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Denumerable Markov Chains

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PREFACE TO THE SECOND EDITION

With the first edition out of print, we decided to arrange for republication of *Denumerable Markov Chains* with additional bibliographic material. The new edition contains a section Additional Notes that indicates some of the developments in Markov chain theory over the last ten years. As in the first edition and for the same reasons, we have resisted the temptation to follow the theory in directions that deal with uncountable state spaces or continuous time. A section entitled Additional References complements the Additional Notes.

J. W. Pitman pointed out an error in Theorem 9-53 of the first edition, which we have corrected. More detail about the correction appears in the Additional Notes. Aside from this change, we have left intact the text of the first eleven chapters.

The second edition contains a twelfth chapter, written by David Griffeath, on Markov random fields. We are grateful to Ted Cox for his help in preparing this material. Notes for the chapter appear in the section Additional Notes.

J.G.K., J.L.S., A.W.K.
March, 1976

PREFACE TO THE FIRST EDITION

Our purpose in writing this monograph has been to provide a systematic treatment of denumerable Markov chains, covering both the foundations of the subject and some topics in potential theory and boundary theory. Much of the material included is now available only in recent research papers. The book's theme is a discussion of relations among what might be called the descriptive quantities associated with Markov chains—probabilities of events and means of random variables that give insight into the behavior of the chains.

We make no pretense of being complete. Indeed, we have omitted many results which we feel are not directly related to the main theme, especially when they are available in easily accessible sources. Thus, for example, we have only touched on independent trials processes, sums of independent random variables, and limit theorems. On the other hand, we have made an attempt to see that the book is self-contained, in order that a mathematician can read it without continually referring to outside sources. It may therefore prove useful in graduate seminars.

Denumerable Markov chains are in a peculiar position in that the methods of functional analysis which are used in handling more general chains apply only to a relatively small class of denumerable chains. Instead, another approach has been necessary, and we have chosen to use infinite matrices. They simplify the notation, shorten statements and proofs of theorems, and often suggest new results. They also enable one to exploit the duality between measures and functions to the fullest.

The monograph divides naturally into four parts, the first three consisting of three chapters each and the fourth containing the last two chapters.

Part I provides background material for the theory of Markov chains. It is included to help make the book self-contained and should facilitate the use of the book in advanced seminars. Part II contains basic results on denumerable Markov chains, and Part III deals with discrete potential theory. Part IV treats boundary theory for both transient and recurrent chains. The analytical prerequisites for the two chapters in this last part exceed those for the earlier parts of the book and are not all included in Part I. Primarily, Part IV presumes that the reader is familiar with the topology and measure theory of compact metric spaces, in addition to the contents of Part I.

Two chapters—Chapters 1 and 7—require special comments. Chapter 1 contains prerequisites from the theory of infinite matrices and some other topics in analysis. In it Sections 1 and 5 are the most important for an understanding of the later chapters. Chapter 7, entitled "Introduction to Potential Theory," is a chapter of motivation and should be read as such. Its intent is to point out why classical potential theory and Markov chains should be at all related.

The book contains 239 problems, some at the end of each chapter except Chapters 1 and 7.

For the most part, historical references do not appear in the text but are collected in one segment at the end of the book.

Some remarks about notation may be helpful. We use sparingly the word "Theorem" to indicate the most significant results of the monograph; other results are labeled "Lemma," "Proposition," and "Corollary" in accordance with common usage. The end of each proof is indicated by a blank line. Several examples of Markov chains are worked out in detail and recur at intervals; although there is normally little interdependence between distinct examples, different instances of the same example may be expected to build on one another.

A complete list of symbols used in the book appears in a list separate from the index.

We wish to thank Susan Knapp for typing and proof-reading the manuscript.

We are doubly indebted to the National Science Foundation: First, a number of original results and simplified proofs of known results were developed as part of a research project supported by the Foundation. And second, we are grateful for the support provided toward the preparation of this manuscript.

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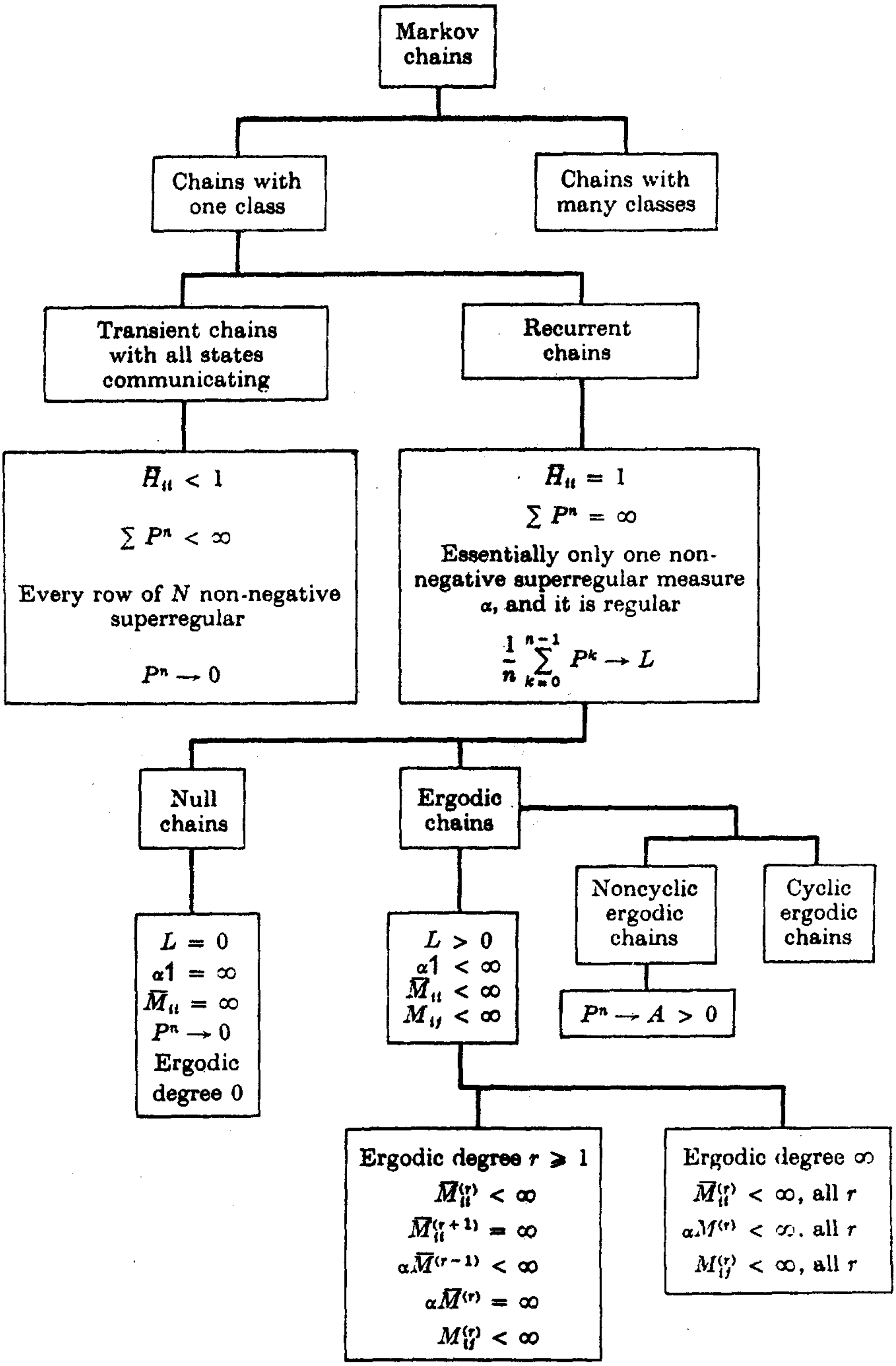
RELATIONSHIPS AMONG MARKOV CHAINS



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CHAPTER 1

PREREQUISITES FROM ANALYSIS

1. Denumerable matrices

The word **denumerable** in the sequel means finite or countably infinite. Let M and N be two non-empty denumerable sets. A **matrix** is a function with domain the set of ordered pairs (m, n), where $m \in M$ and $n \in N$, and with range a subset of the extended real number system—the reals with $+\infty$ and $-\infty$ adjoined. We call the sets M and N **index sets.** The matrix is called a **finite matrix** if both M and N are finite sets.

To say that the m-nth entry of the matrix is x or is equal to x, we mean that the value of the function on the pair (m, n) is x. A matrix is said to be **non-negative** if all of its entries are non-negative, and it is said to be **positive** if all of its entries are positive. We agree to use upper-case italic letters to stand for matrices. If A is a matrix, we denote the m-nth entry of A by A_{mn}. Some examples of matrices are as follows:

(1) If all entries of a matrix are equal to zero, we say that the matrix is the **zero matrix**, denoted by 0.

(2) A matrix for which M and N are the same set is called a **square matrix.** The entries corresponding to $m = n$ are **diagonal entries**; other entries are **off-diagonal entries.**

(3) A square matrix whose off-diagonal entries all equal zero is a **diagonal matrix.** The diagonal matrix obtained from a square matrix A by setting all of its off-diagonal entries equal to zero is denoted A_{dg}.

(4) A diagonal matrix whose diagonal entries are all equal to one is called the **identity matrix**, denoted by I.

(5) A matrix whose second index set contains only one element is called a **column vector.** If we wish to distinguish a column vector from an arbitrary matrix, we shall denote the former by a lower-case italic letter.

(6) A matrix whose first index set contains only one element is called a **row vector**. If we wish to distinguish a row vector from an arbitrary matrix, we shall denote the former by a lower-case Greek letter.

(7) If A is a matrix defined on index sets M and N, define a matrix A^T, called the **transpose** of A, to have index sets N and M and to have entries given by $(A^T)_{nm} = A_{mn}$. The transpose of the transpose of A is simply A.

(8) The column vector all of whose entries are equal to one is denoted $\mathbf{1}$; the row vector with all entries one is $\mathbf{1}^T$. A matrix other than a row or column vector which has all entries equal to one is denoted by E.

(9) If A is an arbitrary matrix and c is a real number, cA is the matrix whose entries are given by $(cA)_{mn} = cA_{mn}$.

(10) The matrix $-A$ is defined to be the matrix $(-1)A$.

(11) A **constant** (column) **vector** is a vector of the form $c\mathbf{1}$ for some extended real number c.

(12) A **bounded vector** is a vector all of whose entries are less than or equal in absolute value to some finite real number c.

Two matrices A and B are equal, written $A = B$, if they have the same index sets and if $A_{mn} = B_{mn}$ for every m and n. Inequalities are defined similarly. For example, $A > B$ if A and B have the same index sets and if $A_{mn} > B_{mn}$ for every m and n. In particular, non-negative matrices are those for which $A \geq 0$, and positive matrices are those for which $A > 0$.

Addition of matrices is defined for matrices A and B having the same index sets M and N. Their **sum** $C = A + B$ has the same index sets, and addition is defined entry-by-entry:

$$C_{mn} = A_{mn} + B_{mn}.$$

The sum $C = A + B$ is **well defined** if no entry of C is given by $\infty - \infty$ or by $-\infty + \infty$. We leave the verification of the following properties of matrices with index sets M and N to the reader:

(1) $A + 0 = A$ for every A.

(2) For every A having all entries finite, $A + (-A) = 0$.

(3) For any matrices A, B, and C,

$$A + (B + C) = (A + B) + C$$

if the indicated sums on at least one side of the equality are well defined.

Up to now, we have imposed no orderings on our index sets, and in fact nothing we have done so far necessitates doing so. We shall define even matrix multiplication shortly in a way that requires no ordering.

There is, however, a standard way of representing matrices as rectangular arrays, and for this purpose one normally orders the index sets with the usual ordering on the non-negative integers. The elements of the index sets are thus numbered 0, 1, 2, ... either up to some integer r if the index set is finite or indefinitely if the index set is infinite. Under such orderings of its index sets, a matrix A is represented as

$$A = \begin{pmatrix} A_{00} & A_{01} & A_{02} & \cdots \\ A_{10} & A_{11} & A_{12} & \cdots \\ A_{20} & A_{21} & A_{22} & \cdots \\ \vdots & & & \ddots \end{pmatrix}.$$

We note that other representations are possible if at least one of the index sets is infinite; such representations come from ordering the index sets with an order type other than that of the non-negative integers. We shall meet another order type with its corresponding representation at the end of this section. We point out, however, that orderings are completely irrelevant as far as the fundamental properties of matrices are concerned, and we shall have little occasion to refer to them again.

For any real number a_m, define a_m^+ and a_m^- by

$$a_m^+ = \max(a_m, 0)$$
$$a_m^- = -\min(a_m, 0).$$

The sum of denumerably many non-negative terms $\sum_{m\in M} a_m^+$ or $\sum_{m\in M} a_m^-$ always exists independently of any ordering on M. Therefore, we say that $\sum_{m\in M} a_m = \sum_{m\in M} a_m^+ - \sum_{m\in M} a_m^-$ is **well defined** if not both of $\sum_{m\in M} a_m^+$ and $\sum_{m\in M} a_m^-$ are infinite.

Definition 1-1: Let A be a matrix with index sets K and M, and let B be a matrix with index sets M and N. Suppose the sums

$$\sum_{m\in M} A_{km}B_{mn}$$

are well defined for every k and every n. Then the **matrix product** $C = AB$ is said to be well defined; its index sets are K and N, and its entries are given by $C_{kn} = \sum_{m\in M} A_{km}B_{mn}$. Matrix multiplication is not defined unless all of these properties hold.

Most of the propositions and theorems about matrices that we shall deal with are statements of equality of matrices $A = B$. Such statements are really just assertions about the equality of corresponding entries of A and B, and a proof that A equals B need only contain an argument that an arbitrary entry of A equals the corresponding entry of B. With this understanding, we see that the proof of the additive properties of matrices is reduced to a trivial repetition of the properties of real numbers. Propositions about multiplication, however, when looked at entry-by-entry involve a new idea.

Let A be a matrix with index sets M and N and let m and n be fixed elements of M and N, respectively. The mth **row** of A is defined to be the restriction of the function A to the domain of pairs (m, s), where s runs through the set N. Similarly the nth **column** of A is defined to be the restriction of the function A to the domain of pairs (t, n), where t runs through the elements of the set M. We note that the mth row of a matrix is a row vector and that the nth column is a column vector. With these conventions matrices can be thought of as sets of rows or as sets of columns, and addition of matrices is simply addition of corresponding rows or columns of the matrices involved. Furthermore, the k-nth entry in the matrix product of A and B is the product of the kth row of A by the nth column of B and is of the form $\sum_{m \in M} \pi_m f_m$, where π is a row vector and f is a column vector. That is, propositions about matrix multiplication, when proved entry-by-entry, may sometimes be proved by considering only the product of a row vector and a column vector.

Because of the correspondence of row vectors to rows and column vectors to columns, we shall agree to call the domain of a row vector or a column vector the elements of a single index set.

Connected with any definition of multiplication are five properties which may or may not be valid for the structure being considered. All five of the properties do hold for the real numbers, and we state them in this context:

(1) Existence and uniqueness of a multiplicative identity. The real number 1 satisfies $c1 = 1c = c$ for every c.

(2) Commutativity: $ab = ba$

(3) Distributivity: $a(b + c) = ab + ac$
$(a + b)c = ac + bc$

(4) Associativity: $a(bc) = (ab)c$

(5) Existence and uniqueness of multiplicative inverses of all non-zero elements.

We can easily settle whether the first two properties hold for matrix multiplication. First, the identity matrix I plays the role of the multiplicative identity, and the identity is clearly unique. Second, commutativity can be expected to fail except in special cases because it is not even necessary for the index sets of two matrices to agree properly after the order of multiplication has been reversed.

The validity of the third property, that of distributivity, is the content of the next proposition.

Proposition 1-2: If A, B, and C are matrices and if AB, AC, and $AB + AC$ are well defined, then $A(B + C) = AB + AC$. Similarly $(D + E)F = DF + EF$ if DF, EF, and $DF + EF$ are all well defined.

PROOF: We prove only the first assertion. We may assume that A is a row vector π and that B and C are column vectors f and g. Then

$$\begin{aligned}\pi f + \pi g &= \sum_{m\in M} \pi_m f_m + \sum_{m\in M} \pi_m g_m \\ &= \sum_{m\in M} (\pi_m f_m + \pi_m g_m) \\ &= \sum_{m\in M} \pi_m (f_m + g_m) \\ &= \pi(f + g).\end{aligned}$$

The fourth and fifth properties are related and nontrivial. Associativity does not always hold, but useful sufficient criteria for its validity are known. For an example of how associativity may fail, let A be a matrix whose index sets are the non-negative integers and whose entries are given by

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & -1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & -1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & -1 & \cdots \\ 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & & & & & \ddots \end{pmatrix}$$

Then

$$1^T(A1) = 0,$$

whereas

$$(1^T A)1 = 1.$$

All the products involved are well defined, but the multiplications do not associate.

We shall not consider the problem of existence of inverses, but uniqueness rests upon associativity. For suppose $AB = BA = AC = CA = I$. Since $AC = I$, we have $B(AC) = B$, and since $BA = I$, we have $(BA)C = C$. Therefore, $B = C$ if and only if $B(AC) = (BA)C$. With this note we proceed with some sufficient conditions for associativity.

Lemma 1-3: Let b_{ij} be a sequence of real numbers nondecreasing with i and with j. Then $\lim_i \lim_j b_{ij} = \lim_j \lim_i b_{ij}$, both possibly infinite.

PROOF: In the extended sense $\lim_i b_{ij} = L_j$ exists and so does $\lim_j b_{ij} = L_i^*$. Now $\{L_j\}$ is nondecreasing, for if $L_j > L_{j+k}$, then for i sufficiently large $b_{ij} > L_{j+k} \geq b_{i,j+k}$, which is impossible. Similarly $\{L_i^*\}$ is nondecreasing, so that $\lim_j L_j = L$ and $\lim_i L_i^* = L^*$ exist in the extended sense. If $L \neq L^*$, we may assume $L^* > L$ and hence L is finite. Then there exists an i such that $L_i^* > L$. Hence

$$\begin{aligned} L_i^* &> L \\ &\geq L_j \quad \text{for all } j \\ &\geq b_{ij} \quad \text{for all } j. \end{aligned}$$

Thus b_{ij} is bounded away from its limit on j, a contradiction.

Following the example of Lemma 1-3, we agree that all limits referred to in the future are on the **extended real line**.

Proposition 1-4: Non-negative matrices associate under multiplication.

PROOF: Since we are interested in each entry separately of a triple product, we may assume that we are to show that $\pi(Af) = (\pi A)f$, where $\pi \geq 0$, $A \geq 0$, $f \geq 0$, π is a row vector, f is a column vector, and the index sets are subsets of the non-negative integers. Then

$$\pi(Af) = \sum_m \sum_n \pi_m A_{mn} f_n$$

and

$$(\pi A)f = \sum_n \sum_m \pi_m A_{mn} f_n.$$

Set $b_{ij} = \sum_{m=0}^{i} \sum_{n=0}^{j} \pi_m A_{mn} f_n = \sum_{n=0}^{j} \sum_{m=0}^{i} \pi_m A_{mn} f_n$ and apply Lemma 1-3 to complete the proof.

If A is an arbitrary matrix we define A^+ and A^- by the equations

$$\begin{aligned} (A^+)_{mn} &= \max\{A_{mn}, 0\} \\ (A^-)_{mn} &= -\min\{A_{mn}, 0\}. \end{aligned}$$

Then $A = A^+ - A^-$, $A^+ \geq 0$, and $A^- \geq 0$. For row and column vectors, the matrices π^+, π^-, f^+, and f^- are defined analogously. We note that if Af is well defined, then so are Af^+ and Af^-. Powers of matrices are defined inductively by $A^0 = I$, $A^n = A(A^{n-1})$. The