# Handbook of Elliptic Integrals for Engineers & Physicists

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# HANDBOOK OF ELLIPTIC INTEGRALS FOR ENGINEERS AND PHYSICISTS

by


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WITH 22 FIGURES


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# Preface.

Engineers and physicists are more and more encountering integrations involving nonelementary integrals and higher transcendental functions. Such integrations frequently involve (not always in immediately recognizable form) elliptic functions and elliptic integrals.

The numerous books written on elliptic integrals, while of great value to the student or mathematician, are not especially suitable for the scientist whose primary objective is the ready evaluation of the integrals that occur in his practical problems. As a result, he may entirely avoid problems which lead to elliptic integrals, or is likely to resort to graphical methods or other means of approximation in dealing with all but the simplest of these integrals.

It became apparent in the course of my work in theoretical aerodynamics that there was a need for a handbook embodying in convenient form a comprehensive table of elliptic integrals together with auxiliary formulas and numerical tables of values. Feeling that such a book would save the engineer and physicist much valuable time, I prepared the present volume.

Although the book is not a text, an attempt has been made to write it in elementary terms so that no previous knowledge of elliptic integrals, theta functions or elliptic functions is needed. A collection of over 3000 integrals and formulas, designed to meet most practical needs, is presented using Legendre's and Jacobi's notations, rather than the less familiar Weierstrassian forms. Many of these formulas are substitutions and recurrence relations for evaluating additional integrals which are not explicitly written. Sufficient explanatory material and cross-references are given to permit the reader to obtain the answers he requires with a minimum of effort.

Short tables of numerical values are given for the elliptic integrals of the first and second kind, for Jacobi's "nome" $q$, for the function denoted by Heuman as $\Lambda_0$, and for $K$ times the Jacobian Zeta function. Tables of the last three functions are useful in the numerical evaluation of elliptic integrals of the third kind.

Particular precautions, of course, have to be taken in a work of this kind to insure accuracy of the formulas. My co-author, Mr. MORRIS

D. FRIEDMAN, undertook the job of verifying each formula. Where ever possible, they were either derived independently in different ways or checked against more than one source. Criticisms of the material contained in the handbook and notice of any errors which may yet appear in it will be sincerely welcomed.

It is impossible to acknowledge properly all the sources to which debt is owed. The bibliography, however, lists many books in which the derivation of some of the formulas can be found or where related material may be obtained. For friendly advice and valuable suggestions, I am under obligation to Professors A. ERDÉLYI, W. MAGNUS and R. C. ARCHIBALD, and to colleagues in the Theoretical Aerodynamics Section, Ames Aeronautical Laboratory, NACA. To my colleague, DORIS COHEN, I am grateful for a critical reading of the manuscript and for many suggestions leading to improvement of exposition and organization. Hearty thanks are also extended to Mrs. ROSE CHIN BYRD and Mrs. MARY T. HUGGINS for assistance in the task of preparing the tables of numerical values and to Mr. DUANE W. DUGAN for help in reading the proofs.

On behalf of both authors I wish finally to express gratitude to Springer-Verlag and to Professor K. KLOTTER of Stanford University who kindly called their attention to our work. The appearance of this book is in no small measure due to their cooperative attitude, their encouragement, and their genuine interest in the promotion of technical publications.

Palo Alto, California. PAUL F. BYRD.

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# List of Symbols and Abbreviations.

The following table comprises a list of the principal symbols and abbreviations used in the handbook. Notations not listed are so well understood that explanation is unnecessary.

| Symbol or Abbreviation | Meaning | Section |
|---|---|---|
| $\alpha^2$ | Parameter of elliptic integral of the third kind | 110 |
| $\mathrm{am}\,(u, k) \equiv \mathrm{am}\,u$ | Amplitude $u$ | 120 |
| $\mathrm{am}^{-1}\,(y, k)$ | Inverse amplitude $y$ | 130 |
| $\mathrm{cd}\,u$ | $\equiv \dfrac{\mathrm{cn}\,u}{\mathrm{dn}\,u}$ | 120 |
| $\mathrm{cd}^{-1}\,(y, k)$ | — | 130 |
| $\mathrm{cn}\,(u, k) \equiv \mathrm{cn}\,u$ | Cosine amplitude $u$; Jacobian elliptic function | 120 |
| $\mathrm{cn}^{-1}\,(y, k)$ | — | 130 |
| $\cos^{-1}\varphi$ | Inverse trigonometric function, often written *arc cos* $\varphi$ | |
| $\mathrm{cs}\,u$ | $\equiv \dfrac{\mathrm{cn}\,u}{\mathrm{sn}\,u}$ | 120 |
| $\mathrm{cs}^{-1}\,(y, k)$ | — | 130 |
| $\mathrm{dc}\,u$ | $\equiv \dfrac{\mathrm{dn}\,u}{\mathrm{cn}\,u}$ | 120 |
| $\mathrm{dc}^{-1}\,(y, k)$ | — | 130 |
| $\mathrm{dn}\,u$ | Delta amplitude $u$; Jacobian elliptic function | 120 |
| $\mathrm{dn}^{-1}\,(y, k)$ | — | 130 |
| $\mathrm{ds}\,u$ | $\equiv \dfrac{\mathrm{dn}\,u}{\mathrm{sn}\,u}$ | 120 |
| $\mathrm{ds}^{-1}\,(y, k)$ | — | 130 |
| $e_1, e_2, e_3$ | Roots of polynomial written in Weierstrassian form | 1030 |
| $E\,(\varphi, k) \equiv E\,(u)$ | Legendre's incomplete elliptic integral of the second kind; ($\varphi = \mathrm{am}\,u$) | |
| $E'\,(\varphi, k) \equiv E\,(\varphi, k')$ | Associated incomplete elliptic integral of the second kind | |
| $E\,(k) \equiv E \equiv E\,(\pi/2, k)$ | Complete elliptic integral of the second kind | 110 |
| $E' \equiv E\,(k')$ | Associated complete elliptic integral of the second kind | |
| $F\,(\varphi, k) \equiv u$ | Incomplete elliptic integral of the first kind | |

| Symbol or Abbreviation | Meaning | Section |
|---|---|---|
| $F(a,b;c;z)$ | $\equiv \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(c)_m m!} z^m$, hypergeometric series | 900 |
| $G$ | $= \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^2} \approx 0.91596559$, Catalan's constant | 615 |
| $g_2, g_3$ | — | 1030 |
| $\Gamma(z)$ | $\equiv \frac{1}{z} \prod_{m=1}^{\infty} \left[\left(1+\frac{1}{m}\right)^z \left(1+\frac{z}{m}\right)^{-1}\right]$, $z \neq 0, -1, -2, \ldots$, Gamma function | |
| $H, H_1$ | Eta functions of Jacobi | 1050 |
| $I_\nu(z)$ | $\equiv \sum_{m=0}^{\infty} \frac{1}{m!\,\Gamma(\nu+m+1)} \left(\frac{z}{2}\right)^{\nu+2m}$ $\equiv e^{-\nu\pi i/2} J_\nu(iz)$, modified Bessel functions of first kind | 560 |
| Im | Imaginary part of a complex quantity | |
| $J_\nu(z)$ | $\equiv \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(\nu+m+1)} \left(\frac{z}{2}\right)^{\nu+2m}$, Bessel functions of first kind | 560 |
| $k$ | Modulus of Jacobian elliptic functions and integrals | 110 |
| $k' = \sqrt{1-k^2}$ | Complementary modulus | 110 |
| $K(k) \equiv K = F(\pi/2, k)$ | Complete elliptic integral of the first kind | 110 |
| $K' \equiv K(k')$ | Associated complete elliptic integral of the first kind | 110 |
| $\Lambda_0(\varphi, k)$ | $\equiv \frac{2}{\pi}[EF(\varphi,k') + KE(\varphi,k') - KF(\varphi,k')]$, Heuman's Lambda function | 150 |
| $\ln z$ | Natural logarithm of $z$ | |
| $m, n$ | Integers, unless otherwise stated | |
| $n!$ | $= 1.2 \ldots n$; $n$ factorial | |
| $\mathrm{nc}\, u$ | $\equiv \frac{1}{\mathrm{cn}\, u}$ | 120 |
| $\mathrm{nc}^{-1}(y, k)$ | — | 130 |
| $\mathrm{nd}\, u$ | $\equiv \frac{1}{\mathrm{dn}\, u}$ | 120 |
| $\mathrm{nd}^{-1}(y, k)$ | — | 130 |
| $\mathrm{ns}\, u$ | $\equiv \frac{1}{\mathrm{sn}\, u}$ | 120 |
| $\mathrm{ns}^{-1}(y, k)$ | — | 130 |
| $\Omega_1, \Omega_2, \Omega_3, \Omega_4, \Omega_5, \Omega_6$ | — | 430 |

| Symbol or Abbreviation | Meaning | Section |
|---|---|---|
| $\wp(u)$ | Weierstrassian elliptic function | 1030 |
| $\Pi(\varphi, \alpha^2, k) \equiv \Pi(u, \alpha^2)$ | Legendre's incomplete elliptic integral of the third kind; ($\varphi = \operatorname{am} u$) | 110 |
| $\Pi(\alpha^2, k) \equiv \Pi(\pi/2, \alpha^2, k)$ | Complete elliptic integral of the third kind | |
| $q$ | $\equiv e^{-\pi K'/K}$, referred to as the *nome* | 1050 |
| $Q_n(z)$ | $\equiv \frac{1}{2^{n+1}} \int_{-1}^{1} (1-t^2)^n (z-t)^{-n-1} dt$, R.P. $(n+1) > 0$, Legendre functions (spherical harmonics) | 560 |
| R.P. | Real part of a complex quantity | |
| $\sigma(u)$ | Weierstrassian Sigma function | 1030 |
| $\operatorname{sd} u$ | $\equiv \frac{\operatorname{sn} u}{\operatorname{dn} u}$ | 120 |
| $\operatorname{sd}^{-1}(y, k)$ | — | 130 |
| $\sin^{-1}\varphi$ | Inverse trigonometric function, often written *arc sin* $\varphi$ | |
| $\operatorname{sn} u$ | Sine amplitude $u$ | 120 |
| $\operatorname{sn}^{-1}(y, k)$ | — | 130 |
| $\operatorname{tn} u \equiv \operatorname{sc} u$ | $\equiv \frac{\operatorname{sn} u}{\operatorname{cn} u}$ | 120 |
| $\operatorname{tn}^{-1}(y, k)$ | — | 130 |
| $\Theta, \Theta_1$ | Jacobi's Theta functions | 1050 |
| $\vartheta_0, \vartheta_1, \vartheta_2, \vartheta_3$ | Elliptic Theta functions | 1050 |
| $y$ | Variable limit of integration in all integrals | |
| $\psi(z)$ | $\equiv \xi - \frac{1}{z} + \sum_{m=1}^{\infty} \frac{z}{m(z+m)}$, digamma function; $\xi$ is Euler's number $\approx 0.5772155665$ | 900 |
| $Z(u, k) \equiv Z(u) \equiv Z(\beta, k)$ | $\equiv E(u) - \frac{E}{K} u$, Jacobian Zeta function; ($\operatorname{am} u = \beta$) | 140 |
| $\zeta(u)$ | Weierstrassian Zeta function | 1030 |
| $(a)_n$ | $= a(a+1)\dots(a+n-1)$, for $n = 1, 2, \dots$ $(a)_0 = 1$, Pochhammer's symbol | |
| $\binom{a}{n} = (-1)^n \frac{(-a)_n}{n!}$ | $= \frac{a(a-1)\dots(a-n+1)}{1.2.3\dots n}$; $\binom{a}{0} = 1$. | |

# Errata and Additions.

Page 13, line 4: Read $\sin^{-1}(1/k \sin\psi)$ for $\sin^{-1}(1/k)$

Page 17, line 4 from Figs. 5 and 6: Read Im for *Im*. In Fig. 7, the line $AB$ of the rectangle should be drawn heavy.

Page 26, Fig. 12: Read $K+2iK'$ for $K+2iK$.

Page 28, line 2 from top, and lines 5, 7, and 10 from bottom: Read Im for *Im*. In Fig. 17, the line $AB$ of the rectangle should be drawn heavy.

Page 118, formula 255.18: Read $y \neq 0$ for $y \neq c$.

Page 214, first term in bracket in formula 361.27: Read $-2k'^2 n$ for $-2k'^2$.

Page 224, footnote 1. The following should be added:

In this table of integrals, the integration is over real $u$. In case IV, however, when $\alpha > 1$, there are singularities on the real axis. We must thus make a remark on the correct interpretation of the results given.

The integrals 415.01—415.06 are interpreted as Cauchy Principal Values. This is also the interpretation in 436.01—436.05 when $\alpha \operatorname{sn} u_1 > 1$. In 415.07—415.08, and also in 436.06 if $\alpha \operatorname{sn} u_1 > 1$, the integrals are interpreted as "generalized" principal values. Thus, for example, the principal part of the integral

$$\int_0^K \frac{du}{(1-\alpha^2 \operatorname{sn}^2 u)^2} \text{ is } \oint_0^K \frac{du}{1-\alpha^2 \operatorname{sn}^2 u} + \frac{\alpha}{2}\frac{\partial}{\partial\alpha}\oint_0^K \frac{du}{1-\alpha^2 \operatorname{sn}^2 u},$$

where the symbol $\oint$ means that the Cauchy Principal Value is to be taken.

Pages 245—248: In this section, an integral such as $\int_0^\varphi\int_0^{\varphi_1} F(\varphi_1,\vartheta)\,d\varphi_1\,d\vartheta$ means $\int_0^\varphi\left(\int_0^{\varphi_1} F(\varphi_1,\vartheta)\,d\vartheta\right)d\varphi_1$.

Page 276, line 2: Read "With Variable Upper Limit" for "Indefinite Integrals".

Pages 282 and 283, formulas 710.06—710.11: Read $\partial/\partial k$ for $d/dk$.

Page 284 formulas 730.00—730.04: Read $\partial/\partial\varphi$ for $d/d\varphi$.

Pages 285 and 286: In formula 732.00, read $\partial/\partial\varphi$ for $d/d\varphi$. In formulas 732.01—732.12, read $\partial/\partial y$ for $d/dy$.

Pages 305, 306 and 315: Read Im for *Im*.

Page 321: The numerical tables are accurate to six places. Calculations for $KZ(\beta, k)$ and $\Lambda_0(\beta, k)$ were made using corrected Legendre tables as source. The values given for $F(\varphi, k)$ and $E(\varphi, k)$ are abbreviations of Legendre's tables.

# ERRATA

P. 2, Line 6 from bottom: For missing upper limit of 2nd integral, read $a$.

P. 6, formula (22):. For missing upper limit of 1st integral, read $y$.

P. 14, formula 117.03: Line 1, read $\Pi[-(\alpha^2-k^2)/(1-\alpha^2),\ k]$ for $\Pi[(\alpha^2-k^2)/(1-\alpha^2),\ k]$; line 3, read $\Pi[\varphi,-(\alpha^2-k^2)/(1-\alpha^2),\ k]$ for $\Pi[(\alpha^2-k^2)/(1-\alpha^2),\ k]$; formulas 117.04 and 117.05, read $\Pi[-k'^2/(\alpha^2-1),\ k']$ for $\Pi[k'^2/(\alpha^2-1),\ k']$; formulas 117.06, read $\Pi[-(k^2-\alpha^2)/\alpha^2,\ k']$ for $\Pi[(k^2-\alpha^2)/\alpha^2,\ k']$.

P. 18, formulas 119.03: read $E(k_1, k')$ for $E(k_1)$, and $F(k_1, k')$ for $K(k_1)$, where, $k_1 = \sqrt{1-a^2k^2}/k'$.

P. 23, 1st formula in 123.07: Read $-2$ for 2; 2nd formula on the right, read cn for dn.

P. 24, last formula in 125.01: Read $\mathrm{cn}(u,k)$ for $\mathrm{nc}(u,k)$.

P. 26, Fig. 12: For distance from axis to upper edge of curve, read $k'$ for $-k'$.

P. 28, formula 129.51: In the brackets, for modulus of sn, read $q$ for $k(q)$.

P. 29, last integral in line 15: For missing upper limit, read $a$.

P. 32, last formula in 133.01: On the left, read $\mathrm{sn}^{-1}(x,k) + \mathrm{cn}^{-1}(y,k)$ for $\mathrm{sn}^{-1}(y,k) + \mathrm{cn}^{-1}(x,k)$.

P. 34, line 3 from bottom: Read 710.06 for 710.04.

P. 37, line 7: Read 710.11 for 710.07.

P. 40, 3rd formula in line 2: Read $(k_1+\alpha_1^2)^2$ for $(k_1+\alpha_1^2)$.

P. 57, formula 218.11: Read $(a^2-b^2)^2$ for $(a^2-b^2)$.

P. 61, in 'box' above formula 221.00: Read $\tan^{-1}(y/b)$ for $\ln^{-1}(y/b)$.

P. 64, formula 225.00: Read $F(\varphi,k)$ for $F(\cos\varphi,k)$. Formula 225.04, under the summation sign, insert factor $2^j$.

P. 67, line 2: First term, read $\dfrac{2}{(y_1-p)^m}$ for 2; second term, read $\dfrac{2}{(y-p)^m}$ for 2.

P. 75, line 3: Read $(p-b)$ for $(p-b($.

P. 84, formula 238.02: Read $\alpha^2$ for $\alpha$.

P. 85, formula 238.17: In denominator on the right, read $t^m$ for $^m$.

P. 86, formula 239.03: Under the integral sign, read $dt$.

P. 90, formula 242.06: On the right, multiply by the factor $(-1)^m$.

P. 91, formula 243.02, in denominator, read $4A$ for $4A^2$; formula 243.06, on the left, read $+A$ for $-A$.

P. 97, formula 250.03: Delete 4 in denominator.

P. 100, formula 251.24: Read $(1-\alpha^2\mathrm{sn}^2u)^m$ for $(1-\alpha^2\mathrm{sn}^2u)$.

P. 134, formula 259.53: Read $1-(\sqrt{3}-2)\,\mathrm{cn}\,u$ for $1+(\sqrt{3}-2)\,\mathrm{cn}\,u$.

P. 140, Formula 261.59, read $2\,\mathrm{ds}\,u$ for $\mathrm{ds}\,u$; formula 261.55 on the right, multiply by factor $1/4^m$.

P. 143, formula 264.04: Delete $(\sqrt{k'})^m$ and $(\sqrt{k'})^{m+j}$.

P. 145, formula 264.54: Delete $(\sqrt{k'})^m$ and $(\sqrt{k'})^{m+j}$; line 3, read $-\alpha$ for $\alpha$.

P. 152, formula 272.54: Read $\sqrt[4]{t^2-1}$ for $\sqrt{t^2-1}$; formula 272.58, read $\sqrt[4]{(t^2-1)^3}$ for $\sqrt{(t^2-1)^3}$.

P. 154, formula 273.53: Read $4/\sqrt{2}$ for $1/\sqrt{2}$.

P. 156, formula 275.00, delete minus sign. Formula 275.03, read $-24$ for $+24$, and $(4-j)!$ for $(m-j)!$ Formula 275.04, multiply right side by $(-1)^m$, and read $-(1-Y^2)^{m-j}$ for $-(Y^2-1)^{m-j}$.

Byrd and Friedman: *Handbook of Elliptic Integrals for Engineers and Physicists.*

P. 157, formula 275.06: Multiply right side by $(-1)^m$.

P. 160, formula 279.00, 1st line, 2nd integral on right: Read $(t^4-a)$ for $(t-a)$.

P. 161: Delete formula 279.50.

P. 171, first line in 'box': Read $\sqrt{2}/a$ for $\sqrt{2/a}$.

P. 177, formula 291.02: Read $ak^2p\,(1+\frac{b}{ap})$ for $ak^2p$.

P. 180, formula 294.01: Under second integral sign, read $du$.

P. 187, formula 297.56: Read $\cosh^m\vartheta$ for $\cosh^m$.

P. 196, formula 318.05: Read $\mathrm{sd}^{2m-1}u$ for $\mathrm{sd}^{2m+1}u$; formula 318.06, read $\mathrm{sd}^{2m}u$ for $\mathrm{sd}^{2m+2}u$; formula 318.03, read $1/2k^3k'^3$ for $1/2k^2k'^2$, and multiply first term in brackets by $(2k^2-1)$.

P. 199, formula 330.04: Read $(a_1+\mathrm{sn}\,u)^{-m}$ for $(a_1+\mathrm{sn}\,u)^m$.

P. 200, formula 332.51: Read $(1-\alpha_1^2)^m$ for $(1-\alpha_1^2)$.

P. 204, formula 339.75: Under summation sign, 2nd term on right, read $k^{2j}$ for $k^{2m}$.

P. 208, formula 348.52: Read $G_{2m+2}$ for $G_m$; formula 348.85, read $(-1)^{m-j+1}$ for $(-1)^m$.

P. 209, formula 351.51: For upper index on 2nd summation sign, read $n$ for $m$.

P. 210, formula 353.01: Read $k^{2(m+n-1)}$ for $k^{2\;m+n-1)}$; 5th and 6th line in formula 355.01, read $\mathrm{nd}^{2(m-1)}u$ for $\mathrm{nd}^{2m-1}u$; 3rd line in formula 356.01, read $\mathrm{cn}^{2(m-1)}u$ for $\mathrm{cn}^{2\;m-1)}u$.

P. 214, 1st term in brackets in formula 361.27: Read $-2k'^2u$ for $-2k'^2$.

P. 215, formula 361.50: On the right, read $1\pm\mathrm{sn}\,u$ for $1+\mathrm{sn}\,u$. Formula 361.54: After the 3rd equality sign in the expression for $f_1$, insert factor $1/2$.

P. 217, formula 362.03: Read $k^2E(u)$ for $E(u)$, and $k^4\mathrm{sn}\,u\,\mathrm{cd}\,u$ for $k^2\,\mathrm{sn}\,u\,\mathrm{cd}\,u$.

P. 232, line 3 in 'box': Read $\Theta$ for $H$, and $\vartheta_0$ for $\vartheta$.

P. 247, formula 531.07: Read $K'(k_4)$ for $K(k_4)$.

P. 254, last integral in line 7: Read $\tau^{2p-1}$ for $\tau^{p-1}$.

P. 256, 1st line in formula 576.00: Read $\sqrt[4]{3}$ for $\sqrt{3}$.

P. 258, 1st line in formula 578.00: Read $\sqrt[4]{3}$ for $\sqrt{3}$.

P. 267, top line and also 1st line above 'box': Read $1<1/\sqrt{n}$ for $1<1/n$.

P. 268, line above 'box': Read $1/\sqrt{n}<Y<\infty$ for $1/n<Y<\infty$.

P. 272, formulas 610.00: and 611.00: Read $k^{2n}$ for $k^{2n+1}$.

P. 275, formula 615.12: Read $k^{m-2}$ for $k^{n-2}$.

P. 286, formula 732.07: Read $\infty>y>1$ for $\infty>y>0$.

P. 292, formula 806.03: Read $\frac{2}{(j-2)!}$ for $\frac{1}{(j-3)!}$.

P. 298, formula 900.05: Read $\binom{-\frac{1}{2}}{m}^2$ for $\binom{-\frac{1}{2}}{m}$; formula 900.08, under summation sign. read $k_1^{2m+2}/(2m+1)^2$ for $k_1^{2m+2}$.

P. 299, formula 902.01: Read $\binom{-\frac{1}{2}}{m}$ for $\left(\frac{-\frac{1}{2}}{m}\right)$.

P. 302, formula 907.03, last term: Read $k^2u^8$ for $u^8$.

P. 303, 3rd line in footnote: Read $1-q^{2m+1}$ for $1+q^{2m+1}$; in last line of footnote, read $1+q^{2m+1}$ for $1-q^{2m+1}$; formula 907.07, read 408 for 498.

P. 308, last line in formulas 1032.03: Read $4t^3$ for $4t^2$.

P. 311, formula 1037.10: In the brackets, replace $a$ by $\alpha$, with $\alpha=\wp^{-1}(-d)$.

P. 312, formula 1037.12: Read $m-1$ for $m\cdot-$.

P. 315, 2nd formula on right in 1051.02: Read $-H(u)$ for $H(u)$, and $-\vartheta_1(v)$ for $\vartheta_1(v)$.

Pages 317-319, formulas 1053.02-1053.05: The Theta functions in all these formulas should be replaced by the logarithm of the Theta functions.

P. 321, 1st formula in 1060.04: Read $\mathrm{dc}\,u\,\mathrm{ns}\,u$ for $\mathrm{dn}\,u\,\mathrm{cs}\,u$.

# Table of Contents.

# Introduction.

Integrals of the form $\int R[t, \sqrt{P(t)}]\, dt$, where $P(t)$ is a polynomial of the third or fourth degree and $R$ is a rational function, have the simplest algebraic integrands that can lead to nonelementary[1] integrals. Equivalent integrals occur in trigonometric and other forms, in pure and applied mathematics. Such integrals are known as elliptic integrals because a special example of this type arose in the rectification of the arc of an ellipse. Although some early work on them was done by FRAGNANO, EULER, LAGRANGE and LANDEN, they were first treated systematically by LEGENDRE, who showed that any elliptic integral may be made to depend on three fundamental integrals which he denoted by $F(\varphi, k)$, $E(\varphi, k)$ and $\Pi(\varphi, n, k)$. These three integrals are called Legendre's canonical elliptic integrals of the first, second and third kind respectively. Legendre's normal forms are not the only standard forms possible, but they have retained their usefulness for over a century.

The elliptic functions of ABEL, JACOBI and WEIERSTRASS are obtained by the inversion of elliptic integrals of the first kind. As is shown in Vol. LV of this series[2], these inverse functions have numerous direct applications in problems of electrostatics, hydromechanics, aerodynamics etc. They are also highly useful for evaluating elliptic integrals, which is the primary concern of this handbook. The more modern theory of WEIERSTRASS is sometimes better suited to certain problems, but for most practical problems Jacobian elliptic functions appear in a more natural way, simplifying the formulas and facilitating numerical evaluation of the results. Thus, for obtaining a comprehensive table of elliptic integrals we find the older notation of JACOBI and LEGENDRE more convenient. (The Weierstrassian functions are treated briefly in the Appendix.)

The general plan of the handbook is as follows: The definitions and other basic information concerning the elliptic integrals and Jacobian

[1] The nonelementary character of elliptic integrals is briefly demonstrated in *Integration in Finite Terms* (Liouville's Theory of Elementary Methods) by J. F. Ritt, Columbia University Press, New York, 1948, pp. 35—37.

[2] F. OBERHETTINGER and W. MAGNUS, *Anwendung der elliptischen Funktionen in Physik und Technik*. (Grundlehren der mathematischen Wissenschaften, Band LV.) Springer-Verlag, 1949.

elliptic functions are given first [Items **100–199**]. Then elliptic integrals in the various algebraic or trigonometric forms in which they are encountered in practice are expressed in terms of integrals involving Jacobian elliptic functions [Items **200–299**]. These latter functions are integrated in Items **300–499**. Specific reference is made in each formula of section **200–299** to the applicable formula in the tables [**300–499**], in which the Jacobian forms are explicitly integrated. We have adopted this procedure because in this way it is possible to give in less space evaluations of a variety of elliptic integrals, particularly those leading to elliptic integrals of the third kind. The remainder of the handbook is devoted to auxiliary formulas and related integrals.

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In our table of integrals involving algebraic integrands, one of the limits of integration is usually taken to be a zero of the polynomial under the radical sign, while the other limit may vary. The use of these tables, however, is not as restrictive as this may appear, because it is easy to evaluate integrals in which both limits are variable. Consider, for example, the integral

$$J = \int_{y_1}^{y} \frac{dt}{\sqrt{(a-t)(t-b)(t-c)}},$$

where $a > y > y_1 > b$. This integral may be reduced by using either **235.00** or **236.00**, since one can write

$$J = \int_{y_1}^{y} \frac{dt}{\sqrt{(a-t)(t-b)(t-c)}} = \int_{b}^{y} \frac{dt}{\sqrt{(a-t)(t-b)(t-c)}} - \int_{b}^{y_1} \frac{dt}{\sqrt{(a-t)(t-b)(t-c)}},$$

or

$$J = \int_{y_1}^{a} \frac{dt}{\sqrt{(a-t)(t-b)(t-c)}} - \int_{y}^{a} \frac{dt}{\sqrt{(a-t)(t-b)(t-c)}}.$$

As another example, take the integral

$$I = \int_{y_1}^{y} \frac{dt}{\sqrt{(t^2-a^2)(t^2-b^2)}},$$

with $\infty > y > y_1 > a$. Here $I$ may also be split into two integrals, e.g.,

$$\int_{y_1}^{y} \frac{dt}{\sqrt{(t^2-a^2)(t^2-b^2)}} = \int_{y_1}^{\infty} \frac{dt}{\sqrt{(t^2-a^2)(t^2-b^2)}} - \int_{y}^{\infty} \frac{dt}{\sqrt{(t^2-a^2)(t^2-b^2)}}.$$

Hence we may employ **215.00**.

The above also applies to the table of integrals involving trigonometric or hyperbolic integrands.

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We give now the following five examples to illustrate in detail how the handbook may be used for rapid evaluation of elliptic integrals encountered in geometrical and physical problems.

*Example I.*

The length of arc of a hyperbola

$$\frac{x^2}{\alpha^2} - \frac{y^2}{\beta^2} = 1 \tag{1}$$

measured from the vertex to any point $(x, y)$ is determined by the integral

$$s = \sqrt{\frac{\alpha^2 + \beta^2}{\beta^2}} \int_0^y \sqrt{\frac{b^2 + t^2}{a^2 + t^2}}\, dt, \tag{2}$$

where

$$a^2 = \beta^2, \qquad b^2 = \frac{\beta^4}{\alpha^2 + \beta^2}, \qquad (b < a). \tag{3}$$

From **221.04** and **313.02**, it is seen that

$$\begin{cases} \displaystyle\int_0^y \sqrt{\frac{b^2 + t^2}{a^2 + t^2}}\, dt = \frac{b^2}{a} \int_0^{u_1} \operatorname{nc}^2 u\, du = \frac{b^2}{a k'^2} \left[k'^2 u - E(u) + \operatorname{dn} u \operatorname{tn} u\right]_0^{u_1} \\ \displaystyle\qquad = \frac{b^2}{a k'^2} \left[k'^2 F(\varphi, k) - E(\varphi, k) + \tan\varphi \sqrt{1 - k^2 \sin^2\varphi}\right], \end{cases} \tag{4}$$

where

$$\begin{cases} k = \sqrt{(a^2 - b^2)/a^2} = \sqrt{\alpha^2/(\alpha^2 + \beta^2)}, \\ \varphi = \operatorname{am} u_1 = \sin^{-1} \sqrt{\dfrac{y^2}{y^2 + b^2}} = \sin^{-1} \sqrt{\dfrac{(\alpha^2 + \beta^2)\, y^2}{\beta^4 + (\alpha^2 + \beta^2) y^2}}; \end{cases} \tag{5}$$

hence, finally

$$s = \sqrt{\alpha^2 + \beta^2} \left[\frac{\beta^2}{\alpha^2 + \beta^2} F(\varphi, k) - E(\varphi, k) + \frac{y}{\beta} \sqrt{\frac{(\alpha^2 + \beta^2)(\beta^2 + y^2)}{\beta^4 + (\alpha^2 + \beta^2)\, y^2}}\right]. \tag{6}$$

*Example II.*

An integral which arises in finding the decrease in lifting pressure near the tip of a sweptback wing flying at supersonic speed has the form

$$\begin{cases} I(X, Y) = \displaystyle\int_{t_0}^{t_1} \frac{dt}{\left(\frac{\beta Y}{X} - t\right)\sqrt{(t_1 - t)(t_1 + t)(1 + t)(t - t_0)}}, \\ \left(-1 < -t_1 < \dfrac{\beta Y}{X} < t_0 < t_1\right). \end{cases} \tag{7}$$

From **257.39**, with $a=t_1$, $b=t_0$, $c=-t_1$, $d=-1$, $m=1$ and $p=\beta Y/X$, we obtain (noting that the integral is complete)

$$(8)\qquad I=\frac{2}{(p-a)\sqrt{(a-c)(b-d)}}\int_0^K \frac{1-\left(\frac{b-a}{b-d}\right)\operatorname{sn}^2 u}{1-\frac{(p-d)(a-b)}{(a-p)(b-d)}\operatorname{sn}^2 u}\,du,$$

where

$$(9)\qquad k^2=\frac{(a-b)(c-d)}{(a-c)(b-d)}=\frac{(t_1-t_0)(1-t_1)}{2t_1(t_0+1)}.$$

Since $1>\frac{(p-d)(a-b)}{(a-p)(b-d)}>k^2$, we use **413.06**. Equation (8) thus becomes

$$(10)\qquad \begin{cases} I=\dfrac{2K}{(p-d)\sqrt{(a-c)(b-d)}}-\dfrac{\pi\Lambda_0(\xi,k)}{\sqrt{(a-p)(p-d)(b-p)(p-c)}}\\[2ex] \quad=\dfrac{2XK}{(\beta Y+X)\sqrt{2(t_0+1)t_1}}-\dfrac{\pi X^2\Lambda_0(\xi,k)}{\sqrt{(X^2t_1^2-\beta^2Y^2)(X+\beta Y)(Xt_0-\beta Y)}},\end{cases}$$

where

$$(11)\qquad \xi=\sin^{-1}\sqrt{\frac{(p-c)(b-d)}{(p-d)(b-c)}}=\sin^{-1}\sqrt{\frac{(\beta Y+t_1X)(t_0+1)}{(\beta Y+X)(t_0+t_1)}}$$

and $\Lambda_0(\xi,k)$ is defined in **150**.

*Example III.*

To obtain the gravitational potential $V$ of a homogeneous solid ellipsoid at the exterior point $(X, Y, Z)$, one is led to the integral

$$(12)\qquad \begin{cases} V=\pi\varrho\alpha\beta\gamma\displaystyle\int_{y_1}^{\infty}\left(1-\frac{X^2}{\alpha^2+t}-\frac{Y^2}{\beta^2+t}-\frac{Z^2}{\gamma^2+t}\right)\times\\[2ex] \qquad\times\dfrac{dt}{\sqrt{(\alpha^2+t)(\beta^2+t)(\gamma^2+t)}}\end{cases}$$

where $\varrho$ is the density and $y_1$ is the largest root of

$$(13)\qquad \frac{X^2}{\alpha^2+y_1}+\frac{Y^2}{\beta^2+y_1}+\frac{Z_2}{\gamma^2+y_1}=1,\quad(\alpha^2>\beta^2>\gamma^2).$$

Equation (12) thus involves the evaluation of four integrals of the form

$$(14)\qquad I=\int_{y_1}^{\infty}\frac{R(t)\,dt}{\sqrt{(t-a)(t-b)(t-c)}},$$

where $a=-\gamma^2$, $b=-\beta^2$, $c=-\alpha^2$, $(y_1>a>b>c)$ and $R(t)$ is a rational function of $t$. From **238.00** we have immediately

$$(15)\qquad \int_{y_1}^{\infty}\frac{dt}{\sqrt{(t-a)(t-b)(t-c)}}=\frac{2}{\sqrt{a-c}}\int_0^{u_1}du=\frac{2}{\sqrt{a-c}}u_1=\frac{2F(\varphi,k)}{\sqrt{a-c}}$$