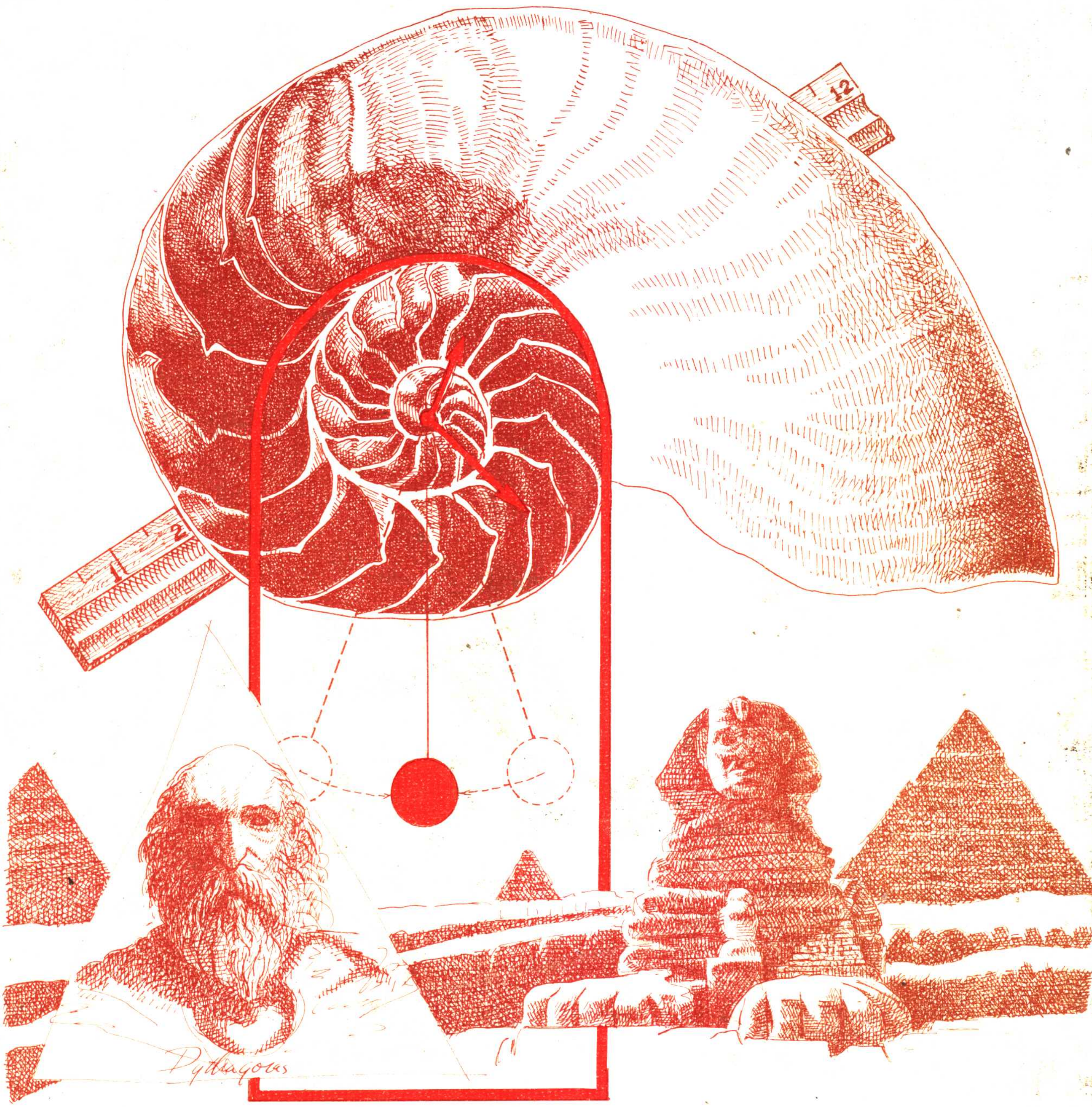
TRIGONOMETRY WITH APPLICATIONS

M. N. MANOUGIAN



TRIGONOMETRY WITH APPLICATIONS

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This text is the result of extensive classroom experimentation and is intended for use in a first course in college trigonometry. The text contains the traditional subject matter with special emphasis being placed on informal presentation, motivation, relevance, and testing.

There are seven chapters and appendices. Each chapter is divided into sections which begin with a statement of objectives. Mathematical concepts are introduced with simple examples, many of which illustrate the applicability of trigonometry to real world situations. Numerous worked-out examples are given to illustrate the concepts presented, and techniques and drills are emphasized throughout the text. Each section is concluded with a large number of drill exercises. The student is made aware that it is by doing mathematics that one learns mathematics. The exercises which are graded in difficulty, are designed to help the student's understanding of the concepts discussed in each section. Following each chapter is a review emphasizing the main ideas discussed in the chapter. These serve as self-tests. Also, at the end of each chapter is a sample test.

In order to help students in their learning process, other features are also included. In each section, the student is asked to respond to simple questions under the heading "Self-test". The answers to these questions are provided at the bottom of the page for instant feedback. This feature is used to ensure student understanding of the material discussed and to prepare them for the exercises. Also, these serve as stopping points for the instructor who wishes to divide each section into subsections. Metric units are used extensively and a brief summary of Metric to English and English to Metric conversions are included in the Appendix. Finally, for better understanding and general interest, historical developments of some of the concepts and biographical sketchs of some of the famous mathematicians are presented.

The book is designed for a one-semester or a one-quarter course. Instructors may wish to create a syllabus and set their own pace and areas of emphasis. Various syllabi may be devised for a course. Chapter One is a review of functions which may be omitted for students with a course in college algebra. The trigonometric functions are introduced as functions whose domain is a subset of all angles. The circular functions are also introduced and the relationship between the two is indicated. Although computations are included, they are de-emphasized in view of the wide use of scientific calculators. In fact, this is why the chapter on logarithms is placed in the Appendix to be used as the need arises.

The author wishes to thank Janice Kartsatos for proofreading and checking the answers to the examples and exercises, the many students at the University of South Florida who have participated in the preparation of the manuscript, and the several prepublication reviewers for their comments and helpful suggestions.

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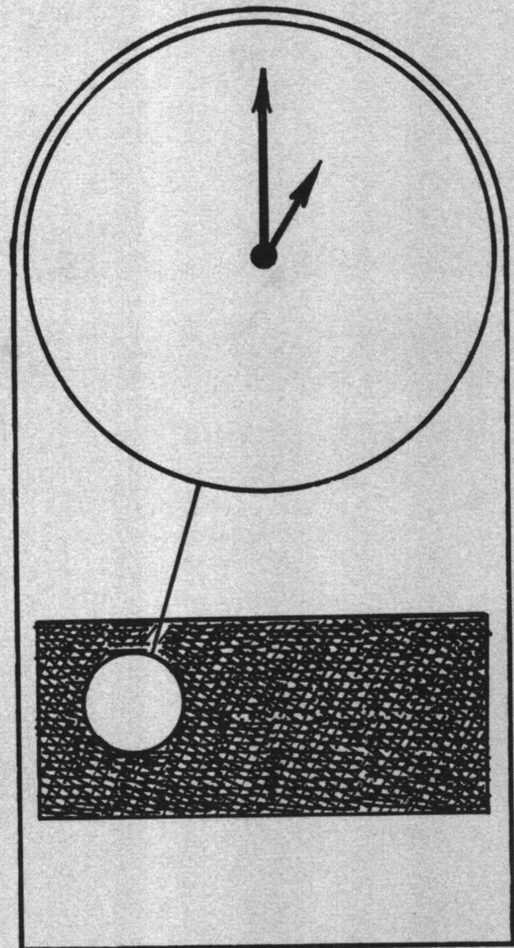
Tampa, Florida

TABLE OF CONTENTS



CHAPTER ONE:
REVIEW OF FUNCTIONS

1.1 THE RECTANGULAR COORDINATE SYSTEM

OBJECTIVES

1. *Define Cartesian product.*
2. *Introduce the concept of one-to-one correspondence.*
3. *Construct the number line.*
4. *Set up the Rectangular coordinate system.*

In this section we present a type of set* called the **Cartesian product**. This set is used to describe the association between two sets of numbers or objects. In everyday life we often encounter associations between two sets. For example,

interest earned and amount invested
time of day and temperature
month of year and the rise in food prices
students in this class and grades
cost and profit
distance traveled by a car and its speed.

RENE DESCARTES (1596-1650)

Descartes' coordinate geometry provided a vehicle whereby points in the plane could be viewed as ordered pairs of numbers and ordered pairs of numbers could be viewed as points in the plane. Thus, coordinate geometry unified algebra and geometry, enabling us to do geometry problems by algebra. The concept of the Cartesian product gives us an analytic way of viewing the geometric concept of dimension. Thus, one-dimensional space is represented by R, two-dimensional space by R^2 and so on.

*For a brief review of sets see Appendix A.1.

It might be helpful to take a moment and make up another list showing an association between two sets. The following two examples give some motivation for introducing the idea of a Cartesian product. This concept will then be used to introduce the Rectangular coordinate system.

EXAMPLE 1. Political analysts study the candidates of different political parties by pairing the leading candidate of one party with the leading candidate of another party. Suppose in a presidential election, the Republicans have three leading candidates, denoted by R_1, R_2, and R_3, and the Democrats have two leading candidates denoted by D_1, and D_2. All possible two-man races can be represented by the set of ordered pairs

$$\{(R_1,D_1),(R_2,D_1),(R_3,D_1),(R_1,D_2),(R_2,D_2),(R_3,D_2)\}$$

It might be helpful for the student to visualize the ordered pairs as the array shown in Table 1. In each ordered pair we have two entries, a Republican candidate and a Democratic candidate in that order.

	D_1	D_2
R_1	(R_1,D_1)	(R_1,D_2)
R_2	(R_2,D_1)	(R_2,D_2)
R_3	(R_3,D_1)	(R_3,D_2)

Table 1

	Cafe-c	Antonio's-a
ballet-b	(b,c)	(b,a)
jazz-j	(j,c)	(j,a)
play-p	(p,c)	(p,a)

Table 2

EXAMPLE 2. Michael and Terri are discussing their date for the evening. Michael suggests the following choices: a ballet performance, a jazz concert, or a play. Terri suggests that after the show they go to the Cafe de Paris restaurant or to Antonio's Pizza Parlor. Assuming that they are going to a show first and eat later, Michael and Terri can spend their evening in one of six ways. We illustrate the different possibilities in the array shown in Table 2.

Using set notation, we let $A = \{b,j,p\}$ and $B = \{c,a\}$. From these two sets, we form a new set called the Cartesian product of A and B and denoted by $A\times B$ (read "A cross B").

$$A\times B=\{(b,c),(j,c),(p,c),(b,a),(j,a),(p,a)\}$$

For instance, one of the entries of the set $A\times B$ is (j,a), which represents the possibility of going to the jazz concert first and then going to Antonio's Pizza Parlor. The set $A\times B$ consists of all possible pairs of elements in which the first element belongs to A and the second element belongs to B.

A pair of elements in which the order is specified is called an **ordered pair of elements**. The symbol (j,a) denotes an ordered pair of elements and the set $A\times B$ consists of ordered pairs of elements. The element j is called the **first component** of the element,and the element a is called the **second component** of the ordered pair (j,a).

In general, the **Cartesian product** (or the **cross product**) of two sets is defined as follows:

> **DEFINITION 1.1** If A and B are two nonempty sets, then the *Cartesian product* of A and B, denoted by $A \times B$, is the set of all ordered pairs (a,b) such that $a \in A$ and $b \in B$. That is
>
> $$A \times B = \{(a,b) \mid a \in A \text{ and } b \in B\}$$

Since the order is important, the ordered pair (a,b) is generally different from the ordered pair (b,a). Thus, in Example 2, $A \times B \neq B \times A$. In fact, if Michael and Terri are starving, it makes a big difference whether they consider $A \times B$ or $B \times A$!

> ***Self-test.*** If $A = \{0,1\}$ and $B = \{x,y\}$, then:
>
> (a) $A \times B =$ ______________________
>
> (b) $B \times A =$ ______________________
>
> (c) $A \times A =$ ______________________

Now, consider the set of all real numbers R. The Cartesian product $R \times R$, which may be denoted by R^2, is the set of all possible ordered pairs of real numbers.

$$R \times R = \{(x,y) \mid x,y \in R\}$$

Before we discuss a pictorial representation of R^2 let us consider the number line. A convenient way to picture the real numbers is to represent them geometrically as the set of all points on a straight line. This will serve as an aid to understanding relations among real numbers. We shall assume the following:

For each real number, there corresponds one and only one point on the line, and conversely, for each point on the line, there corresponds one and only one real number.

This property between two sets, the set of real numbers and the set of points on a line, is called a **one-to-one correspondence**. For example, the two sets $A = \{a,b,c\}$ and $B = \{0,1,2\}$ can be put into a one-to-one correspondence, since for every element of A we can assign a unique element of B and vice versa. The sets $C = \{x,y,z\}$ and $D = \{0,1\}$ cannot be placed in a one-to-one correspondence. For example, we may assign x to 1 and z to 0, but there will be no element left in D that can be assigned to y in C.

ANSWERS:

(a) $A \times B = \{(0,x),(0,y),(1,x),(1,y)\}$ (b) $B \times A = \{(x,0),(x,1),(y,0),(y,1)\}$

(c) $A \times A = \{(0,0),(0,1),(1,0),(1,1)\}$

We now illustrate the correspondence between the set of real numbers and the set of points on a line. Consider a line L and select:

(a) A point O called the **origin** and associate it with the number zero.
(b) A convenient point to the right of zero to represent the number 1.
(c) The **positive direction** as the direction traveled when going from zero to one. The opposite direction will be called the **negative direction** of the line when going from zero to the left.

The line segment from 0 to 1 is called the unit segment and its length is chosen as the unit length. The line L is then called a **number line** (also called a **directed line** or **coordinate line**). Ordinarily (but not necessarily), such a line is drawn horizontally, and the choice of the representation of the 1 to the right of 0 is arbitrary (see Figure 1).

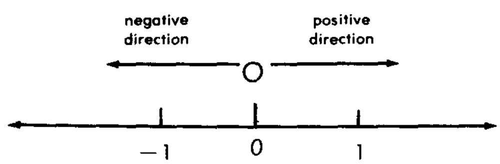


Figure 1

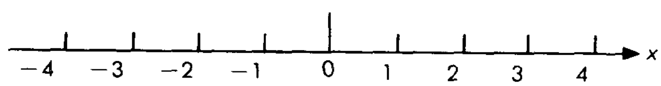


Figure 2

Next, we reproduce the unit length successively on both sides of 0 and 1 on the line L to obtain the graphical representation of the set of integers (see Figure 2). Note that the negative numbers are measured in the negative direction from 0.

Now we can construct a pictorial representation of R^2. We will use a **Rectangular** (or **Cartesian**) **coordinate system** to set up a correspondence between R^2 and the Cartesian plane, with each point in the plane corresponding to an ordered pair in R^2.

In order to set up a Rectangular coordinate system, we draw a horizontal line, which we call the **x-axis**. Then we draw a vertical line called the **y-axis** (see Figure 3). The point of intersection of the axes is called the **origin** and the two lines are called the **coordinate axes**. Usually the number zero is represented by the origin and a number scale is marked on each of the axes. On the *x*-axis, the positive numbers are to the right and on the *y*-axis, the positive numbers extend upward. The coordinate axes divide the plane into four **quadrants**. The four quadrants are:

$$\begin{aligned} &\text{Quadrant I:} && \{(x,y) \mid x>0,\ y>0\} \\ &\text{Quadrant II:} && \{(x,y) \mid x<0,\ y>0\} \\ &\text{Quadrant III:} && \{(x,y) \mid x<0,\ y<0\} \\ &\text{Quadrant IV:} && \{(x,y) \mid x>0,\ y<0\} \end{aligned}$$

To avoid ambiguities, the coordinate axes are excluded from the quadrants.

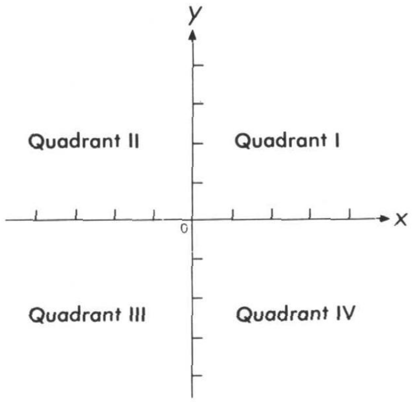


Figure 3

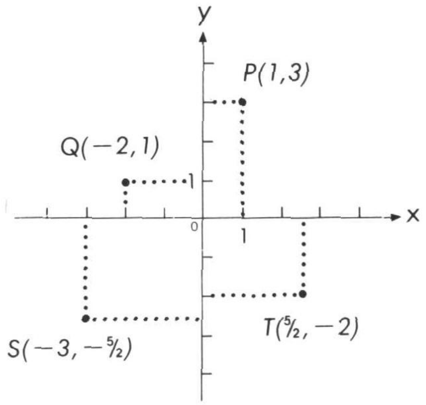


Figure 4

Looking at Figure 4, we see that the ordered pairs of numbers (1,3), (−2,1), $(-3, -\frac{5}{2})$ and $(\frac{5}{2}, -2)$ are represented by the points *P*, *Q*, *S*, and *T* in quadrants I, II, III, and IV, respectively. The ordered pair (1,3) corresponds to the (unique) point one unit to the right of the *y*-axis (one unit in the positive *x*-direction) and three units above the *x*-axis (three units in the positive *y*-direction). The **coordinates** of the point *P* are the components 1 and 3 in the ordered pair of numbers (1,3). The first component is also referred to as the **abscissa** and the second component is called the **ordinate**. The word coordinates was first used by Leibniz.*

---

* GOTTFRIED WILHELM LEIBNIZ

Gottfried Wilhelm Leibniz (1646-1716), philosopher and metaphysician was born in Leipzig, Germany, the son of a professor of philosophy. Leibniz is considered one of the greatest thinkers of modern times, proficient in law, theology, geology, and history, as well as serving as a diplomat. Leibniz wrote with ease in five different languages, Greek, French, Latin, English, and his native German.He received his bachelor's degree in philosophy at age seventeen and had a doctorate in law when he was twenty. During his lifetime, he invented one of the first calculating machines and founded the Academy of Sciences in Berlin.

In mathematics, Leibniz contributed immensely to the development of calculus and symbolic logic. In spite of his accomplishments, his life ended sadly, troubled in his final years by ill health and an ongoing bitter dispute with Newton over priority for the discovery of the calculus.

Self-test. Referring to the points in Figure 4, find:

(a) Abscissa of Q = ____________

(b) Ordinate of Q = ____________

(c) Abscissa of S = ____________

(d) Ordinate of S = ____________

It is important to note that to each element in R^2*, there corresponds exactly one point in the plane, and to each point in the plane there corresponds exactly one element in* R^2.

Now we consider some subsets of R^2.

EXAMPLE 3. Let $A=\{1,2,3\}$ and $B=\{4,5\}$. Find each of the following:

(a) $A \times B$ (b) $B \times A$

Solution. We use Definition 1.1

$$\begin{aligned} \text{(a)} \quad A \times B &= \{(a,b) \mid a \in A \text{ and } b \in B\} \\ &= \{(1,4),(1,5),(2,4),(2,5),(3,4),(3,5)\} \\ \text{(b)} \quad B \times A &= \{(b,a) \mid b \in B \text{ and } a \in A\} \\ &= \{(4,1),(4,2),(4,3),(5,1),(5,2),(5,3)\} \end{aligned}$$

In Figure 5, the set of circled points represents $A \times B$, while the set of points with squares represents $B \times A$.

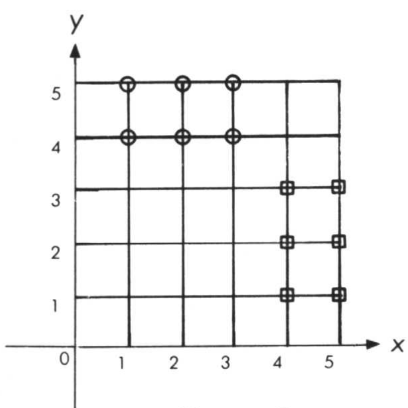


Figure 5

ANSWERS:

(a) −2 (b) 1 (c) −3 (c) −5/2

EXAMPLE 4. Let $A=\{t \mid -2 \leq t < 3\}$ and $B=\{2\}$. Find each of the following:

(a) $A \times B$ (b) $B \times A$

Solution. We use Definition 1.1

(a) $A \times B=\{(x,y) \mid -2 \leq x < 3 \text{ and } y=2\}$

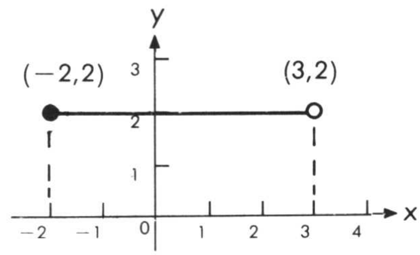


Figure 6

A geometric representation (or **graph**) of $A \times B$ is shown in Figure 6. Every element on the line between point $(-2,2)$ and $(3,2)$ represents an element of $A \times B$. The point $(-2,2) \in A \times B$. This is indicated by the shaded circle. On the other hand, $(3,2) \notin A \times B$, indicated by an unshaded circle.

(b) $B \times A=\{(x,y) \mid x=2 \text{ and } -2 \leq y < 3\}$

A geometric representation of $B \times A$ is shown in Figure 7.

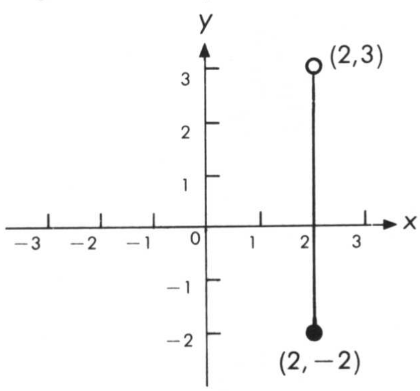


Figure 7

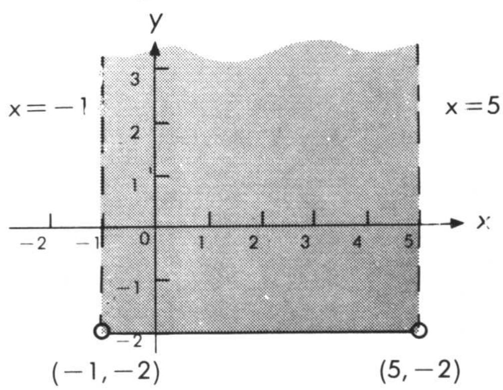


Figure 8

EXAMPLE 5. Let $A=\{t \mid |t-2| < 3\}$ and $B=\{t \mid t \geq -2\}$. Find each of the following:

(a) $A \times B$ (b) $B \times A$

Solution. We use Definition 1.1

(a)
$$\begin{aligned} A \times B &= \{(x,y) \mid x \in A \text{ and } y \in B\} \\ &= \{(x,y) \mid |x-2| < 3 \text{ and } y \geq -2\} \\ &= \{(x,y) \mid -1 < x < 5 \text{ and } y \geq -2\} \end{aligned}$$

A geometric representation of $A \times B$ is shown in Figure 8. The shaded region, including the solid line joining the points $(-1,-2)$ and $(5,-2)$, represents elements that belong to $A \times B$. Note that points on the broken lines $x=-1$ and $x=5$ do not belong to $A \times B$.