy, at the ball of the foot. There he founded the "Society of Pythagoreans," a community of men and women sharing quasi-religious rituals, dietary laws, and devotion to mathematics as the key to understanding nature. Though the actual community lasted only a few decades, its doctrines continued to influence Greek thought for much longer. Centuries later various thinkers around the Mediterranean were still calling themselves "Pythagoreans" and professing the Pythagorean belief in the primacy of mathematics among the sciences. To an extent the Society has affected Western thought down to the present, for Plato (c.427–347) was strongly influenced by Pythagorean ideas. ("All philosophy is a series of footnotes to Plato."–A. N. Whitehead.)

The Pythagoreans accepted Thales' program of making geometry a deductive science. To this end their greatest contribution was a dramatic discovery that helped set the standard of proof. Thales had deduced his theorems by a combination of logic and intuitive reflection. The Pythagoreans discovered that logic and intuition can disagree!

Here's what happened. Let AB and CD be two straight line segments (see Figure 1). We will say that a straight line segment XY is a "common measure"

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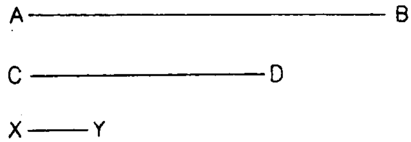

Figure 1

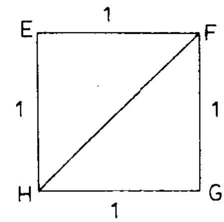

Figure 2

of *AB* and *CD* if there are whole numbers m and n so that *XY* laid end-over-end m times is the same length as *AB* and *XY* laid end-over-end n times is the same length as *CD*. For example if *AB* were a yard long and *CD* 10 inches, a segment *XY* of 2 inches would be a common measure with $m = 18$ and $n = 5$; for laying *XY* end-over-end eighteen times would produce a length of 36 inches, the same as *AB*, and laying *XY* end-over-end five times would produce a length of 10 inches, the same as *CD*. It was intuitively evident to the early Pythagoreans (and as I write it is intuitively plausible to me) that a common measure can be found for *any* pair of segments—though of course it may be necessary to take *XY* quite small in order to measure both *AB* and *CD* exactly. Since $AB/CD = (m \cdot XY)/(n \cdot XY) = m/n$, a "rational" number (that is, a ratio of whole numbers), what their intuition predicted was that the quotient of two lengths would always come out rational.

Now take a square with side equal to 1 and draw a diagonal (see Figure 2). Applying the Theorem of Pythagoras (p. viii) to the right triangle *FGH* we get $FH^2 = FG^2 + GH^2 = 1^2 + 1^2 = 2$, so $FH = \sqrt{2}$ and therefore the quotient FH/FG of the two lengths *FH* and *FG* is equal to $\sqrt{2}/1 = \sqrt{2}$ also. If the early Pythagoreans had been correct that the quotient of two lengths is always rational, $\sqrt{2}$ would then be rational. But one of the later Pythagoreans (probably Hippasos of Metapontion,[2] after 430 B.C.) discovered, by an argument not based (primarily) on intuition, that $\sqrt{2}$ is *not* rational.

The proof went something like this. Any rational number can be "reduced to lowest terms," that is, expressed by whole numbers having no whole number factor (other than 1) in common; for example $360/75 = 24/5$ and 24 and 5 have on common factor. Therefore if $\sqrt{2}$ were rational it would be possible to express it as $\sqrt{2} = p/q$ where p and q are whole numbers with no common factor. Squaring both sides gives $2 = p^2/q^2$, and multiplying both sides by q^2 gives $2q^2 = p^2$. This means p^2 is even, because it is twice another whole number. The Pythagoreans has previously proven that only even numbers have even squares,[3] so they knew that, since p^2 is even, p must be even also. This has two consequences:

(1) p is twice some other whole number (this is what being "even" means) which we can call "r," so $p = 2r$; and

(2) q is odd, for we said p and q have no common factor, and an even q would have a factor 2 in common with p.

We will pursue (1). Substituting $2r$ for p in the equation $2q^2 = p^2$ (above), we get $2q^2 = (2r)^2$ or $2q^2 = 4r^2$. Dividing both sides by 2 gives $q^2 = 2r^2$ so q^2, being twice a whole number, is even. As before this implies that q is even (only even numbers have even squares). But we just said in (2) that q is odd! As the hypothesis that $\sqrt{2}$ is rational has led to this contradiction, logic forces us to conclude that $\sqrt{2}$ is not rational.[4]

At this point the Pythagoreans were perplexed. They were *sure*, on intuitive grounds, that $\sqrt{2}$, being the quotient of two lengths, is a rational number. On the other hand they were equally sure, on grounds of logic and computation, that $\sqrt{2}$ is *not* a rational number!

Had the mathematical world decided to accept intuition as more reliable than logic the future of mathematics would have been quite different; but it did decide in favor of logic,[5] and mathematicians ever since have been trained to revere logic and mistrust intuition. (I think this has something to do with the generalization that mathematicians are "cold" people.)

To say mathematicians consider intuition unreliable, however, is not to say they have banished it from mathematics. On the contrary, the basic assumptions from which any branch of mathematics proceeds—the "axioms"—are accepted, without proof, primarily because of intuitive appeal. And intuition plays a big role in the discovery of theorems as well, or mathematicians would be spending most of their time trying to prove false statements. It's just that intuitive evidence is not accepted as conclusive.

The Pythagorean heritage is what modern mathematicians call "rigor," a habit of mind characteristic of mathematics. Every effort is made to insulate the subject from its down-to-earth origins. Terms are defined and principles formulated with constant vigilance against unstated assumptions. Theorems are derived by logic alone.

In the 5th century B.C., before mathematics was made rigorous, mathematicians had already constructed long chains of geometric theorems in which each theorem was deduced, informally, from those before it. Each chain started with generalizations from experience which of course were not proven.

As the scope of these chains grew there emerged the daring idea that it might be possible to link them together into a single network, anchored to a small number of generalizations from experience, which would contain a broad inventory of elementary geometric knowledge. And toward the end of the century—in fact, about the same time it was proven that $\sqrt{2}$ is not rational—a mathematician named Hippokrates of Chios[6] accomplished exactly this, in a book he called the *Elements*.

Later, while the rigorization of mathematics was underway, other comprehensive geometric networks[7] were forged. Each was called the *Elements*, and presumably each, by having simpler axioms, tighter logic, or more theorems, was an improvement on its predecessors. The series culminated in the famous *Elements* of Euclid, completed about 300 B.C.

Euclid's *Elements* is a single deductive network of 465 theorems that includes not only an enormous amount of elementary geometry, but generous helpings of algebra and number theory as well. Its organization and level of logical rigor were such that it soon became geometry's standard text. In fact it so completely superseded previous efforts that they all disappeared.

The *Elements*—from now on the title will refer to Euclid's book only—is the most successful textbook ever written. It has gone through more than a thousand editions and was used well into the last century (here and there, it is used even today). More importantly, it is the paradigm that scientists have been emulating ever since its appearance. It is the archetypal scientific treatise. To study the form and limitations of the *Elements*, therefore, is to poke through the entrails of the whole scientific enterprise.

Given the stature of this work, surprisingly little is known about its author. Scholars even hesitate to conjecture Euclid's dates, except to say that he "flourished" about 300 B.C. Just then the center of scientific and mathematical activity was shifting from Athens to Alexander the Great's new city Alexandria at the mouth of the Nile. Euclid lived in Alexandria, where he was a professor of mathematics at the Museum (the university). Beyond this all that is known about Euclid is contained in two anecdotes. In one a beginning student of geometry asks him, "What shall I get by learning these things?" Euclid responds by calling a servant and saying, "Give him a coin, since he must make gain out of what he learns." In the other the king, Ptolemy I,[8] asks him, "Is there in geometry any shorter way than the *Elements*?"—to which Euclid replies, "There is no royal road to geometry."

Material Axiomatic Systems

The *Elements* is the oldest example we possess of what is now called a "material axiomatic system." Before we examine the *Elements* itself, with all the explaining and restructuring of Euclid's text our study will involve, I think it would be wise to discuss material axiomatic systems in general (in this and the next two sections) and to illustrate Euclid's proof techniques in a nongeometric context (in the following section).

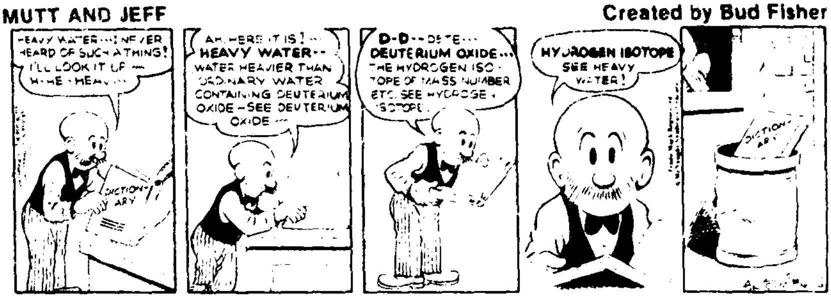

Figure 3. Courtesy of P. S. de Beaumont.

Pattern for a Material Axiomatic System[9]

(1) The basic technical terms of the discourse are introduced and their meanings explained. These basic terms are called *primitive terms*.
(2) A list of primary statements about the primitive terms is given. In order for the system to be significant to the reader, he or she must find these statements acceptable as true based on the explanations given in (1). These primary statements are called *axioms*.
(3) All other technical terms are defined by means of previously introduced terms. Technical terms which are not primitive terms are accordingly called *defined terms*.
(4) All other statements of the discourse are logically deduced from previously accepted or established statements. These derived statements are called *theorems*.

Notice there are two kinds of technical terms. The meanings of the "defined" terms (item (3)) are prescribed by reference to terms (of either type) previously introduced; at least in relation to those earlier terms, therefore, the defined terms are completely unambiguous. But unfortunately it is not possible to achieve unambiguity for *all* terms: dictionaries are, after all, circular. (See Figure 3, or try Jeff's experiment yourself with a word like "alive" or "straight" that cycles back quickly.) Thus it is necessary to accept some terms into the system (usually from everyday speech) without benefit of precise definition; these are the "primitive" terms of item (1). Of course every effort is made to indicate the sense in which each primitive term is to be taken, but no amount of explaining can guarantee that everyone will understand them in exactly the same way.

Similarly there are two kinds of statements. Just as one cannot define every term, one cannot deduce every statement. Accordingly, the statements of item (2) are accepted without deductive proof, on grounds that are outside the official structure of the system. (Within the system they are viewed simply as assumptions.) These statements provide a starting point from which all the other statements (item (4)) are logically deduced.

For many people a sticking place is that phrase, "logically deduced." Before we proceed to an example of a material axiomatic system, therefore, I think we should spend some time talking about logic—at first in general, then as we will encounter it in this book.

Logic

Every rational discussion involves the making of inferences. What *kinds* of inferences are allowed depends on who the participants are and what subject is being discussed. In this sense each type of discussion has its own special logic. For example, the sort of evidence that physicists accept as strong confirmation of a theory is rejected as totally inadequate by mathematicians trying to prove a theorem; in turn, the esoteric reasoning mathematicians sometimes employ is utterly worthless to literary critics analyzing a novel. (Indeed, there are forms of argument employed regularly in mathematics that are applicable to *nothing* else.[10])

Usually, however, the term "logic" is used in a more general sense, to refer to principles of reasoning that the various special logics are presumed to have in common. The belief is that this common logic would be acceptable and potentially useful to participants in *any* rational discussion. Of course there's no way of checking this without polling the entire planet, or at least scrutinizing its more than 3,000 languages, but since Greek concepts are so much a part of the Western heritage it seems safe to say there is a widely shared logic at least among people with Western-style educations.

Though this traditional logic does not include the special techniques of modern mathematics, it does include all the forms of argument used by mathematicians in Euclid's time. In fact, today many people, hearing the term "logic," can think of little *except* the principles of reasoning used by Euclid, because the only time they have ever heard logic discussed explicitly (rather than taken for granted) was in a high school geometry course.

Throughout this book, even when we take up non-Euclidean geometry, Euclid's logic is all we will ever need. We have good reason, therefore, to feel confident about the soundness of our logic. It is safely within traditional logic, and has been embedded in the fabric of Western thought for more than 2,000 years.

Nonetheless it is wise to take *all* logic with a grain of salt. It is vulnerable to doubt, on at least two counts.

I'll let the author of *Alice's Adventures in Wonderland* and *Through the Looking-Glass* tell you about the first.

> Achilles had overtaken the Tortoise, and had seated himself comfortably on its back.
>
> "So you've got to the end of our race-course?" said the Tortoise. "Even though it *does* consist of an infinite series of distances? I thought some wiseacre[11] or other had proved that the thing couldn't be done?"
>
> "It *can* be done," said Achilles. "It *has* been done! *Solvitur ambulando*. You see the distances were constantly *diminishing*: and so—"
>
> "But if they had constantly been *increasing*?" the Tortoise interrupted. "How then?"
>
> "Then I shouldn't be *here*," Achilles modestly replied; "and *you* would have got

several times round the world, by this time!"

"You flatter me-*flatten*, I mean," said the Tortoise; "for you *are* a heavy weight, and *no* mistake! Well now, would you like to hear of a race-course, that most people fancy they can get to the end of in two or three steps, while it *really* consists of an infinite number of distances, each one longer that the previous one?"

"Very much indeed!" said the Grecian warrior, as he drew from his helmet (few Grecian warrior possessed *pockets* in those days) an enormous note-book and a pencil. "Proceed! And speak *slowly*, please! *Shorthand* isn't invented yet!"

"That beautiful First Theorem of Euclid!" the Tortoise murmured dreamily. "You admire Euclid?"

"Passionately! So far, at least, as one *can* admire a treatise that won't be published for some centuries to come!"

"Well, now, let's take a little bit of the argument in that First Theorem—just *two* steps, and the conclusion drawn from them. Kindly enter them in your note-book. And, in order to refer to them conveniently, let's call them A, B, and Z:

(A) Things that are equal to the same are equal to each other.
(B) The two sides of this triangle are things that are equal to the same.
(Z) The two sides of this triangle are equal to each other.

"Readers of Euclid will grant, I suppose, that Z follows logically from A and B, so that anyone who accepts A and B as true, *must* accept Z as true?"

"Undoubtedly! The youngest child in a high school—as soon as high schools are invented, which will not be until some two thousand years later—will grant *that*."

"And if some reader had *not* yet accepted A and B as true, he might still accept the *sequence* as a *valid* one, I suppose?"

"No doubt such a reader might exist. He might say 'I accept as true the hypothetical proposition that, if A and B be true, Z must be true; but I *don't* accept A and B as true.' Such a reader would do wisely in abandoning Euclid, and taking to football."

"And might there not *also* be some reader who would say 'I accept A and B as true, but I *don't* accept the hypothetical'?"

"Certainly there might be. *He*, also, had better take to football."

"And *neither* of these readers," the Tortoise continued, "is *as yet* under any logical necessity to accept Z as true?"

"Quite so," Achilles assented.

"Well, now, I want you to consider *me* as a reader of the *second* kind, and to force me, logically, to accept Z as true."

"A tortoise playing football would be—" Achilles was beginning.

"—an anomaly, of course," the Tortoise hastily interrupted. "Don't wander from the point. Let's have Z first, and football afterwards!"

"I'm to force you to accept Z, am I?" Achilles said musingly. "And your present position is that you accept A and B, but you *don't* accept the hypothetical—"

"Let's call it C," said the Tortoise.

"—but you don't accept:

(C) If A and B are true, Z must be true."

"That is my present position," said the Tortoise.

"Then I must ask you to accept C."

"I'll do so," said the Tortoise, "as soon as you've entered it in that note-book of yours. What else have you got in it?"

"Only a few memoranda," said Achilles, nervously fluttering the leaves: "a few

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