# STUDIES IN MATHEMATICS AND ITS APPLICATIONS

VOLUME 11

*Editors:*

J. L. LIONS, *Paris*

# STOCHASTIC CONTROL BY FUNCTIONAL ANALYSIS METHODS

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NORTH-HOLLAND PUBLISHING COMPANY – AMSTERDAM • NEW YORK • OXFORD

ISBN: 0 444 86329 x

*Publishers:*
NORTH-HOLLAND PUBLISHING COMPANY
AMSTERDAM • NEW YORK • OXFORD

*Sole distributors for the U.S.A. and Canada:*
ELSEVIER SCIENCE PUBLISHING COMPANY, INC.
52 VANDERBILT AVENUE, NEW YORK, N.Y. 10017

**Library of Congress Cataloging in Publication Data**

Bensoussan, Alain.
Stochastic control by functional analysis methods.

(Studies in mathematics and its applications ; v. 11)
Bibliography: p.
1. Control theory. 2. Stochastic analysis.
I. Title. II. Series.
QA402.3.B433 629.8'312 81-19900
ISBN 0-444-86329-X AACR2

PRINTED IN THE NETHERLANDS

# INTRODUCTION

Our objective in this work is to give a presentation of some basic results of stochastic control. It is thus a text intended to advanced students and researchers willing to learn the theory.

Stochastic control covers a broad area of disciplines and problems. It is also a field in full development, and some important aspects remain to be cleaned up. That is why in presenting stochastic control a choice is necessary. We have emphasized this choice in the title. The theory of partial differential equations, the semi-group theory, variational and quasi variational inequalities play a very important role in solving problems of stochastic control. We have tried to use them as much as possible, since they bring tools and results which are very important, specially for computational purposes, and which cannot be obtained in an other way, namely regularity results, and weak solution concepts. The books by W. Fleming - R. Rishel [1], A. Friedman [1], N. Krylov [1], A. Bensoussan - J.L. Lions [1], [2] show already the importance of the techniques of Functional Analysis. W. Fleming - Rishel, Friedman besides covering many other topics, rely mostly on the classical theory of P.D.E. We try to emphasize here the importance of variational methods. Naturally, the present text has a lot in common with the books of J.L. Lions and the A. But here we have tried to simplify as much as possible the presentation, in particular leaving aside the most technical problems, which are treated there.

Also the books by J.L. Lions and the A. are devoted to variational and quasi variational inequalities. In the book of Krylov, one will find the study of the general Bellman equation, i.e., when the control enters into the drift as well as into the diffusion term. We do not treat this general case here, although it is certainly one of the nicest accomplishments of P.D.E. techniques in stochastic control. Quite fundamental results have been obtained for the general Bellman equation by P.L. Lions [1], [2], and more specialized ones by L.C. Evans - A. Friedman [1], H. Brezis - L.C. Evans [1]. More recently R. Jensen - P.L. Lions [1] have introduced new important ideas of approximation. To report on that work would have gone beyond the objectives of the present text, and requested too much material.

There are many important other topics that we have not considered here. We have not reported on the developments of the so called "probabilistic approach" initiated by C. Striebel [1], [2] and R. Rishel [1] and developed extensively by M. Davis - P. Varayia [1], M. Davis [1] and many other authors. A good report can be found in N. El Karoui [1], (see also Lepeltier - Marchal [1]).

This approach is of course fundamental for very general processes, which are not Markov processes. It is certainly the most general one and very satisfactory from the probabilistic point of view. But for the applications, where the processes are mostly Markov, it seems less convenient than the analytic approach, especially for computational purposes. Also it requires technical developments which again would have gone beyond the scope of this text.

The interested reader should consult besides the litterature, which has been briefly mentionned the recent book by I.Gikhman - A. Skorokhod [1].

Another very important area, which is in full development is the theory of non linear filtering and control under partial observation. Important results have been obtained recently by several authors in non linear filtering, T. Allinger - S.K. Mitter [1], E. Pardoux [1], and exploited for the control under partial observation by W. Fleming - E. Pardoux [1], W. Fleming [1]. Stochastic P.D.E. play a very important role in this direction, and probably the field will progress fast (see E. Pardoux [2], M. Viot [1], W. Fleming - M. Viot [1]). For the control of stochastic distributed parameter systems see A.V. Balakrishnan [1], A. Bensoussan [1], [2], A. Bensoussan - M. Viot [1], R. Curtain - A.J. Pritchard [1], S. Tzapestas [1].

We consider in this work some stochastic control problems in discrete time, but mostly as an approximation to continuous time stochastic control. We refer to the books by D. Bertsekas [1], D. Bertsekas - S. Shreve [1], E. Dynkin - A. Yushkevitch [1] for many more details. In a related direction, we have not discussed the numerical techniques which are used to solve stochastic control problems. We refer to J.P. Quadrat [1], P.L. Lions - B. Mercier [1] and to the book of H.J. Kushner [1].

Let us also mention the theory of large stochastic systems, with several players, the problems of identification, adaptive control, stochastic realization, stochastic stability etc., as interesting and important areas of research.

In Chapter I we present the elements of Stochastic Calculus and Stochastic Differential Equations, in Chapter II the theory of partial differential equations, and in Chapter III the Martingale problem. This permits to deal with the various formulations of diffusion processes and to interpret the solution of elliptic and parabolic equations as functionals on the trajectory of the diffusion process (in a way similar to the well known method of characteristics for 1st order linear P.D.E.). This allows us also to show the Markov semi-group property of diffusions.

In Chapter IV we present the theory of Stochastic Control with complete information (when the control affects only the drift term). We study the Hamilton-Jacobi-Bellman equation, interpret its solution as a value function and solve the stochastic control problem in the stationary as well as non stationary case. We also present a semi group approach to stochastic control for general Markov processes.

In Chapter V, we present the theory of filtering and prediction for linear stochastic differential equations, which leads to the Kalman filter. We show that the problem reduces to quadratic optimization problems, for which a decoupling argument yields the filter and the Riccati equation.

In Chapter VI, we present the variational approach to stochastic control, in two situations, one with complete observation and one with uncomplete observation. We discuss also the separation principle.

Chapter VII is devoted to optimal stopping problems which are solved by the theory of variational inequalities. We also develop a semi group approach in the case of general Markov processes.

In Chapter VIII we present the theory of impulsive control and its solution by the method of quasi variational inequalities. Also a semi group formulation is given.

We have tried to be self contained as such as possible, and have avoided too technical topics. Some basics on probability and Functional Analysis are the only requirements in order to read this book. Nevertheless, we recall the results that we need. We have restricted ourselves to stationary diffusions stopped at the exit of a domain, since one can explain the ideas in the simplest form in that context. But of course the methodology carries over to many other processes, like diffusions with reflexion, diffusions with jumps, random evolutions etc. see A. Bensoussan - J.L. Lions [2], A. Bensoussan - P.L. Lions [1], A. Bensoussan - J.L. Menaldi [1]). When presenting the semi group approach we have kept a certain degree of generality, although we have not tried to describe all the examples which are covered in this approach (cf. M. Robin [1] for many examples like semi Markov processes, jump processes, ...).

# ACKNOWLEDGMENTS

Many of the ideas presented here owe a lot to discussions and joint research with colleagues in France or abroad. I would like to thank particularly J. Frehse, A. Friedman, M. Goursat, J.L. Joly, J.L. Lions, P.L. Lions, J.L. Menaldi, U. Mosco, G. Papanicolaou, E. Pardoux, J.P. Quadrat, M. Robin, M. Viot with whom I had certainly the most fruitful exchanges and from whom I learnt a lot.

The material covered here has been presented first as a course at the University of Illinois, during the fall 1980. I would like to thank particularly P. Kokotovic who originated the idea of a course and suggested the writing of a text. I would like to thank also the members of the Control Group, at Coordinated Science Laboratory, University of Illinois, in particular Prof. Cruz, Perkins for their help. Prof. B. Hajek at the University of Illinois read carefully the manuscript and made very useful comments. I would like to thank him very much for that.

I would like to thank very much Professors Lions, Papanicolaou and Rockafellar for accepting to publishing this book in their series with North Holland as well as the publisher for his agreement.

Mrs Kurinckx realized a very nice typing of the manuscript. I would like to thank her very much for this excellent job.

# CONTENTS

# CHAPTER I

# STOCHASTIC CALCULUS AND STOCHASTIC DIFFERENTIAL EQUATIONS

## INTRODUCTION

This chapter is devoted to the presentation of the stochastic dynamic systems which will be used throughout this work, namely those whose evolution is described by stochastic differential equations. This requires a stochastic calculus and the concept of stochastic integral, originated by K. Ito. The model looks like

$$dy = g(y,t)dt + \sigma(y,t)dw(t)$$

and $g$ is called the drift term, $\sigma$ the diffusion term. This model generalizes the model of ordinary differential equations

$$\frac{dy}{dt} = g(y,t)$$

and expresses the fact that the velocity is perturbed by a random term of mean $0$.

In the standard set up (strong solution) one assumes lipschitz properties of $g$, $\sigma$ with respect to the space variable. It is important for the applications to control to weaken the concept of solution in order to assume only measurability and boundedness of the drift term. This is achieved through Girsanov transformation.

We have kept the presentation to what is essential within the scope of this text. But, aside basic preliminaries in Probability theory, we give complete proofs. We refer to the comments for indications on the natural extensions.

Basic references for this chapter are the books by J. NEVEU [1], I. GIKHMAN-ASKOROKHOD [2], A. FRIEDMAN [1], D. STROOCK - S.R.S. VARADHAN [1], E.B. DINKIN [1].

# 1. PRELIMINARIES

## 1.1. Random variables

Let $\Omega$ be a set. A $\sigma$-algebra $\mathcal{A}$ on $\Omega$ is a set of subsets of $\Omega$ such that

(1.1) $\forall\ A_i$ , $i \in I$ , $A_i \in \mathcal{A}$ , (I countable) then

$\bigcap_i A_i$ , $\bigcup_i A_i \in \mathcal{A}$

(1.2) if $A \in \mathcal{A}$ , $\complement A \subset \mathcal{A}$

(1.3) $\emptyset$ , $\Omega \in \Omega$

The elements of $\mathcal{A}$ are called *events*. The set $(\emptyset,\Omega)$ is a $\sigma$-algebra. It is contained in all $\sigma$-algebras on $\Omega$. It is called the *trivial $\sigma$-algebra*.

A probability on $(\Omega,\mathcal{A})$ is a positive measure on $\mathcal{A}$ , with total mass 1, i.e., a map $A \to P(A)$ from $\mathcal{A}$ into $[0,1]$ such that

$$P(\Omega) = 1$$

$$P(\bigcup_n A_n) = \sum_n P(A_n) \text{ if the } A_n \text{ are disjoint.}$$

When $P(A) = 1$, one says that A is *almost certain* ($\Omega$ is the certain event).

The triple $(\Omega,\mathcal{A},P)$ is called a *probability space*.

If $\mathcal{B} \subset \mathcal{A}$, and is also a $\sigma$-algebra we say that $\mathcal{B}$ is a *sub $\sigma$-algebra* of $\mathcal{A}$ .

On R (the set of real numbers), the open intervals generate a $\sigma$-algebra on R, which is called the *Borel $\sigma$-algebra on* R.

On a product space $X^1 \times X^2$, if $\mathcal{X}^1, \mathcal{X}^2$ are σ-algebras on $X^1, X^2$ respectively, the *product σ-algebra* $\mathcal{X}^1 \times \mathcal{X}^2$ is the σ-algebra generated by the events of the form $A^1 \times A^2$ where $A^1 \in \mathcal{X}^1$, $A^2 \in \mathcal{X}^2$.

Hence the Borel σ-algebra on $R^n$ is the σ-algebra generated by open cubes.

The concept carries over to an infinite set of spaces $X^i$, $i \in I$. The product σ-algebra $\otimes_I \mathcal{X}^i$, is generated by the events of the form $\prod_I A^i$, $A^i \in \mathcal{X}^i$, $A^i = X^i$ except for a finite number of i.

A *random variable* is a *measurable* map $\Omega \xrightarrow{f} R$, i.e., if $\mathcal{B}$ denotes the Borel σ-algebra on R, $f^{-1}(B) \in \mathcal{A}$, $\forall B \in \mathcal{B}$.

If $f_i$, $i \in I$ are random variables, there is a smallest σ-algebra for which all maps $f_i$ are measurable. It is called the σ-algebra generated by the family $f_i$, and denoted by $\sigma(f_i, i \in I)$. It is also the product σ-algebra $\otimes_I \sigma(f_i)$.

Note that if $f_k$ are random variables and $f_k(\omega) \to f(\omega)$ $\forall \omega$, then f is a R.V. Also a positive random variable is the increasing limit of a sequence of piecewise constant positive R.V. namely

$$f_n = \sum_{q=1}^{n2^n} \frac{q-1}{2^n} \chi_{\{\frac{q-1}{2^n} \le f < \frac{q}{2^n}\}} + n\chi_{\{f \ge n\}} .$$

We will need some results concerning extensions of probabilities,for which we refer to J. NEVEU [1].

Let Ω be a set. We call *algebra* a set of subsets of Ω, satisfying properties (1.1), (1.2), (1.3) except that in (1.1) the set I is not countable, but only *finite*.

We say that a class $\mathcal{C}$ of subsets of Ω is *compact*, if for any sequence $\{C_n, n \ge 1\}$ in $\mathcal{C}$ such that $\bigcap_n C_n = \emptyset$, there exists an integer N such that $\bigcap_{n \le N} C_n = \emptyset$.

*Theorem 1.1*. *Let* $\mathcal{A}^0$ *be an algebra on* $\Omega$ *and* $P^0$ *be a finitely additive function on* $\mathcal{A}^0$ *with values in* $[0,1]$, *such that* $P^0(\Omega) = 1$. *Assume that there exists a compact class* $\mathcal{C}$ *such that* $\forall A \in \mathcal{A}^0$

$$P^0(A) = \lim_{n\to\infty} P^0(C_n) \text{ , with } C_n \in \mathcal{C} \cap \mathcal{A}^0 \text{ ,}$$

*and* $C^n \subset A$. *Then* $P^0$ *can be extended in a unique way to the* $\sigma$*-algebra* $\mathcal{A}$ *generated by* $\mathcal{A}^0$, *as a probability on* $\Omega, \mathcal{A}$. □

One can obtain from Theorem 1.1,

*Theorem 1.2*. *Let us consider an arbitrary family* $(\Omega_i, \mathcal{A}_i)$ *where* $i \in I$, *of Polish spaces* (1), *provided with their Borel* $\sigma$*-algebras. Assume that for* $I_1$ *finite, there exists a probability* $P^{I_1}$ *on* $(\prod_{i\in I_1} \Omega_i, \otimes_{I_1} \mathcal{A}_i)$ *and that the family is compatible in the following sense, if* $I_1 \subset I_2$ *are finite then the restriction of* $P^{I_2}$ *to* $\prod_{i\in I_1} \Omega_i$ *coincides with* $P^{I_1}$. *Then there exists one and only one probability* $P$ *on* $(\prod_{i\in I} \Omega_i, \otimes_I \mathcal{A}_i)$ *which extends the family* $P^{I_1}$. □

Let us also state the following useful result (cf. Dynkin [1]). Let $E, \mathcal{E}$ be a topological space provided with its Borel $\sigma$-algebra. A set $\mathcal{H}$ of functions $f(x)$, $x \in E$ which are *non negative* is called a $\lambda$-system if the following conditions are satisfied

(1.4) $\quad 1 \in \mathcal{H}$

(1.5) $\quad$ if $f_1, f_2 \in \mathcal{H}$ and $c_1, c_2$ are numbers such that $c_1 f_1 + c_2 f_2 \geq 0$ , then $c_1 f_1 + c_2 f_2 \in \mathcal{H}$

(1.6) $\quad$ if $f_n \in \mathcal{H}$ and $f_n \uparrow f$, then $f \in \mathcal{H}$.

Then we have

(1) A Polish space is a complete denumerable metric space.

*Theorem 1.3. If a λ-system contains all bounded non negative continuous functions, then it also contains all non negative Borel functions.* □

## 1.2. Expectation. Conditional expectation

Let $f$ be a R.V., $f \geq 0$. One can define

$$Ef = \int f(\omega)dP(\omega) \quad \text{(possibly } +\infty) .$$

We say that $f_1 = f_2$ a.s., if $P(\omega | f_1 \neq f_2) = 0$ .

In general, we do not distinguish between R.V. which are a.s. equal. This relation is an equivalence relation among random variables. One defines the spaces $L^p(\Omega,\mathcal{A},P)$ $1 \leq p < \infty$ of equivalence classes of R.V. such that

$$\int_\Omega |f(\omega)|^p dP(\omega) < \infty$$

and $L^\infty$ is the space of R.V. which are a.s. bounded. We recall that

$$f_k \to f \quad \text{a.s.} \quad \text{if} \quad P(\omega : f_k \to f) = 1$$

$$f_k \overset{P}{\to} f \quad \text{(in probability) if} \quad P(\omega \,|\, |f_k(\omega) - f(\omega)| > \varepsilon) \to 0 \quad \text{as } k \to \infty, \ \forall\, \varepsilon > 0$$

$$f_k \to f \quad \text{in } L^p \quad \text{(usual meaning)} .$$

Let us consider $L^2(\Omega,\mathcal{A},P)$; it is a *Hilbert space*. If $\mathcal{B}$ is a sub-σ-algebra of $\mathcal{A}$, then the subset of $L^2(\Omega,\mathcal{A},P)$ made of R.V. which are $\mathcal{B}$ measurable is a closed sub vector space of $L^2(\Omega,\mathcal{A},P)$, hence a sub Hilbert space denoted by $L^2(\Omega,\mathcal{B},P)$.

If $f \in L^2(\Omega,\mathcal{A},P)$, one defines

$$E^{\mathcal{B}} f = \text{projection of } f \text{ on } L^2(\Omega,\mathcal{B},P) .$$

It is called the *conditional expectation* of $f$ with respect to $\mathcal{B}$ .