SCHAUM'S OUTLINE OF

Modern Introductory DIFFERENTIAL EQUATIONS

with

Laplace Transforms • Numerical Methods
Matrix Methods • Eigenvalue Problems

by

RICHARD BRONSON, Ph.D.

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SCHAUM'S OUTLINE SERIES
McGRAW-HILL BOOK COMPANY

New York, St. Louis, San Francisco, Düsseldorf, Johannesburg, Kuala Lumpur, London, Mexico, Montreal, New Delhi, Panama, São Paulo, Singapore, Sydney, and Toronto

To Ignace and Gwendolyn Bronson
Samuel and Rose Feldschuh

07-008009-7

7 8 9 10 11 12 13 14 15 SH SH 7 9

Library of Congress Cataloging in Publication Data

Bronson, Richard.
Schaum's outline of modern introductory differential equations.

(Schaum's outline series)
1. Differential equations. I. Title.
II. Title: Modern introductory differential equations.
[QA372.B855] 515'.35 73-19601

ISBN 0-07-008009-7

Preface

Over the past twenty years there have been significant developments in the field of differential equations. The advent of high-speed computers has made solutions by numerical techniques feasible and has resulted in a host of new methods. The systems approach favored in many present-day engineering problems lends itself to both matrix methods and Laplace transforms.

This book outlines, with many solved problems, both the classical theory of differential equations and the more modern techniques currently available. The only prerequisite for any of the topics treated is calculus. As a supplement to standard textbooks, or as a textbook in its own right, it should prove useful for undergraduate courses and for independent study.

Chapters 1 through 21 and Chapters 37 through 39 cover the classical material, including separable and exact equations, solutions of linear equations with constant coefficients by the characteristic equation method, variation of parameters and the method of undetermined coefficients, infinite series solutions, and boundary-value and Sturm-Liouville problems. In contrast, Chapters 22 through 36 deal with the newer techniques currently in vogue, in particular, Laplace transforms, matrix methods, and numerical techniques. This last subject, because of its great practical importance, has been developed more completely than is usual at this level.

Each chapter of the book is divided into three parts. The first outlines the salient points, drawing attention to potential difficulties and pointing out subtleties that could be easily overlooked. The second part consists of completely worked-out problems which clarify the material presented in the first part and which, on occasion, also expand on that development. Finally, there is a section of problems with answers through which the student can test his understanding of the material.

I should like to thank the many individuals who helped make this book a reality. The valuable suggestions by Joseph Klein and Jack Mieses for Chapters 22 through 27, and those of Mabel Dukeshire, are all warmly acknowledged. Particular thanks are due Raymond Raggi who programmed most of the numerical methods and David Beckwith of the Schaum's staff for his splendid editing. Finally, my greatest debt is to my wife Evelyn who besides doing most of the typing contributed substantially to the editing and proofreading phases of this project.

RICHARD BRONSON

Fairleigh Dickinson University
October 1973

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Chapter 1

Basic Concepts

1.1 ORDINARY DIFFERENTIAL EQUATIONS

A *differential equation* is an equation involving an unknown function and its derivatives.

Example 1.1. The following are differential equations involving the unknown function y.

$$\frac{dy}{dx} = 5x + 3 \tag{1.1}$$

$$e^y \frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 = 1 \tag{1.2}$$

$$4\frac{d^3y}{dx^3} + (\sin x)\frac{d^2y}{dx^2} + 5xy = 0 \tag{1.3}$$

$$\left(\frac{d^2y}{dx^2}\right)^3 + 3y\left(\frac{dy}{dx}\right)^7 + y^3\left(\frac{dy}{dx}\right)^2 = 5x \tag{1.4}$$

$$\frac{\partial^2 y}{\partial t^2} - 4\frac{\partial^2 y}{\partial x^2} = 0 \tag{1.5}$$

A differential equation is an *ordinary differential equation* if the unknown function depends on only one independent variable. If the unknown function depends on two or more independent variables, the differential equation is a *partial differential equation.*

Example 1.2. Equations (*1.1*) through (*1.4*) are examples of ordinary differential equations, since the unknown function y depends solely on the variable x. Equation (*1.5*) is a partial differential equation, since y depends on both the independent variables t and x.

In this book we will be concerned solely with ordinary differential equations.

1.2 ORDER AND DEGREE

The *order* of a differential equation is the order of the highest derivative appearing in the equation.

Example 1.3. Equation (*1.1*) is a first-order differential equation; (*1.2*), (*1.4*), and (*1.5*) are second-order differential equations. (Note in (*1.4*) that the order of the highest *derivative* appearing in the equation is two.) Equation (*1.3*) is a third-order differential equation.

The *degree* of a differential equation that can be written as a polynomial in the unknown function and its derivatives is the power to which the highest-order derivative is raised.

Example 1.4. Equation (*1.4*) is a differential equation of degree three, since the highest-order derivative, in this case the second, is raised to the third power. Equations (*1.1*) and (*1.3*) are examples of first-degree differential equations.

Not every equation can be classified by degree. For instance, *(1.2)* has no degree, as it cannot be written as a *polynomial* in the unknown function and its derivatives (because of the term e^y).

1.3 LINEAR DIFFERENTIAL EQUATIONS

An nth-order ordinary differential equation in the unknown function y and the independent variable x is *linear* if it has the form

$$b_n(x)\frac{d^ny}{dx^n} + b_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + b_1(x)\frac{dy}{dx} + b_0(x)y = g(x) \qquad (1.6)$$

The functions $b_j(x)$ $(j = 0, 1, 2, \ldots, n)$ and $g(x)$ are presumed known and depend only on the variable x. Differential equations that cannot be put into the form *(1.6)* are *nonlinear.*

Example 1.5. Equation *(1.1)* is a first-order linear equation, with $b_1(x) = 1$, $b_0(x) = 0$, and $g(x) = 5x + 3$. Equation *(1.3)* is a third-order linear equation, with $b_3(x) = 4$, $b_2(x) = \sin x$, $b_1(x) = 0$, $b_0(x) = 5x$, and $g(x) = 0$. Equations *(1.2)* and *(1.4)* are nonlinear.

1.4 NOTATION

The expressions y', y'', y''', $y^{(4)}$, $\ldots$, $y^{(n)}$ are often used to represent, respectively, the first, second, third, fourth, $\ldots$, nth derivatives of y with respect to the independent variable under consideration. Thus, y'' represents d^2y/dx^2 if the independent variable is x, but represents d^2y/dp^2 if the independent variable is p. If the independent variable is time, usually denoted by t, primes are often replaced by dots. Thus, $\dot{y}$, $\ddot{y}$, and $\dddot{y}$ represent dy/dt, d^2y/dt^2, and d^3y/dt^3, respectively.

Observe that parentheses are used in $y^{(n)}$ to distinguish it from the nth power, y^n.

Solved Problems

In the following problems, classify each differential equation as to order, degree (if appropriate), and linearity. Determine the unknown function and the independent variable.

1.1. $y''' - 5xy' = e^x + 1.$

Third-order: the highest-order derivative is the third. *First-degree*: the equation has the form required in Section 1.2, and the third derivative is raised to the first power. *Linear*: $b_3(x) = 1$, $b_2(x) = 0$, $b_1(x) = -5x$, $b_0(x) = 0$, $g(x) = e^x + 1$. The unknown function is y; the independent variable is x.

1.2. $t\ddot{y} + t^2\dot{y} - (\sin t)\sqrt{y} = t^2 - t + 1.$

Second-order: the highest-order derivative is the second. *No degree*: because of the term $\sqrt{y}$, the equation cannot be written as a polynomial in y and its derivatives. *Nonlinear*: the equation cannot be put into the form *(1.6)*. The unknown function is y; the independent variable is t.

1.3. $s^2\dfrac{d^2t}{ds^2} + st\dfrac{dt}{ds} = s.$

Second-order. First-degree: the equation is a polynomial in the unknown function t and its derivatives (with coefficients in s), and the second derivative is raised to the first power. *Nonlinear*: $b_1 = st$, which depends on both s and t. The unknown function is t; the independent variable is s.

1.4. $$5\left(\frac{d^4b}{dp^4}\right)^5 + 7\left(\frac{db}{dp}\right)^{10} + b^7 - b^5 = p.$$

Fourth-order. Fifth-degree: the equation has the form required in Section 1.2, and the fourth derivative is raised to the fifth power. *Nonlinear.* The unknown function is b; the independent variable is p.

1.5. $$y\frac{d^2x}{dy^2} = y^2 + 1.$$

Second-order. First-degree. Linear: $b_2(y) = y$, $b_1(y) = 0$, $b_0(y) = 0$, and $g(y) = y^2 + 1$. The unknown function is x; the independent variable is y.

Supplementary Problems

For the following differential equations, determine (*a*) order, (*b*) degree (if appropriate), (*c*) linearity, (*d*) unknown function, and (*e*) independent variable.

1.6. $(y'')^2 - 3yy' + xy = 0.$

1.7. $x^4y^{(4)} + xy''' = e^x.$

1.8. $t^2\ddot{s} - t\dot{s} = 1 - \sin t.$

1.9. $y^{(4)} + xy''' + x^2y'' - xy' + \sin y = 0.$

1.10. $\dfrac{d^nx}{dy^n} = y^2 + 1.$

1.11. $\left(\dfrac{d^2r}{dy^2}\right)^2 + \dfrac{d^2r}{dy^2} + y\dfrac{dr}{dy} = 0.$

1.12. $\left(\dfrac{d^2y}{dx^2}\right)^{3/2} + y = x.$

1.13. $\dfrac{d^7b}{dp^7} = 3p.$

1.14. $\left(\dfrac{db}{dp}\right)^7 = 3p.$

Answers to Supplementary Problems

1.6.	(*a*) 2	(*b*) 2	(*c*) nonlinear	(*d*) y	(*e*) x
1.7.	(*a*) 4	(*b*) 1	(*c*) linear	(*d*) y	(*e*) x
1.8.	(*a*) 2	(*b*) 1	(*c*) linear	(*d*) s	(*e*) t
1.9.	(*a*) 4	(*b*) none	(*c*) nonlinear	(*d*) y	(*e*) x
1.10.	(*a*) n	(*b*) 1	(*c*) linear	(*d*) x	(*e*) y
1.11.	(*a*) 2	(*b*) 2	(*c*) nonlinear	(*d*) r	(*e*) y
1.12.	(*a*) 2	(*b*) none	(*c*) nonlinear	(*d*) y	(*e*) x
1.13.	(*a*) 7	(*b*) 1	(*c*) linear	(*d*) b	(*e*) p
1.14	(*a*) 1	(*b*) 7	(*c*) nonlinear	(*d*) b	(*e*) p

Chapter 2

Solutions

2.1 DEFINITION OF SOLUTION

A *solution* of a differential equation in the unknown function y and the independent variable x on the interval $\mathcal{J}$ is a function $y(x)$ that satisfies the differential equation identically for all x in $\mathcal{J}$.

Example 2.1. Is $y(x) = c_1 \sin 2x + c_2 \cos 2x$, where c_1 and c_2 are arbitrary constants, a solution of $y'' + 4y = 0$?

Differentiating y, we find:

$$y' = 2c_1 \cos 2x - 2c_2 \sin 2x \qquad y'' = -4c_1 \sin 2x - 4c_2 \cos 2x$$

Hence,

$$\begin{aligned} y'' + 4y &= (-4c_1 \sin 2x - 4c_2 \cos 2x) + 4(c_1 \sin 2x + c_2 \cos 2x) \\ &= (-4c_1 + 4c_1) \sin 2x + (-4c_2 + 4c_2) \cos 2x \\ &= 0 \end{aligned}$$

Thus, $y = c_1 \sin 2x + c_2 \cos 2x$ satisfies the differential equation for all values of x and is a solution on the interval $(-\infty, \infty)$.

Example 2.2. Determine whether $y = x^2 - 1$ is a solution of $(y')^4 + y^2 = -1$.

Note that the left side of the differential equation must be nonnegative for every real function $y(x)$ and any x, since it is the sum of terms raised to the second and fourth powers, while the right side of the equation is negative. Since no function $y(x)$ will satisfy this equation, the given differential equation has no solution.

We see that some differential equations have infinitely many solutions (Example 2.1), whereas other differential equations have no solutions (Example 2.2). It is also possible that a differential equation has exactly one solution. Consider $(y')^4 + y^2 = 0$, which for reasons identical to those given in Example 2.2 has only the solution $y \equiv 0$.

2.2 PARTICULAR AND GENERAL SOLUTIONS

A *particular solution* of a differential equation is any one solution. The *general solution* of a differential equation is the set of all solutions.

Example 2.3. The general solution to the differential equation in Example 2.1 can be shown to be (see Chapters 11 and 12) $y = c_1 \sin 2x + c_2 \cos 2x$. That is, every particular solution of the differential equation has this general form. A few particular solutions are: (*a*) $y = 5 \sin 2x - 3 \cos 2x$ (choose $c_1 = 5$ and $c_2 = -3$), (*b*) $y = \sin 2x$ (choose $c_1 = 1$ and $c_2 = 0$), and (*c*) $y \equiv 0$ (choose $c_1 = c_2 = 0$).

The general solution of a differential equation cannot always be expressed by a single formula. As an example consider the differential equation $y' + y^2 = 0$, which has two particular solutions $y = \frac{1}{x}$ and $y \equiv 0$. Linear differential equations are special in this regard and their general solutions are discussed in Chapter 11.

2.3 INITIAL-VALUE PROBLEMS. BOUNDARY-VALUE PROBLEMS

A differential equation along with subsidiary conditions on the unknown function and its derivatives, all given at the same value of the independent variable, constitutes an *initial-value problem*. The subsidiary conditions are *initial conditions*. If the subsidiary conditions are given at more than one value of the independent variable, the problem is a *boundary-value problem* and the conditions are *boundary conditions*.

Example 2.4. The problem $y''+2y'=e^x$; $y(\pi)=1$, $y'(\pi)=2$ is an initial-value problem, since the two subsidiary conditions are both given at $x=\pi$. The problem $y''+2y'=e^x$; $y(0)=1$, $y(1)=1$ is a boundary-value problem, since the two subsidiary conditions are given at the different values $x=0$ and $x=1$.

A *solution* to an initial-value or boundary-value problem is a function $y(x)$ that both solves the differential equation (in the sense of Section 2.1) and satisfies all given subsidiary conditions.

Example 2.5. Determine whether any of the functions (*a*) $y_1=\sin 2x$, (*b*) $y_2(x)=x$, or (*c*) $y_3(x)=\frac{1}{2}\sin 2x$ is a solution to the initial-value problem $y''+4y=0$; $y(0)=0$, $y'(0)=1$. (*a*) $y_1(x)$ is a solution to the differential equation and satisfies the first initial condition $y(0)=0$. However, $y_1(x)$ does not satisfy the second initial condition ($y_1'(x)=2\cos 2x$; $y_1'(0)=2\cos 0=2\neq 1$); hence it is not a solution to the initial-value problem. (*b*) $y_2(x)$ satisfies both initial conditions but does not satisfy the differential equation; hence $y_2(x)$ is not a solution. (*c*) $y_3(x)$ satisfies the differential equation and both initial conditions; therefore, it is a solution to the initial-value problem.

Solved Problems

2.1. Determine whether $y(x)=2e^{-x}+xe^{-x}$ is a solution of $y''+2y'+y=0$.

Differentiating $y(x)$, it follows that

$$y'(x) = -2e^{-x}+e^{-x}-xe^{-x} = -e^{-x}-xe^{-x}$$

$$y''(x) = e^{-x}-e^{-x}+xe^{-x} = xe^{-x}$$

Substituting these values into the differential equation, we obtain

$$y''+2y'+y = xe^{-x}+2(-e^{-x}-xe^{-x})+(2e^{-x}+xe^{-x}) = 0$$

Thus, $y(x)$ is a solution.

2.2. Is $y(x)\equiv 1$ a solution of $y''+2y'+y=x$?

From $y(x)\equiv 1$ it follows that $y'(x)\equiv 0$ and $y''(x)\equiv 0$. Substituting these values into the differential equation, we obtain

$$y''+2y'+y = 0+2(0)+1 = 1\neq x$$

Thus, $y(x)\equiv 1$ is not a solution.

2.3. Show that $y=\ln x$ is a solution of $xy''+y'=0$ on $\mathcal{I}=(0,\infty)$ but is not a solution on $\mathcal{I}=(-\infty,\infty)$.

On $(0,\infty)$ we have $y'=1/x$ and $y''=-1/x^2$. Substituting these values into the differential equation, we obtain

$$xy''+y' = x\left(-\frac{1}{x^2}\right)+\frac{1}{x} = 0$$

Thus, $y=\ln x$ is a solution on $(0,\infty)$.

Note that $y = \ln x$ could not be a solution on $(-\infty, \infty)$, since the logarithm is undefined for negative numbers and zero.

2.4. **Show that $y = 1/(x^2 - 1)$ is a solution of $y' + 2xy^2 = 0$ on $\mathcal{I} = (-1, 1)$ but not on any larger interval containing $\mathcal{I}$.**

On $(-1, 1)$, $y = 1/(x^2 - 1)$ and its derivative $y' = -2x/(x^2 - 1)^2$ are well-defined functions. Substituting these values into the differential equation, we have

$$y' + 2xy^2 = -\frac{2x}{(x^2-1)^2} + 2x\left[\frac{1}{x^2-1}\right]^2 = 0$$

Thus, $y = 1/(x^2 - 1)$ is a solution on $\mathcal{I} = (-1, 1)$.

Note, however, that $1/(x^2 - 1)$ is not defined at $x = \pm 1$ and therefore could not be a solution on any interval containing either of these two points.

2.5. **Find the solution to the initial-value problem $y' + y = 0$; $y(3) = 2$, if the general solution to the differential equation is known to be (see Chapter 8) $y(x) = c_1 e^{-x}$, where c_1 is an arbitrary constant.**

Since $y(x)$ is a solution of the differential equation for every value of c_1, we seek that value of c_1 which will also satisfy the initial condition. Note that $y(3) = c_1 e^{-3}$. To satisfy the initial condition $y(3) = 2$, it is sufficient to choose c_1 so that $c_1 e^{-3} = 2$, that is, to choose $c_1 = 2e^3$. Substituting this value for c_1 into $y(x)$, we obtain $y(x) = 2e^3 e^{-x} = 2e^{3-x}$ as the solution of the initial-value problem.

2.6. **Find a solution to the initial-value problem $y'' + 4y = 0$; $y(0) = 0$, $y'(0) = 1$, if the general solution to the differential equation is known to be (see Chapter 12) $y(x) = c_1 \sin 2x + c_2 \cos 2x$.**

Since $y(x)$ is a solution of the differential equation for all values of c_1 and c_2 (see Example 2.1), we seek those values of c_1 and c_2 that will also satisfy the initial conditions. Note that $y(0) = c_1 \sin 0 + c_2 \cos 0 = c_2$. To satisfy the first initial condition, $y(0) = 0$, we choose $c_2 = 0$. Furthermore, $y'(x) = 2c_1 \cos 2x - 2c_2 \sin 2x$; thus, $y'(0) = 2c_1 \cos 0 - 2c_2 \sin 0 = 2c_1$. To satisfy the second initial condition, $y'(0) = 1$, we choose $2c_1 = 1$, or $c_1 = \frac{1}{2}$. Substituting these values of c_1 and c_2 into $y(x)$, we obtain $y(x) = \frac{1}{2} \sin 2x$ as the solution of the initial-value problem. (See Example 2.5.)

2.7. **Find a solution to the boundary-value problem $y'' + 4y = 0$; $y\left(\frac{\pi}{8}\right) = 0$, $y\left(\frac{\pi}{6}\right) = 1$, if the general solution to the differential equation is $y(x) = c_1 \sin 2x + c_2 \cos 2x$.**

Note that

$$y\left(\frac{\pi}{8}\right) = c_1 \sin\left(\frac{\pi}{4}\right) + c_2 \cos\left(\frac{\pi}{4}\right) = c_1(\tfrac{1}{2}\sqrt{2}) + c_2(\tfrac{1}{2}\sqrt{2})$$

To satisfy the condition $y\left(\frac{\pi}{8}\right) = 0$, we require

$$c_1(\tfrac{1}{2}\sqrt{2}) + c_2(\tfrac{1}{2}\sqrt{2}) = 0$$

Furthermore,

$$y\left(\frac{\pi}{6}\right) = c_1 \sin\left(\frac{\pi}{3}\right) + c_2 \cos\left(\frac{\pi}{3}\right) = c_1(\tfrac{1}{2}\sqrt{3}) + c_2(\tfrac{1}{2})$$

To satisfy the second condition, $y\left(\frac{\pi}{6}\right) = 1$, we require

$$\tfrac{1}{2}\sqrt{3}\,c_1 + \tfrac{1}{2}c_2 = 1 \qquad (2)$$

Solving (1) and (2) simultaneously, we find

$$c_1 = -c_2 = \frac{2}{\sqrt{3}-1}$$

Substituting these values into $y(x)$, we obtain

$$y(x) = \frac{2}{\sqrt{3}-1}(\sin 2x - \cos 2x)$$

as the solution of the boundary-value problem.

2.8. Find a solution to the boundary-value problem $y'' + 4y = 0$; $y(0) = 1$, $y(\pi/2) = 2$, if the general solution to the differential equation is known to be $y(x) = c_1 \sin 2x + c_2 \cos 2x$.

Since $y(0) = c_1 \sin 0 + c_2 \cos 0 = c_2$, we must choose $c_2 = 1$ to satisfy the condition $y(0) = 1$. Since $y\left(\frac{\pi}{2}\right) = c_1 \sin \pi + c_2 \cos \pi = -c_2$, we must choose $c_2 = -2$ to satisfy the second condition, $y\left(\frac{\pi}{2}\right) = 2$. Thus, to satisfy both boundary conditions simultaneously, we must require c_2 to equal both one and minus two, which is impossible. Therefore, there does not exist a solution to this problem.

2.9. Determine c_1 and c_2 so that $y(x) = c_1 \sin 2x + c_2 \cos 2x + 1$ will satisfy the conditions $y\left(\frac{\pi}{8}\right) = 0$ and $y'\left(\frac{\pi}{8}\right) = \sqrt{2}$.

Note that

$$y\left(\frac{\pi}{8}\right) = c_1 \sin\left(\frac{\pi}{4}\right) + c_2 \cos\left(\frac{\pi}{4}\right) + 1 = c_1(\tfrac{1}{2}\sqrt{2}) + c_2(\tfrac{1}{2}\sqrt{2}) + 1$$

To satisfy the condition $y\left(\frac{\pi}{8}\right) = 0$, we require $c_1(\tfrac{1}{2}\sqrt{2}) + c_2(\tfrac{1}{2}\sqrt{2}) + 1 = 0$, or equivalently,

$$c_1 + c_2 = -\sqrt{2} \qquad (1)$$

Since $y'(x) = 2c_1 \cos 2x - 2c_2 \sin 2x$,

$$y'\left(\frac{\pi}{8}\right) = 2c_1 \cos\left(\frac{\pi}{4}\right) - 2c_2 \sin\left(\frac{\pi}{4}\right)$$

$$= 2c_1(\tfrac{1}{2}\sqrt{2}) - 2c_2(\tfrac{1}{2}\sqrt{2}) = \sqrt{2}\,c_1 - \sqrt{2}\,c_2$$

To satisfy the condition $y'\left(\frac{\pi}{8}\right) = \sqrt{2}$, we require $\sqrt{2}\,c_1 - \sqrt{2}\,c_2 = \sqrt{2}$, or equivalently,

$$c_1 - c_2 = 1 \qquad (2)$$

Solving (1) and (2) simultaneously, we obtain $c_1 = -\tfrac{1}{2}(\sqrt{2}-1)$ and $c_2 = -\tfrac{1}{2}(\sqrt{2}+1)$.

2.10. Determine c_1 and c_2 so that $y(x) = c_1 e^{2x} + c_2 e^{x} + 2 \sin x$ will satisfy the conditions $y(0) = 0$ and $y'(0) = 1$.

Because $\sin 0 = 0$, $y(0) = c_1 + c_2$. To satisfy the condition $y(0) = 0$, we require

$$c_1 + c_2 = 0 \qquad (1)$$

From
$$y'(x) = 2c_1 e^{2x} + c_2 e^{x} + 2\cos x$$
we have $y'(0) = 2c_1 + c_2 + 2$. To satisfy the condition $y'(0) = 1$, we require $2c_1 + c_2 + 2 = 1$, or
$$2c_1 + c_2 = -1 \tag{2}$$
Solving (*1*) and (*2*) simultaneously, we obtain $c_1 = -1$ and $c_2 = 1$.

Supplementary Problems

2.11. Which of the following functions are solutions of the differential equation $y'' - y = 0$? (*a*) e^x, (*b*) $\sin x$, (*c*) $4e^{-x}$, (*d*) 0, (*e*) $\frac{1}{2}x^2 + 1$.

2.12. Which of the following functions are solutions of the differential equation $y'' - 4y' + 4y = e^x$? (*a*) e^x, (*b*) e^{2x}, (*c*) $e^{2x} + e^x$, (*d*) $xe^{2x} + e^x$, (*e*) $e^{2x} + xe^x$.

In Problems 2.13–2.22, find c_1 and c_2 so that $y(x) = c_1 \sin x + c_2 \cos x$ will satisfy the given conditions. Determine whether the given conditions are initial conditions or boundary conditions.

2.13. $y(0) = 1, \quad y'(0) = 2.$

2.14. $y(0) = 2, \quad y'(0) = 1.$

2.15. $y\left(\frac{\pi}{2}\right) = 1, \quad y'\left(\frac{\pi}{2}\right) = 2.$

2.16. $y(0) = 1, \quad y\left(\frac{\pi}{2}\right) = 1.$

2.17. $y'(0) = 1, \quad y'\left(\frac{\pi}{2}\right) = 1.$

2.18. $y(0) = 1, \quad y'(\pi) = 1.$

2.19. $y(0) = 1, \quad y(\pi) = 2.$

2.20. $y(0) = 0, \quad y'(0) = 0.$

2.21. $y\left(\frac{\pi}{4}\right) = 0, \quad y\left(\frac{\pi}{6}\right) = 1.$

2.22. $y(0) = 0, \quad y'\left(\frac{\pi}{2}\right) = 1.$

In Problems 2.23–2.27, find values of c_1 and c_2 so that the given functions will satisfy the prescribed initial conditions.

2.23. $y(x) = c_1 e^x + c_2 e^{-x} + 4\sin x; \quad y(0) = 1, \quad y'(0) = -1.$

2.24. $y(x) = c_1 x + c_2 + x^2 - 1; \quad y(1) = 1, \quad y'(1) = 2.$

2.25. $y(x) = c_1 e^x + c_2 e^{2x} + 3e^{3x}; \quad y(0) = 0, \quad y'(0) = 0.$

2.26. $y(x) = c_1 \sin x + c_2 \cos x + 1; \quad y(\pi) = 0, \quad y'(\pi) = 0.$

2.27. $y(x) = c_1 e^x + c_2 x e^x + x^2 e^x; \quad y(1) = 1, \quad y'(1) = -1.$

Answers to Supplementary Problems

2.11. (*a*), (*c*), (*d*)

2.12. (*a*), (*c*), (*d*)

2.13. $c_1 = 2$, $c_2 = 1$; initial conditions

2.14. $c_1 = 1$, $c_2 = 2$; initial conditions

2.15. $c_1 = 1$, $c_2 = -2$; initial conditions

2.16. $c_1 = c_2 = 1$; boundary conditions

2.17. $c_1 = 1$, $c_2 = -1$; boundary conditions

2.18. $c_1 = -1$, $c_2 = 1$; boundary conditions

2.19. no values; boundary conditions

2.20. $c_1 = c_2 = 0$; initial conditions

2.21. $c_1 = \dfrac{-2}{\sqrt{3}-1}$, $c_2 = \dfrac{2}{\sqrt{3}-1}$; boundary conditions

2.22. no values; boundary conditions

2.23. $c_1 = -2$, $c_2 = 3$

2.24. $c_1 = 0$, $c_2 = 1$

2.25. $c_1 = 3$, $c_2 = -6$

2.26. $c_1 = 0$, $c_2 = 1$

2.27. $c_1 = 1 + \dfrac{3}{e}$, $c_2 = -2 - \dfrac{2}{e}$