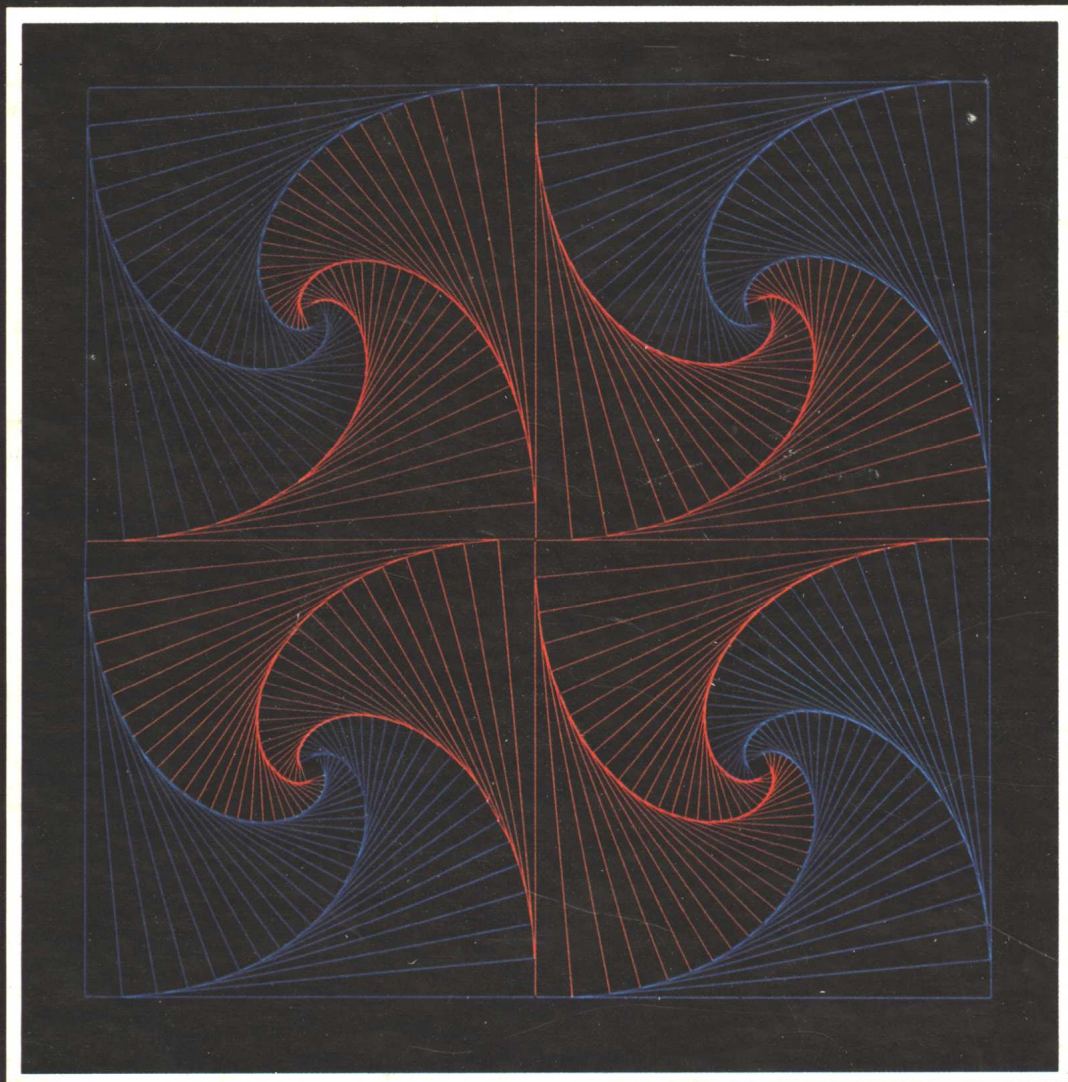


# Fundamental Mathematics

A Cultural Approach



A. Richard Polis  
Earl M. L. Beard

# **Fundamental Mathematics**

## A Cultural Approach

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## DEDICATION

To our wives Sandra and Joan  
and to our children Adam,  
Daniel, David, Ethan, and Sara.

# Preface

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Our purpose in writing this book is to stimulate mathematical ideas in a number of different ways. The text contains both standard introductory mathematics presented in the traditional manner as well as a great deal of material not often found in a text at this level. We have integrated discussions that normally would be found only in outside readings. These discussions are concentrated in the *capsules* and serve two distinct functions.

One type of capsule is devoted to the role of mathematics in our culture. The material in these capsules illustrates how the study of mathematics crosses boundaries into the areas of art, social science, and the natural sciences. Here we see the evidence and influence of mathematics in such diverse areas as: politics; advertising; philosophy; puzzles; language; design; curve stitching; the art of Escher; gambling; chemistry; psychology; artificial intelligence; and criminology. Each of these capsules leads naturally to a discussion of mathematics and its place in our culture.

The second type of capsule is devoted to the development of mathematics and mathematical thought in our civilization. These capsules are basically historical. They outline the development of mathematics and discuss the lives and works of some of the great mathematicians. These capsules present the human aspect of mathematics by treating subjects such as: Great Women Mathematicians; George Cantor; Pierre De Fermat; Fibonacci; Archimedes; René Descartes; Evariste Galois; Carl Friedrich Gauss; Léonard Euler; Euclid; Pythagoras; and Blaise Pascal.

Throughout this book we have tried to emphasize the way in which mathematics develops as well as the applications of mathematics in our

complex society. There are hundreds of exercises and problems solved step by step within the text to help students master the concepts.

The assumption underlying the spirit of this book is that non-math majors can handle mathematical concepts better on various levels than on one level alone. To accomplish this we have included material that should be of interest to many students and have written in a conversational style.

Finally, we have tried to make the book flexible. We have purposely included more material than an instructor could cover in a full year of class time. However, much of the material can be read independently by the student. The size of the book allows an instructor to cut sections and chapters without sacrificing the course. By selecting chapters and sections this book can be used as the basis for a quarter, a semester, a two-quarter, or a two-semester course.

This book can be used in a fundamental mathematics course or in a general course for college students. Education majors will find in this book both the mathematics and the cultural background of mathematics that are essential to a successful teacher. There is a great deal of material on the arts, humanities, and social sciences that is specifically designed to interest the average college student in mathematics. We trust that students, in the process of learning from this book, will enjoy reading it. We certainly enjoyed writing it.

A. Richard Polis  
Earl M. L. Beard

# TO THE INSTRUCTOR

This book evolved from work we have been doing with students majoring in liberal arts, fine arts, and education, as well as mathematics. The material in the text allows for a wide range of teaching styles and content.

A traditional course in college mathematics would involve material from the chapters on Numeration, Rationals, Reals, Euclidean Geometry, Algebraic Ideas, Probability, and Statistics. Student interest would be maintained by treating the capsules as outside reading.

A nontraditional course might use Logic, Number Theory, Group Theory, Linear Systems and Flow Charting, Transformations, Geometry, Combinatorial Topology, and Recreational Geometry. Such a course might use the capsules to stimulate class discussion in areas not normally associated with mathematics courses. The capsules can serve as an introduction to topics of interest to either instructor or students.

The instructor can rearrange topics to meet personal preferences. A major feature of this book is the amount of flexibility it allows either for classes, individuals, outside work, or discussion. A course might be organized using topics and sections from some or all of the chapters. For example:

Chapter 1: Fallacies or Logic in Advertising and Politics  
Number Patterns and Inductive Reasoning  
The Nature of Reasoning

Chapter 2: Set Description  
Subsets, Complementation, and Venn Diagrams  
Set Equality and Equivalence Relations  
Union and Intersection of Sets  
Equality in Mathematics and the Social Sciences

**Chapter 3: Body Language and Special Number Words**  
Early Systems of Numeration  
Positional Systems

**Chapter 4: Polygonal Numbers**  
Number Patterns and Problems  
Fibonacci Numbers

**Chapter 6: Language and Mathematics**  
Open Sentences and Solution Sets  
Relations and Functions  
Linear Equations and Applications

**Chapter 7: Groups**  
The Art and Mathematics of Design  
Rings and Fields

**Chapter 8: Artificial Intelligence**  
Flow Charting  
Linear Systems and Applications  
Dances and Tours

Each section, apart from the capsules, is prefaced with a list of objectives. These can be used to highlight ideas, to give students direction, and to help the instructor in planning lectures or examinations. The instructor should feel free to add, omit, or delete objectives.

How you personally used the various features of this book, along with your comments and suggestions, are both welcome and solicited.

A. R. P.  
E. M. L. B.



# TO THE STUDENT

Our intention in writing this text was to make mathematics clear, interesting, and important to you the student. To accomplish this we have written a great deal of mathematics that might be considered to have high external interest. We hope that you find the material helpful, well organized, interesting, and—at least sometimes—provocative.

At the beginning of all lettered sections you will find a list of objectives. We suggest you read them before *and* after you have read the section. You should “skim” a section before you read it thoroughly. By skimming first, you will get the big picture and can begin to fill in the gaps by reading the material slowly and carefully. Always have paper and pencil in hand when you read. You should work out all examples *yourself*, even when we have presented a step-by-step analysis. When you have a question for the instructor, you should phrase your questions like this: “What does line six, page  $x$  mean?” or “The fifth step in example 3 is not clear. How did they get that result?”

We hope this text becomes a helpful tool that you find enjoyable as well as clarifying. We hope that it increases your knowledge and interest in the world of mathematics. That world is all around you. It’s just a matter of looking. . . .

A. R. P.  
E. M. L. B.

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