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Generalized Functions in Mathematical Physics

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Translated from the Russian
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Preface

As physics advances, its theoretical statements require ever "higher" mathematics. In this connection it is well worth quoting what the eminent English theoretical physicist Paul Dirac said in 1930 (Dirac [1]) when he gave a theoretical prediction of the existence of antiparticles:

> It seems likely that this process of increasing abstraction will continue in the future and that advance in physics is to be associated with a continual modification and generalisation of the axioms at the base of the mathematics rather than with a logical development of any one mathematical scheme on a fixed foundation.

Subsequent development of theoretical physics, particularly of quantum field theory, fully corroborated this view. Again in this connection we quote the apt words of N.N. Bogoliubov. In 1963 he said: "The basic concepts and methods of quantum field theory are becoming more and more mathematical."

The construction and investigation of mathematical models of physical phenomena constitute the subject of mathematical physics.

Since the time of Newton the search for and study of mathematical models of physical phenomena—the problems of mathematical physics—have made it necessary to resort to a wide range of mathematical tools and have thus stimulated the development of various areas of mathematics. Traditional (classical) mathematical physics had to do with the problems of classical physics: mechanics, hydrodynamics, acoustics, diffusion, heat conduction, potential theory, electrodynamics, optics and so forth. These problems all reduced to boundary-value problems for differential equations (the equations of mathematical physics). The basic mathematical tool for investigating such problems is the theory of differential equations and allied fields of mathematics: integral equations, the calculus of variations, approximate and numerical methods. With the advent of quantum physics, the range of mathematical tools expanded considerably through the use of the theory of operators, the theory of generalized functions, the theory of functions of complex variables, topological and algebraic methods, computational mathematics and computers. All these theories were pressed into service *in addition* to the traditional tools of mathematics. This intensive interaction of theoretical physics and mathematics gradually brought to the fore a new domain, that of modern mathematical physics.

To summarize, then: modern mathematical physics makes extensive use of the latest attainments of mathematics, one of which is the theory of generalized functions. The present monograph is devoted to a brief exposition of the fundamentals of that theory and of certain of its applications to mathematical physics.

At the end of the 1920's Paul Dirac (see Dirac [3]) introduced for the first time in his quantum mechanical studies the so-called delta function (δ function), which has the following properties:

$$\delta(x) = 0, \quad x \neq 0; \qquad \int \delta(x)\,\varphi(x)\,dx = \varphi(0), \quad \varphi \in C. \qquad (*)$$

It was soon pointed out by mathematicians that from the purely mathematical point of view the definition is meaningless. It was of course clear to Dirac himself that the δ function is not a function in the classical meaning and, what is important, it operates as an operator (more precisely, as a functional) that relates, via formula $(*)$, to each continuous function φ a number $\varphi(0)$, which is its value at the point 0. It required quite a few years and the efforts of many mathematicians[§] in order to find a mathematically proper definition of the delta function, of its derivatives and, generally, of a generalized function.

The foundations of the mathematical theory of generalized functions were laid by the Soviet mathematician S.L. Sobolev in 1936 (see Sobolev [1]) when he successfully applied generalized functions to a study of the Cauchy problem for hyperbolic equations. After World War II, the French mathematician L. Schwartz attempted, on the basis of an earlier created theory of linear locally convex topological spaces[§§], a systematic construction of a theory of generalized functions and explained it in his well-known monograph entitled *Théorie des distributions* [1] (1950-51). From then on the theory of generalized functions was developed intensively by many mathematicians. This precipitate development of the theory of generalized functions received its main stimulus from the requirements of mathematical and theoretical physics, in particular, the theory of differential equations and quantum physics. At the present time, the theory of generalized functions has advanced substantially and has found numerous applications in physics and mathematics, and more and more is becoming a workaday tool of the physicist, mathematician and engineer[§§§]. Generalized functions possess a number of remark-

§ See the pioneering works of Bochner [1] and Hadamard [1].

§§ See Dieudonné and Schwartz [1].

§§§ See Arsac [1], Bogoliubov, Logunov and Todorov [1], Bogoliubov, Medvedev and Polivanov [1], Bogoliubov and Shirkov [1], Bremermann [1], Ehrenpreis [1], Garding [1], Garsoux [1], Gelfand and Shilov [1], Gelfand able properties that extend the capabilities of classical mathematical analysis; for example, any generalized function turns out to be infinitely differentiable (in the generalized meaning), convergent series of generalized functions may be differentiated termwise an infinite number of times, there always exists the Fourier transform of a generalized function, and so on. For this reason, the use of generalized function techniques substantially expands the range of problems that can be tackled and leads to appreciable simplifications that make elementary operations automatic.

The present monograph is an expanded version of a course of lectures that the author has been delivering to students, postgraduates, and associates of the Moscow Physics and Technology Institute and the Steklov Mathematical Institute.

I take this opportunity to thank all my associates for their constructive criticism that has helped to improve the presentation. In particular I wish to thank my pupils Yu. N. Drozhzhinov, V.V. Zharinov and R. Kh. Galeev.

The first Russian edition of this book was sold out in a short time. In preparing the second edition, I have taken into account a number of remarks, and part of the material has been reworked and supplemented. Inaccuracies and misprints have been corrected. Significant changes have been introduced into the portion devoted to the theory of holomorphic functions with nonnegative imaginary part in tubular regions over acute cones (Secs. 16-18). This part embodies new results.

V. S. Vladimirov

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and Vilenkin [1], Hörmander [1], Jost [1], Malgrange [1], Palamodov [1], Reed and Simon [1], Schwartz [1, 2], Sobolev [1, 2], Streater and Wightman [1], Treves [1], Vladimirov [1, 2], Zemanian [1], and others.

Contents

Symbols and Definitions

0.1 We denote the *points* of an n-dimensional real space $\mathbb{R}^n$ by $x, y, \xi, \ldots; x = (x_1, x_2, \ldots, x_n)$. The points of an n-dimensional complex space $\mathbb{C}^n$ are given as $z, \zeta, \ldots; z = (z_1, z_2, \ldots, z_n) = x + iy$; $x = Re\, z$ is the real part of z and $y = Im\, z$ is the imaginary part of z, $\bar{z} = x - iy$ is the complex conjugate of z. In the usual manner we introduce in $\mathbb{R}^n$ and $\mathbb{C}^n$ the *scalar products*

$$(x, \xi) = x_1\xi_1 + \ldots + x_n\xi_n, \quad \langle z, \zeta\rangle = z_1\bar{\zeta}_1 + \ldots + z_n\bar{\zeta}_n$$

and the *norms* (*lengths*)

$$|x| = \sqrt{(x, x)} = (x_1^2 + \ldots + x_n^2)^{1/2},$$
$$|z| = \sqrt{\langle z_1, z\rangle} = (|z_1|^2 + \ldots + |z_n|^2)^{1/2}.$$

0.2 *Open sets* in $\mathbb{R}^n$ are denoted by $\mathcal{O}, \mathcal{O}', \ldots$; $\partial\mathcal{O}$ is the boundary of $\mathcal{O}$, or $\partial\mathcal{O} = \bar{\mathcal{O}} \setminus \mathcal{O}$. We will say that the set A *is compact in the open set* $\mathcal{O}$ (or *is strictly contained in* $\mathcal{O}$) if A is bounded and its closure $\bar{A}$ lies in $\mathcal{O}$; we then write $A \Subset \mathcal{O}$.

The following designations are used: $U(x_0; R)$ is an *open ball* of radius R with centre at the point x_0; $S(x_0; R) = \partial U(x_0; R)$ is a *sphere* of radius R with centre at the point x_0; $U_R = U(0; R)$, $S_R = S(0; R)$.

We use $\Delta(A, B)$ to denote the *distance* between the sets A and B in $\mathbb{R}^n$, that is,

$$\Delta(A, B) = \inf_{x \in A,\, y \in B} |x - y|.$$

We use A^ε to denote the ε-*neighbourhood* of a set A, $A^\varepsilon = A + U_\varepsilon$ (Fig. 1a). If $\mathcal{O}$ is an open set, then $\mathcal{O}_\varepsilon$ designates the set of those points of $\mathcal{O}$ which are separated from $\partial\mathcal{O}$ by a distance greater than ε (Fig. 1b):

$$\mathcal{O}_\varepsilon = [x : x \in \mathcal{O}, \Delta(x, \partial\mathcal{O}) > \varepsilon].$$

We use int A to denote the set of interior points of the set A.

The *characteristic function of a set* A is the function $\theta_A(x)$ which is equal to 1 when $x \in A$ and is equal to 0 when $x \bar{\in} A$. The characteristic function $\theta_{[0, \infty)}(x)$ of the semiaxis $x \geqslant 0$ is called the *Heaviside unit function* and is denoted $\theta(x)$ (Fig. 2):

$$\theta(x) = 0, \quad x < 0, \qquad \theta(x) = 1, \quad x \geqslant 0.$$

We write $\theta_n(x) = \theta(x_1) \ldots \theta(x_n)$.

The set A is said to be *convex* if for any points x' and x'' in A the line segment joining them, $tx' + (1-t)x''$, $0 \leqslant t \leqslant 1$, lies entirely in A.

We will use ch A to denote the *convex hull* of a set A.

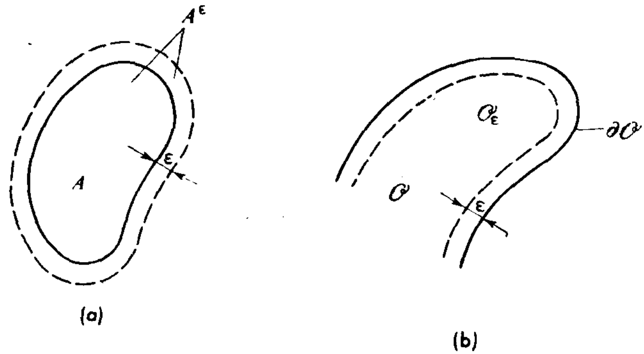


Figure 1

A real function $f(x) < +\infty$ is said to be *convex* on the set A if for any points x' and x'' in A such that the line segment $tx' + (1-t)x''$ joining them lies entirely in A the following inequality holds (Fig. 2b):

$$f(tx' + (1-t)x'') \leqslant tf(x') + (1-t)f(x'')$$

The function $f(x)$ is said to be *concave* if the function $-f(x)$ is convex.

0.3 The Lebesgue integral of a function f over an open set $\mathcal{O}$ is given as

$$\int_{\mathcal{O}} f(x)\,dx, \qquad \int_{\mathbb{R}^n} f(x)\,dx = \int f(x)\,dx.$$

The collection of all (complex-valued, measurable) functions f specified on $\mathcal{O}$ for which the norm

$$\|f\|_{\mathcal{L}^p(\mathcal{O})} = \begin{cases} \left[\int_{\mathcal{O}} |f(x)|^p\,dx\right]^{1/p}, & 1 \leqslant p < \infty, \\ \operatorname*{vrai\,sup}_{x\in\mathcal{O}} |f(x)|, & p = \infty, \end{cases}$$

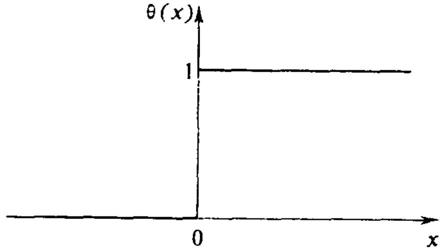
is finite will be denoted as $\mathcal{L}^p(\mathcal{O})$, $1 \leqslant p \leqslant \infty$; we write $\|\ \| = \|\ \|_{\mathcal{L}^2(\mathbb{R}^n)}$, $\mathcal{L}^p(\mathbb{R}^n) = \mathcal{L}^p$.

If $f \in \mathcal{L}^p(\mathcal{O}')$ for any $\mathcal{O}' \Subset \mathcal{O}$, then f is said to be *p-locally summable* in $\mathcal{O}$ (for $p = 1$, we say it is *locally summable in* $\mathcal{O}$).

The collection of p-locally summable functions in $\mathcal{O}$ is denoted $\mathcal{L}^p_{\text{loc}}(\mathcal{O})$, $\mathcal{L}^p_{\text{loc}}(\mathbb{R}^n) = \mathcal{L}^p_{\text{loc}}$.

A measurable function is said to be *finite* in $\mathcal{O}$ if it vanishes almost everywhere outside a certain $\mathcal{O}' \Subset \mathcal{O}$. The set of all functions in $\mathcal{L}^p(\mathcal{O})$ that are finite in $\mathcal{O}$ is denoted $\mathcal{L}^p_0(\mathcal{O})$.

(a)



(b)

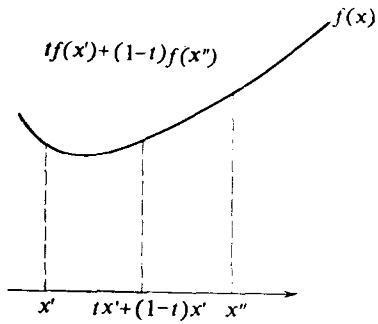


Figure 2

0.4 Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ be a multi-index, that is to say its components α_j are nonnegative integers. We have the following symbolism:

$$\alpha! = \alpha_1!\,\alpha_2!\ldots\alpha_n!, \qquad x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2}\ldots x_n^{\alpha_n},$$

$$\binom{\beta}{\alpha} = \binom{\beta_1}{\alpha_1}\binom{\beta_2}{\alpha_2}\cdots\binom{\beta_n}{\alpha_n} = \frac{\alpha!}{\beta!\,(\alpha-\beta)!},$$

$$|\alpha| = \alpha_1 + \alpha_2 + \ldots \alpha_n.$$

Let $D = (D_1, D_2, \ldots, D_n)$, $D_j = \frac{\partial}{\partial x_j}$, $j = 1, 2, \ldots, n$. Then

$$D^\alpha f(x) = \frac{\partial^{|\alpha|} f(x)}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n}}.$$

It may sometimes happen that $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ will be used to denote a multi-index with components of any sign: $\alpha_j \gtreqless 0$.

0.5 We use $C^k(\mathcal{O})$ to denote the set of all functions $f(x)$ that are continuous in $\mathcal{O}$ together with all derivatives $D^\alpha f(x)$, $|\alpha| \leqslant k$; $C^\infty(\mathcal{O})$ is the collection of all functions infinitely differentiable in $\mathcal{O}$. The set of all functions $f(x)$ in $C^k(\mathcal{O})$ for which all derivatives $D^\alpha f(x)$, $|\alpha| \leqslant k$, admit continuous extension onto $\overline{\mathcal{O}}$ will be denoted by $C^k(\overline{\mathcal{O}})$. We introduce the norm in $C^k(\overline{\mathcal{O}})$ for $k < \infty$ via the formula

$$\|f\|_{C^k(\overline{\mathcal{O}})} = \sup_{\substack{x \in \overline{\mathcal{O}} \\ |\alpha| \leqslant k}} |D^\alpha f(x)|.$$

We also write $C(\mathcal{O}) = C^0(\mathcal{O})$, $C(\overline{\mathcal{O}}) = C^0(\overline{\mathcal{O}})$.

The *support* of a function $f(x)$ continuous in $\mathcal{O}$ is the closure, in $\mathcal{O}$, of those points where $f(x) \neq 0$; the support of f is denoted by $\operatorname{supp} f$. If $\operatorname{supp} f \Subset \mathcal{O}$, then f is finite in $\mathcal{O}$ (compare with Sec. 0.3).

We denote the collection of functions, finite in $\mathcal{O}$, of the class $C^k(\mathcal{O})$ by $C_0^k(\mathcal{O})$; $C_0(\mathcal{O}) = C_0^0(\mathcal{O})$. Finally, the set of all functions of the class $C^k(\overline{\mathcal{O}})$ that vanish on $\partial\mathcal{O}$ together with all derivatives up to order k inclusive will be denoted by $\overset{\circ}{C}{}^k(\overline{\mathcal{O}})$; $\overset{\circ}{C}(\overline{\mathcal{O}}) = \overset{\circ}{C}{}^0(\overline{\mathcal{O}})$. We write $C^k(\mathbb{R}^n) = C^k$; $C_0^k(\mathbb{R}^n) = C_0^k$; $\overset{\circ}{C}{}^k(\overline{\mathbb{R}}^n) = \overset{\circ}{C}{}^k$, $\overset{\circ}{C} = \overset{\circ}{C}{}^0$ ($\overset{\circ}{C}{}^k$ is the set of functions in C^k that vanish at infinity together with all their derivatives up to order k inclusive).

0.6 Symbolism: (a, b) is a bilinear form (linear in a and b separately); $\langle a, b\rangle$ is a linear-antilinear form (linear in a and antilinear in b):

$$\langle \alpha a_1 + b a_2, \lambda b_1 + \mu b_2\rangle = \alpha\bar{\lambda}\langle a_1, b_1\rangle + \alpha\bar{\mu}\langle a_1, b_2\rangle + \beta\bar{\lambda}\langle a_2, b_1\rangle + \beta\bar{\mu}\langle a_2, b_2\rangle;$$

$\sigma_n = \int\limits_{|s|=1} ds = \dfrac{2\pi^{n/2}}{\Gamma(n/2)}$ is the surface area of a unit sphere in $\mathbb{R}^n$;

A^T is the transpose of the matrix A.

We denote the uniform convergence of a sequence of functions, $\{\varphi_n(x)\}$, to a function $\varphi(x)$ on a set A thus:

$$\varphi_n(x) \overset{x \in A}{\Longrightarrow} \varphi(x), \qquad n \to \infty;$$

if $A = \mathbb{R}^n$, then instead of $\overset{x \in \mathbb{R}^n}{\Longrightarrow}$ we write $\overset{x}{\Longrightarrow}$.

The sections are numbered in a single sequence. Each section is made up of subsections, the numbers of which are included in any reference to a section. Formulas are numbered separately in each subsection; they contain the number of the formula and of the subsection. When reference is made to a formula in a different section, the number of that section is also given.

Chapter I

Generalized Functions and Their Properties

The exposition of the theory of generalized functions given in this chapter is tailored to the needs of theoretical and mathematical physics.

1 Basic and Generalized Functions

1.1 Introduction A generalized function is a generalization of the classical notion of a function. On the one hand, this generalization permits expressing in a mathematically proper form such idealized concepts as the density of a material point, the density of a point charge or dipole, the spatial density of a simple or double layer, the intensity of an instantaneous point source, the magnitude of an instantaneous force applied to a point, and so forth. On the other hand, the notion of a generalized function can reflect the fact that in reality one cannot measure the value of a physical quantity at a point but can only measure the mean values within sufficiently small neighbourhoods of the point and then proclaim the limit of the sequence of those mean values as the value of the physical quantity at the given point.

This can be explained by attempting to determine the density set up by a material point of mass 1. Assume that the point is the origin of coordinates. In order to determine the density, we distribute (or, as one often says, smear) the unit mass uniformly inside a sphere of radius ε centered at 0. We then obtain the mean density $f_\varepsilon(x)$ that is equal (see Fig. 3) to

$$f_\varepsilon(x) = \begin{cases} \dfrac{1}{\frac{4}{3}\pi\varepsilon^3} & \text{if } |x| < \varepsilon, \\ 0 & \text{if } |x| > \varepsilon. \end{cases}$$

We are interested in the density at $\varepsilon = +0$. To begin with, for the desired density (which we denote by $\delta(x)$) we take the point limit of the sequence of mean densities $f_\varepsilon(x)$ as $\varepsilon \to +0$, that is, the function

$$\delta(x) = \begin{cases} +\infty \text{ if } x = 0, \\ 0 \quad \text{if } x \neq 0. \end{cases} \tag{1.1}$$

Of the density it is natural to require that the integral of the density over the entire space yield the total mass of substance, or

$$\int \delta(x)\, dx = 1. \tag{1.2}$$

But for the function $\delta(x)$ defined by (1.1), $\int \delta(x)\, dx = 0$. This means that the function does not restore the mass (it does not satisfy the requirement (1.2)) and therefore cannot be taken as

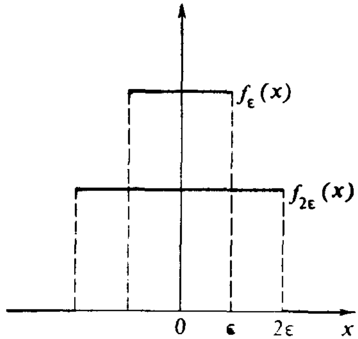


Figure 3

the desired mass. Thus the point limit of a sequence of mean densities $f_\varepsilon(x)$ is unsuitable for our purposes. What is the way out?

Let us now find a somewhat different limit of the sequence of mean densities $f_\varepsilon(x)$, the so-called *weak limit*. It will readily be seen that for any continuous function $\varphi(x)$

$$\lim_{\varepsilon \to +0} \int f_\varepsilon(x)\, \varphi(x)\, dx = \varphi(0). \tag{1.3}$$

Formula (1.3) states that the weak limit of a sequence of functions $f_\varepsilon(x)$, $\varepsilon \to +0$, is a functional $\varphi(0)$ (and not a function!) that relates to every continuous function $\varphi(x)$ a number $\varphi(0)$, which is its value at the point $x = 0$. It is this functional that we take for our sought-for density $\delta(x)$. And this is the famous *delta function* of Dirac. So now we can write

$$f_\varepsilon(x) \xrightarrow{\text{weak}} \delta(x), \qquad \varepsilon \to +0,$$

and we understand by this the limiting relation (1.3). The value of the functional δ on the function φ (the number $\varphi(0)$) will be denoted thus:

$$(\delta, \varphi) = \varphi(0). \tag{1.4}$$

This equality yields the exact meaning of formula (*) (see Preface). The role of the "integral" $\int \delta(x)\, \varphi(x)\, dx$ is played here by the quantity (δ, φ), which is the value of the functional δ on the function φ.

Let us now check to see that the functional δ restores the total mass. Indeed, as we have just said, the role of the "integral" $\int \delta(x)\,dx$ is handled by the quantity $(\delta, 1)$, which, by virtue of (1.4), is equal to the value of the function identically equal to 1 at the point $x = 0$, that is, $(\delta, 1) = 1$.

Also, generally, if masses μ_k are concentrated at distinct points x_k, $k = 1, 2, \ldots, N$, then the density that corresponds to such a mass distribution should be regarded as equal to

$$\sum_{1 \leqslant k \leqslant N} \mu_k\, \delta(x - x_k). \tag{1.5}$$

The expression (1.5) is a linear functional that associates with each continuous function $\varphi(x)$ a number

$$\sum_{1 \leqslant k \leqslant N} \mu_k\, \varphi(x_k).$$

Thus, the density corresponding to a point distribution of masses cannot be described within the framework of the classical concept of a function; to describe it requires resorting to entities of a more general mathematical nature, linear (continuous) functionals.

1.2 The space of basic functions $\mathcal{D}(\mathcal{O})$ In the case of the delta function we have already seen that it is determined by means of continuous functions as a linear (continuous) functional on those functions. Continuous functions are said to be *basic functions* for the delta function. It is this viewpoint that serves as the basis for defining an arbitrary generalized function as a continuous linear functional on a collection of sufficiently "good" so-called basic functions. Clearly, the smaller the set of basic functions, the more continuous linear functionals there are on it. On the other hand, the supply of basic functions should be sufficiently large. In this subsection we introduce the important space of basic functions $\mathcal{D}(\mathcal{O})$ for any open set $\mathcal{O} \subset \mathbb{R}^n$.

Included in the set of basic functions $\mathcal{D}(\mathcal{O})$ are all finite functions infinitely differentiable in $\mathcal{O}$; $\mathcal{D}(\mathcal{O}) = C_0^\infty(\mathcal{O})$ (see Sec. 0.5). We define *convergence* in $\mathcal{D}(\mathcal{O})$ as follows. A sequence of functions $\varphi_1, \varphi_2, \ldots$ in $\mathcal{D}(\mathcal{O})$ converges to the function φ (in $\mathcal{D}(\mathcal{O})$) if there exists a set $\mathcal{O}' \Subset \mathcal{O}$ such that $\operatorname{supp} \varphi_k \subset \mathcal{O}'$ and for every α

$$D^\alpha \varphi_k(x) \overset{x \in \mathcal{O}}{\rightrightarrows} \mathcal{D}^\alpha \varphi(x), \qquad k \to \infty.$$

We then write: $\varphi_k \to \varphi$, $k \to \infty$ in $\mathcal{D}(\mathcal{O})$.

A linear set $\mathscr{D}(\mathcal{O})$ equipped with convergence is called the *space of basic functions* $\mathscr{D}(\mathcal{O})$, and we have the following symbolism: $\mathscr{D} = \mathscr{D}(\mathbb{R}^n)$, $\mathscr{D}(a, b) = \mathscr{D}((a, b))$.

Clearly, if $\mathcal{O}_1 \subset \mathcal{O}_2$, then also $\mathscr{D}(\mathcal{O}_1) \subset \mathscr{D}(\mathcal{O}_2)$, and from the convergence in $\mathscr{D}(\mathcal{O}_1)$ there follows convergence in $\mathscr{D}(\mathcal{O}_2)$.

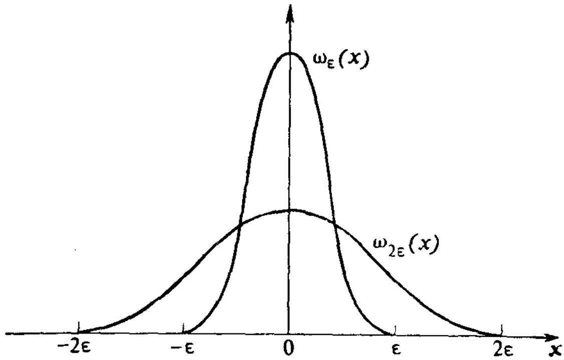


Figure 4

An instance of a nonzero basic function is the "cap" in Fig. 4:

$$\omega_\varepsilon(x) = \begin{cases} C_\varepsilon e^{-\frac{\varepsilon^2}{\varepsilon^2 - |x|^2}}, & |x| \leqslant \varepsilon, \\ 0, & |x| > \varepsilon. \end{cases}$$

In what follows, the function ω_ε will play the part of an averaging function; and so we shall regard the constant C_ε as such that

$$\int \omega_\varepsilon(x)\, dx = 1, \quad \text{that is,} \quad C_\varepsilon \varepsilon^n \int_{|\xi|<1} e^{-\frac{1}{1-|\xi|^2}}\, d\xi = 1.$$

The following lemma yields other instances of basic functions.

Lemma *For any set A and any number $\varepsilon > 0$ there is a function $\eta_\varepsilon \in C^\infty$ such that*

$$\eta_\varepsilon(x) = 1, \quad x \in A^\varepsilon; \qquad \eta_\varepsilon(x) = 0, \quad x \notin A^{3\varepsilon};$$
$$0 \leqslant \eta_\varepsilon(x) \leqslant 1, \qquad |\mathscr{D}^\alpha \eta_\varepsilon(x)| \leqslant K_\alpha \varepsilon^{-|\alpha|}.$$

Proof. Let $\theta_{A^{2\varepsilon}}$ be a characteristic function of the set $A^{2\varepsilon}$ (see Sec. 0.2). Then the function

$$\eta_\varepsilon(x) = \int \theta_{A^{2\varepsilon}}(y)\, \omega_\varepsilon(x-y)\, dy = \int_{A^{2\varepsilon}} \omega_\varepsilon(x-y)\, dy,$$

where ω_ε is the "cap", has the required properties. The proof is complete.