# WALSH FUNCTIONS IN SIGNAL AND SYSTEMS ANALYSIS AND DESIGN

Edited by

**SPYROS G. TZAFESTAS**

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# SERIES EDITOR'S FOREWORD

This Benchmark Series in Electrical Engineering and Computer Science is aimed at sifting, organizing, and making readily accessible to the reader the vast literature that has accumulated. Although the series is not intended as a complete substitute for a study of this literature, it will serve at least three major critical purposes. In the first place, it provides a practical point of entry into a given area of research. Each volume offers an expert's selection of the critical papers on a given topic as well as his views on its structure, development, and present status. In the second place, the series provides a convenient and time-saving means for study in areas related to but not contiguous with one's principal interests. Last, but by no means least, the series allows the collection, in a particularly compact and convenient form, of the major works on which present research activities and interests are based.

Each volume in the series has been collected, organized, and edited by an authority in the area to which it pertains. In order to present a unified view of the area, the volume editor has prepared an introduction to the subject, has included his comments on each article, and has provided a subject index to facilitate access to the papers.

We believe that this series will provide a manageable working library of the most important technical articles in electrical engineering and computer science. We hope that it will be equally valuable to students, teachers, and researchers.

*Walsh Functions in Signal and Systems Analysis and Design* is edited by S. G. Tzafestas of the National Technical University in Athens, Greece. It consists of thirty-eight papers and editor's comments on Walsh functions—their definition, generation, computation and application to system analysis, identification, and design.

JOHN B. THOMAS

# PREFACE

Walsh functions constitute a set of orthogonal functions that have received increasing attention in recent years in a variety of engineering areas such as communication, signal processing, system analysis, and control. Since Walsh functions and transforms are naturally more suited for digital computation, an effort has been made to gradually replace the Fourier transform by Walsh-type transforms. Actually, the Walsh function field has experienced a significant development, and a large amount of theoretical and applied results are presently available.

This volume is the outcome of the editor's feeling that a cohesive book is needed to bring together the major efforts and results of the technical literature on Walsh functions. Thirty-eight papers have been reprinted and additional works are mentioned in the editor's comments.

The book is organized into eight parts. Each part contains original works that have appeared in various technical journals. Existing textbooks on Walsh functions cover only specialized aspects and present the theory from particular angles of attack, while *Walsh Functions in Signal and Systems Analysis and Design* involves a large diversity of results and explains how these results have been achieved. My effort here has been to compose a volume containing a well balanced spectrum of works carried out since the appearance of Harmuth's pioneering paper in 1968 (Paper 1). It is hoped that the book will provide a useful source to researchers and practitioners in the field of Walsh and block pulse functions and their applications.

SPYROS G. TZAFESTAS

# CONTENTS BY AUTHOR

# CONTENTS

## PART VI: BLOCK-PULSE FUNCTIONS



## PART VII: MISCELLANEOUS PROPERTIES: WALSH-TO-FOURIER CONVERSION



## PART VIII: WALSH TRANSFORM APPLICATIONS

# INTRODUCTION

*Walsh functions* belong to the class of piece-wise constant basis functions (PCBF) that have been developed in the twentieth century and have played an important role in scientific and engineering applications. The mathematical techniques of studying functions, signals, and systems through series expansions in orthogonal-complete sets of basis functions are now a standard tool in all branches of science and engineering. Actually, the signals involved in Morse telegraphy are PCBFs, but no mathematical study of these signals was made prior to the beginning of the twentieth century.

The origin of the mathematical study of PCBFs is due to Alfred Haar (1910; 1912), who used a set of functions bearing his name. These functions have not found much use in comparison to the Walsh and block pulse functions considered in this book (Harmuth, 1969; Ahmed and Rao, 1975; Beauchamp, 1975; Prasada Rao, 1983; Tzafestas, 1983; Prasada Rao and Tzafestas, 1985). The development and utilization of Walsh functions has been strongly influenced by the parallel developments in digital electronics and computer science and engineering. Efforts to replace Fourier transforms by Walsh-type transforms have been made in communication, signal processing, image processing, pattern recognition, and so forth. The entrance of Walsh functions into the systems and control field was only about a decade ago, the developments since then occurring rapidly.

The basic definitions of Walsh functions and the ways of generating them by hardware devices are given in Part I. Algorithms for Walsh transform computation, as well as hardware implementations of these algorithms, are discussed in Part II, which includes the works on the sampling principle expressed in terms of the frequency in the Walsh domain (sequency).

The analysis of dynamic systems via Walsh function expansions is presented in Part III. Lumped parameter, distributed parameter, linear, nonlinear, multivariable, time delay, and integral equation-type systems are considered. The work on system parameter estimation via Walsh functions is presented in Part IV where the types of systems considered in Part III are studied.

The problems of designing optimal controllers, observers, and filters via Walsh functions are discussed in Part V. A diversity of results are included, covering most cases studied thus far. Lumped parameter, distributed parameter, and time delay systems are studied.

*Introduction*

Part VI discusses the application of block pulse functions to systems analysis, identification, and control. Block pulse functions give results similar to those obtained through Walsh functions and offer considerable computational advantage, Block functions can be used for systems discretization in the same way that the bilinear transformation and the state transition matrix are used. A comparison of their relative characteristics is provided by Sinha and Zhou (1983). The one-shot operational matrices of Prasada Rao and Palanisamy (Paper 15) play an important role in reducing the errors when using Walsh and block pulse functions in high-order system analysis and design. Also noteworthy are the multidimensional block pulse functions that are useful in the study of distributed parameter systems described by partial differential equations (Nath and Lee, 1983; Prasada Rao and Srinivasan, 1980).

The properties of Walsh transforms and the process of Walsh-to-Fourier transform conversion are studied in Part VII. Aspects such as Walsh transform expression in terms of function derivatives, relation of cyclic autocorrelation with Walsh-Hadamard transform, and relation between arithmetic and logical autocorrelation functions are considered.

A representative set of Walsh transform applications is provided in Part VIII. These applications include biomedical signal (such as the EEG signal) processing, image coding, speech coding, statistical analysis, Boolean function classification, neutron transport theory, nuclear reactor test input design, and system-response time measurement.

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Part I

# WALSH SIGNAL DEFINITION AND GENERATION

# Editor's Comments on Papers 1, 2, and 3



The foundations of the Walsh functions field were made by Rademacher (1922), Walsh, (1923), Fine (1949, 1950), Paley (1952), and Kaczmarz and Steinhaus (1951). The engineering approach to the study and utilization of these functions was originated by Harmuth (1969*a*, 1969*b*), who introduced the concept of sequency to represent the associated, generalized frequency defined as one-half the mean rate of zero crossings. The variety of Walsh function definitions is due to the existence of different orderings. In the sequency ordering (or Walsh ordering), which is popular in communication engineering, Walsh functions are ordered according to the zero crossings (or sign changes). This sequency ordering implies that the *i*th Walsh function *wal* $(i, t)$ has *i* zero crossings in the interval $t \epsilon [0,1]$, and, obviously, is directly related to the sequency concept. The Paley ordering (Paley, 1952) is characterized by the fact that in this form Walsh functions are represented by products of Rademacher functions, which lead to useful, recursive, Walsh-signal generation algorithms. A third ordering was proposed by Henderson (1964, 1970) and is merely Paley's ordering in reversed binary. Henderson's ordering is computationally attractive and occurs when one computes fast Walsh transforms (FWT) without sorting. Yuen (1972) called the index *i*, in *wal* $(i, t)$, the "zequency" of *wal* $(i, t)$ to show that it represents both the generalized frequency and the number of zeros of *wal* $(i, t)$.

Paper 1 provides the background of Walsh transform theory, defines and uses the sequency concept, and introduces the *sal* and *cal* functions, which are analogous to the *sine* and *cosine* functions. Paper 2 presents a set of terms and definitions that standardize the analysis of Walsh functions. These definitions are nonambiguous and can assist workers and authors of papers to present their results in a unifying way. Paper 3 provides the logic design and implementation of a useful, programmable Walsh function generator.

Other definitions of Walsh functions have been given by Lackey and Meltzer (1971) through the Rademacher function products, and by Butin (1972) using polynomial expansions. The Butin definition leads to an easy way of deriving certain Walsh function properties, for example, their connection with the shifting theorem or the Gray code. Proof of the Lackey and Meltzer technique for selecting the combination of Rademacher functions generating a desired Walsh function is given in a paper by Davies (1972). In the same paper a digital Walsh-function generation scheme is also developed. Durrani and Nightingale (1971) provide sequential circuits for the generation of three sets of discrete Walsh functions without reference to any alternative sequency representation. Kitai and Siemens (1972) developed a sequency-ordered, Walsh-function generation scheme, which employs a Gray code counter and is free from hazards. Tzafestas, Frangakis, and Pimenidis (1976) gave a unified presentation of some Walsh function definitions, and on the basis of these definitions they designed global, digital, Walsh-function generating circuits.

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# 1


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# A Generalized Concept of Frequency and Some Applications

HENNING F. HARMUTH, MEMBER, IEEE

***Abstract*—The system of sine and cosine functions has been distinguished historically in communications. Whenever the term frequency is used, reference is made implicitly to these functions; hence the generally used theory of communication is based on the system of sine and cosine functions. In recent years other complete systems of orthogonal functions have been used for theoretical investigations as well as for equipment design. Analogs to Fourier series, Fourier transform, frequency, power spectra, and amplitude, phase, and frequency modulation exist for many systems of orthogonal functions. This implies that theories of communication can be worked out on the basis of these systems. Most of these theories are of academic interest only. However, for the complete system of the orthogonal Walsh functions, the implementation of circuits by modern semiconductor techniques appears to be competitive in a number of applications with the implementation of circuits for the system of sine and cosine functions.**

## Introduction

THE SYSTEM of sine and cosine functions plays a distinguished role in communications. There are a number of historical and practical reasons for this. From the theoretical point of view, one of the major reasons is that Fourier series and Fourier transform permit the representation of a large class of functions by a superposition of sine and cosine functions. This representation makes it possible to apply the concept of frequency, which was originally defined for sine and cosine only, to other functions.

In recent years more general classes of complete systems of orthogonal functions have been used for theoretical investigations as well as equipment design.[1]–[7],[27] Furthermore, semiconductor devices have made it practical to use linear time-variable circuits instead of linear time-invariant ones. While sine and cosine have indisputable advantages for linear time-invariant circuits they often lead to unnecessary complications if used to analyze time-variable circuits.

The purpose of this paper is to show that analogs exist to Fourier series, Fourier transform, frequency, power spectra, and amplitude, phase, and frequency modulation for many systems of orthogonal functions. For one such system, that of the Walsh functions, the experimental level has been reached for the analogs to frequency multiplex telephony and other applications.[1] Hence, we will be mainly concerned with these Walsh functions.[8]–[16]

## Generalized Fourier Transform

It is well known that the analog to the Fourier series exists for many systems of orthogonal functions. Examples are series expansions in Bessel functions, spherical functions, orthogonal polynomials, etc. It is also known that analogs to the Fourier transform exist for many systems of functions.[17],[18] However, most of the generally used complete systems of orthogonal functions are defined by linear differential equations of second order, and it can be shown that the generalized Fourier transform is in this case essentially the same as the ordinary Fourier transform. Consider, e.g., the differential equation of the Legendre polynomials:

$$(1 - x^2)y'' - 2xy' + j(j + 1) = 0. \tag{1}$$

The generalized Fourier transform of a function that vanishes outside a finite interval $x_1 < x < x_2$ consists of a superposition of Legendre polynomials with large value of $j$ and small values of $x$. The differential equation (1) reduces in this case to that of the sine and cosine functions, and the Legendre–Fourier transform to the ordinary Fourier transform except for scale factors.[19]

Walsh functions can be defined by a difference equation rather than a differential equation, and the generalized Fourier transform or Walsh–Fourier transform is quite different from the ordinary Fourier transform.[15] Such a transform has been known for some time,[2] but its

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[1] Circuits of filters, modulators, Walsh function generators, etc., for telephony multiplex systems are shown in previous papers.[28] More general filters are discussed in a paper by Pichler.[29] Radio transmission by Walsh carriers is treated in "Über Funkverkehr mit Sequenzteilung an Stelle von Frequenz- oder Zeitteilung," Defense Department, Federal Republic of Germany Project Rept. T-675-L-203. Speech analysis was done by M. Tasto in a thesis, "Analyse von Zeitfunktionen durch Mäandertransformation und durch Fourier-Transformation," Technische Hochschule Darmstadt, Institut für allgemeine Nachrichtentechnik. The results of this thesis are presently used by M. Boesswetter of the same institute for development of a sequency channel vocoder. Experimental work on filters is further done by Prof. H. Lueg, Technische Hochschule Aachen. Filter and modulator circuits are also shown in Harmuth.[26] A thelphony multiplex system using Walsh functions as carriers has been developed by H. Lüke and R. Maile of AEG-Telefunken Research Institute. Schemes for applying the dyadic correlation function $\int F(t)G(t + T)\,dt$ instead of the usual one, for producing Walsh-shaped instead of sine-shaped electromagnetic waves in the visible-light region, for high-resolution and shape-recognizing Walsh-wave radar, etc., are in the theoretical stage.

[2] The Walsh–Fourier transform, in a different form from the one used here, is due to N. J. Fine. The term "Fine transform" is sometimes used.

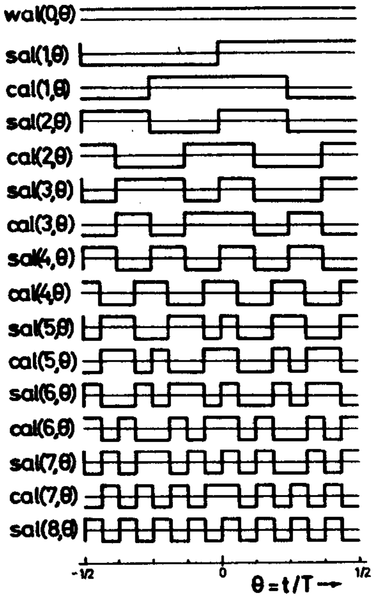


Fig. 1. Walsh functions wal $(0, \theta)$, cal $(i, \theta)$, and sal $(i, \theta)$.

form was not suitable for a generalization of the concept of frequency since the Walsh functions were ordered in a sequence that followed from their original definition by products of Rademacher functions.[20] The difference equation yields them in the sequence shown by Fig. 1. In this figure they are ordered according to the number of sign changes or zero crossings in the half-open interval $-\frac{1}{2} \leqq \theta < \frac{1}{2}$. The functions cal $(i, \theta)$ and sal $(i, \theta)$ have $2i$ zero crossings in this interval; $i = 1, 2, \cdots$. Furthermore, all functions cal $(i, \theta)$ equal $+1$ and all functions sal $(i, \theta)$ change from $-1$ to $+1$ for $\theta = 0$.[3]

Using the notation[4]

$$\mathrm{wal}\,(2i, \theta) = \mathrm{cal}\,(i, \theta),$$

$$\mathrm{wal}\,(2i - 1, \theta) = \mathrm{sal}\,(i, \theta), \quad -\tfrac{1}{2} \leqq \theta < \tfrac{1}{2}$$

$$\mathrm{wal}\,(2i, \theta) = \mathrm{wal}\,(2i - 1, \theta) \equiv 0, \quad \theta < -\tfrac{1}{2}, \quad \theta \geqq +\tfrac{1}{2}$$

one may write the difference equation of the Walsh functions in the following form:[5]

$$\mathrm{wal}\,(2k + q, \theta) = \mathrm{wal}\,(-1)^{[k/2]+q}[k, 2\theta + \tfrac{1}{2}) + (-1)^{k+q}\,\mathrm{wal}\,(k, 2\theta - \tfrac{1}{2})]$$

$$\mathrm{wal}\,(0, \theta) = 1, \quad -\tfrac{1}{2} \leq \theta < +\tfrac{1}{2}$$

$$= 0, \quad \theta < -\tfrac{1}{2}, \quad \theta \geq +\tfrac{1}{2}$$

$$q = 0 \quad \text{or} \quad 1; \qquad k = 0, 1, 2, \cdots.$$

[3] The functions $-\mathrm{sal}\,(1, \theta)$ and $+\mathrm{sal}\,(2^k, \theta)$, $k = 1, 2, \cdots$, are the Rademacher functions.

[4] The notations sal $(i, \theta)$ and cal $(i, \theta)$ have the advantage of showing the similarity to sine and cosine. The notation wal $(k, \theta)$ is frequently better suited for computation. For instance, the multiplication theorems (22) reduce to one theorem wal $(i \oplus k, \theta) =$ wal $(i, \theta)$ wal $(k, \theta)$. In analogy, the multiplication theorems (21) for sine and cosine reduce to one theorem if complex notation is used.

[5] $[k/2]$ means the largest integer smaller than or equal to $k/2$.

For explanation consider the Walsh function wal $(0, \theta)$ of Fig. 1. Shifting it by $\frac{1}{2}$ to the left into the interval $-1 \leqq \theta < 0$ yields wal $(0, \theta + \frac{1}{2})$ and compressing it by a factor 2 into the interval $-\frac{1}{2} \leqq \theta < 0$ yields wal $(0, 2\theta + \frac{1}{2})$. Similarly, wal $(0, 2\theta - \frac{1}{2})$ is obtained by shifting wal $(0, \theta)$ to the right and compressing it into the interval $0 \leqq \theta < \frac{1}{2}$. For $q = 0$ we obtain the sum $(-1)^{0+0}$[wal $(0, 2\theta + \frac{1}{2}) + (-1)^{0+0}$ wal $(0, 2\theta - \frac{1}{2})$] which evidently equals wal $(0, \theta)$. For $q = 1$ we obtain the sum $(-1)^{0+1}$ [wal $(0, >\theta + \frac{1}{2}) + (-1)^{0+1}$ wal $(0, 2\theta - \frac{1}{2})$] which equals wal $(1, \theta) =$ sal $(1, \theta)$. Similarly, one obtains from wal $(1, \theta)$ the shifted and compressed functions wal $(1, 2\theta + \frac{1}{2})$ and wal $(1, 2\theta - \frac{1}{2})$. The sum $(-1)^{0+0}$[wal $(1, 2\theta + \frac{1}{2}) + (-1)^{1+0}$ wal $(1, 2\theta - \frac{1}{2})$] yields wal $(2, \theta) =$ cal $(1, \theta)$, and the sum $(-1)^{0+1}$[wal $(1, 2\theta + \frac{1}{2}) + (-1)^{1+1}$ wal $(1, 2\theta - \frac{1}{2})$] yields wal $(3, \theta) =$ sal $(2, \theta)$.

A Walsh–Fourier series expansion of a function $F(\theta)$ defined in the interval $-\frac{1}{2} \leqq \theta < \frac{1}{2}$ has the following form:

$$F(\theta) = a(0)\,\mathrm{wal}\,(0, \theta) + \sum_{i=1}^{\infty} [a_c(i)\,\mathrm{cal}\,(i, \theta) + a_s(i)\,\mathrm{sal}\,(i, \theta)], \tag{2}$$

where

$$a(0) = \int_{-1/2}^{1/2} F(\theta)\,\mathrm{wal}\,(0, \theta)\,d\theta = \int_{-1/2}^{1/2} F(\theta)\,d\theta,$$

$$a_c(i) = \int_{-1/2}^{1/2} F(\theta)\,\mathrm{cal}\,(i, \theta)\,d\theta,$$

$$a_s(i) = \int_{-1/2}^{1/2} F(\theta)\,\mathrm{sal}\,(i, \theta)\,d\theta.$$

The set of functions required for the Walsh–Fourier transform may be derived by stretching the functions of Fig. 1 by a factor $\xi$ and denoting the stretched functions cal $(i, \theta)$ and sal $(i, \theta)$ by cal $(i/\xi, \theta)$ and sal $(i/\xi, \theta)$. If $\xi$ and $i$ approach infinity in such a way that the limit

$$\lim_{\xi, i \to \infty} i/\xi = \mu \tag{3}$$

exists, one obtains the system of Walsh meander functions $\{$cal $(\mu, \theta)$, sal $(\mu, \theta)\}$ which are defined in the interval $-\infty < \theta < +\infty$ for all real non-negative values of $\mu$. It is useful to extend the definition to negative values of $\mu$ by making cal $(\mu, \theta)$ a symmetric and sal $(\mu, \theta)$ a skew symmetric function of $\mu$:

$$\mathrm{cal}\,(-\mu, \theta) = \mathrm{cal}\,(\mu, \theta), \ \mathrm{sal}\,(-\mu, \theta) = -\mathrm{sal}\,(\mu, \theta). \tag{4}$$

The derivation of the functions cal $(\mu, \theta)$, sal $(\mu, \theta)$ from the Walsh functions cal $(i, \theta)$, sal $(i, \theta)$ is discussed in a more mathematical fashion by Pichler.[15]

The Walsh–Fourier transform of a function $F(\theta)$ and its inverse have the following form:

$$a_c(\mu) = \int_{-\infty}^{\infty} F(\theta)\,\mathrm{cal}\,(\mu, \theta)\,d\theta, \qquad a_s(\mu) = \int_{-\infty}^{\infty} F(\theta)\,\mathrm{sal}\,(\mu, \theta)\,d\theta, \tag{5}$$