Pseudodifferential Operators and Applications

AMERICAN MATHEMATICAL SOCIETY

PROCEEDINGS OF SYMPOSIA IN PURE MATHEMATICS
OF THE AMERICAN MATHEMATICAL SOCIETY
VOLUME 43

PROCEEDINGS OF THE SYMPOSIUM ON PSEUDODIFFERENTIAL OPERATORS
AND FOURIER INTEGRAL OPERATORS WITH APPLICATIONS TO
PARTIAL DIFFERENTIAL EQUATIONS
HELD AT THE UNIVERSITY OF NOTRE DAME
NOTRE DAME, INDIANA
APRIL 2–5, 1984

EDITED BY

FRANCOIS TRÈVES

Prepared by the American Mathematical Society
with partial support from National Science Foundation grant DMS-8318439

1980 *Mathematics Subject Classification*. 22E30, 32F99, 35-XX, 42B20, 43A80, 47-02, 47D30, 47G05, 58GXX, 81C12, 85B40, 85P05.

Library of Congress Cataloging in Publication Data

Symposium on Pseudodifferential Operators & Fourier Integral Operators with Applications to Partial Differential Equations (1984: University of Notre Dame)
Pseudodifferential operators and applications.
(Proceedings of symposia in pure mathematics; v. 43)
Proceedings of a symposium held at the University of Notre Dame, Apr. 2–5, 1984.
Bibliography: p.
1. Pseudodifferential operators–Congresses. 2. Fourier integral operators–Congresses. 3. Differential equaitons, Partial–Congresses. I. Trèves, François, 1930– . II. American Mathematical Society. III. Title. IV. Series.
QA329.7.S96 1984 515.7'242 85-1419
ISBN 0-8218-1469-9

Preface

This volume gathers a number of papers devoted to microlocal analysis, that is to say, to the precise application of Fourier transforms to analysis on manifolds. Mostly these articles are the written versions of lectures given by their authors at the AMS Symposium held at the University of Notre Dame from April 2 to April 5, 1984. In some instances the version is an abbreviated one; in one or two cases the content of the written paper differs from that of the lecture delivered, but covers a related topic.

Microlocalization is the most powerful tool of linear analysis to have emerged since distribution theory; it is the natural continuation of the latter. Its core is the theory of pseudodifferential and Fourier integral operators. Its central tenet is that the natural context for the study of PDE is the cotangent bundle—a claim that would not have excessively surprised the mathematicians of the nineteenth century or the physicists of the twentieth. In fact, physics has anticipated much of the new developments in the direction of microlocal analysis, often with less rigor and greater daring.

The successes of microlocalization have been truly splendid. They have made the analytic foundation of the Atyah–Singer formula simple, helped us penetrate deeper into the difficult study of uniqueness of solutions, and helped us understand why and when boundary problems like the oblique derivative or the $\bar{\partial}$-Neumann problem could be solved. It has allowed us to assert the existence or nonexistence of solutions to large classes of linear PDE and apply the vivid language of geometrical optics to describe their singularities. That same blend of geometrical optics and hard analysis has led to a greatly improved mathematical theory of diffraction. And hyperfunction theory has revitalized the study of overdetermined systems in the analytic category. Recently, the range of microlocal analysis has widened even more: Its advancing edge now tackles inverse scattering, the analysis of tunneling, the study of Cauchy–Riemann structures. It is even beginning to percolate into that most forbidding realm—nonlinear PDE. Although the truly severe nonlinearities, particularity those connected with shocks, seem at present out of reach, there is reason to believe that the future (and the near future, at that) will witness good progress in this direction.

Each of the topics I have just alluded to were discussed by the lecturers at the Symposium in Notre Dame; most of them, and many more, are studied in the articles in this volume. They bear testimony to the vitality and scope of the Fourier transform method.

FRANCOIS TRÈVES

Table of Contents

Proceedings of Symposia in Pure Mathematics
Volume 43 (1985)

Propagation of Local Analyticity for the Euler Equation

S. ALINHAC AND G. METIVIER

The following is the text of a talk given by the first author at the AMS meeting in Notre-Dame, Indiana on the joint paper *Propagation de l'analyticité locale pour les solutions de l'équation d'Euler* (to appear in Arch. Rational Mech. Anal.)

1. Introduction and results. Let $x \in \mathbb{R}^n$, $t \in [0, T]$, and consider the motion of an incompressible nonviscous fluid given by the equations

$$\partial u/\partial t + (u \cdot \nabla)u = \nabla p + f, \quad \operatorname{div} u = 0, \quad u(x,0) = u_0(x),$$

where $u(x,t) = (u_1(x,t),\ldots,u_n(x,t))$ is the velocity, p is the (unknown) pressure, and f and u_0 are given.

Assume that a sufficiently regular solution (u, p) is given globally in the strip $\mathbb{R}^n_x \times [0, T]$. More precisely, assume that, for some $\mu > n/2$ (say $\mu \in \mathbb{N}$), $u \in C^0([0, T], (H^{\mu+1}(\mathbb{R}^n_x))^n)$, $p \in C^0([0, T], H^{\mu+1}(\mathbb{R}^n_x))$, and $u_t \in C^0([0, T], (H^{\mu}(\mathbb{R}^n_x))^n)$. Then the geometry of the fluid lines $x'(t) = u(x(t), t)$ is well defined. If $\phi_{t,s}(x)$ is the solution of these equations, with $\phi_{s,s}(x) = x$, then $\phi_{t,s}\colon \mathbb{R}^n_x \to \mathbb{R}^n_x$ is a diffeomorphism.

Let $\Omega \subset \mathbb{R}^n$ be an open set, and let $\mathscr{C}$ be the tube with basis Ω:

$$\mathscr{C} = \{(x,t), 0 \leqslant t \leqslant T, x \in \phi_{t,0}(\Omega)\}.$$

We prove the following

THEOREM. *If u_0 is analytic in Ω and f in $\mathscr{C}$, then u is analytic in $\mathscr{C}$.*

REMARK. The same theorem holds for u on $U \times [0, T]$, with $u \cdot \nu = 0$ on $\partial U \times [0, T]$.

2. Some related results. (a) The existence of smooth enough solutions for the Euler equation has been established by many authors: see Kato [12], Ebin-Marsden [11], Temam [15], etc.

1980 *Mathematics Subject Classification*. Primary 35L75.

(b) For the case of globally analytic data (on a compact manifold or on a bounded set), existence has been proved by Baouendi–Goulaouic [4]–[6] and Delort [10].

(c) Recently, G. Metivier [14] has obtained local existence for analytic pseudodifferential operators and analytic data.

(d) A priori regularity results such as ours have been proved for the Euler equation by Bardos [7], Bardos–Benachour–Zerner [8], and Benachour [9] in the case of globally analytic data.

(e) For general (nonlinear) hyperbolic equations or systems, propagation of analyticity has been proved by Alinhac–Metivier [1, 2].

(f) Generalizations to nonhomogeneous fluids and boundary-value problems have been obtained by Le Bail [13].

What is the new feature of the present problem? The problem has a pseudodifferential character. More precisely, let $\pi = 1 - \operatorname{grad} \Delta^{-1} \operatorname{div}$ be the projector on divergence-free fields, orthogonally to gradients; then u satisfies

$$\partial u/\partial t + \pi(u \cdot \nabla u) = \pi f, \qquad u(x,0) = u_0(x),$$

which can be thought of as a (nonlinear) hyperbolic pseudodifferential system (whatever that means). The main difficulty here is that π being nonlocal, zones where u is not smooth contribute to the value of $\pi(u \cdot \nabla u)$ in $\mathscr{C}$. The quantitative control of this contribution will be the crucial step in the proof (Lemma 3.5).

3. Some ideas of the proof. The proof is carried out by taking x-derivatives of the equation and estimating them recursively with the aid of an energy inequality (control of t-derivatives being then given by the equation).

For fixed (forever) μ and any p, α, β, let $|\alpha| = p + 1$, $|\beta| \leqslant \mu$, and $v = \partial_x^{\alpha+\beta} u$. Then

$$\begin{aligned} \partial v/\partial t + u \cdot \nabla v &= \partial_x^{\alpha+\beta}(u \cdot \nabla u) - u \cdot \nabla \partial_x^{\alpha+\beta} u + \partial_x^{\alpha+\beta} \nabla p + \partial_x^{\alpha+\beta} f \\ &= g_1 + g_2 + g_3 = g. \end{aligned}$$

3.1. Classical papers (Friedman, Morrey–Nirenberg, etc.) and experience in the field show that "nested" open sets in Ω (which can be assumed smooth without loss of generality) have to be considered.

If $\delta(x) \geqslant 0$ is locally the distance to $\partial\Omega$, let $\Omega_\delta = \{x \in \Omega,\ \delta(x) \geqslant \delta\}$, $\Omega_{t,\delta} = \phi_{t,0}(\Omega_\delta)$, etc. We will estimate, at time t, the H^μ-norm in x of $\partial_x^{\alpha+\beta} u$ in $\Omega_{t,\delta}$. Let

$$X_p(t,\delta) = \sup_{|\alpha|=p} \|\partial_x^\alpha u(t,\cdot)\|_{H^\mu(\Omega_{t,\delta})}$$

and

$$Y_p = \sup_{1 \leqslant q \leqslant p}\ \sup_{t \in [0,T]}\ \sup_{0 < \delta \leqslant \delta_0} \left\{ \frac{e^{-\lambda(q-1)t}(\varepsilon\delta)^{q-1}}{m_{q-1}} X_q(t,\delta) \right\},$$

where $\varepsilon > 0$ ($\varepsilon \ll 1$) and $\lambda \gg 1$ have to be chosen, and $m_p = ctep!/(p+1)^2$ (see [1]).

The final aim of the recursion argument will be to prove $Y_p \leqslant H$, $\forall p$, which implies analyticity in Ω.

3.2. *The energy inequality.* Recall the following classical inequality. If $\partial_t v + u \cdot \nabla v = g$, then

$$\|v(t,\cdot)\|_{L^2(\Omega_{t,\delta})} \leqslant \|v(0,\cdot)\|_{L^2(\Omega_\delta)} + 2\int_0^t \|g(s,\cdot)\|_{L^2(\Omega_{s,\delta})}\, ds.$$

3.3. *Estimates for g_1 and g_3.* Since f is analytic, the estimate of g_3 is straightforward.

The estimate of g_1 can be obtained by using the machinery of [1]. The key facts are, of course, that H^μ is an algebra, and the special choice of m_p, which allows us to keep the same ε, λ in the recursion.

We summarize the results in the following lemma.

LEMMA. *There exist C, $\varepsilon_0 > 0$, and $\delta_0 > 0$ such that for all $0 < \varepsilon \leqslant \varepsilon_0$, $\lambda \geqslant 0$, $0 < \delta \leqslant \delta_0$, $t \in [0, T]$, we have ($i = 1$ or 3)*

$$\|g_i(t,\cdot)\|_{L^2(\Omega_{t,\delta})}$$

$$(3.3) \qquad \leqslant C(p+1)\left\{ X_{p+1}\left(t, \delta\left(1 - \frac{1}{p+1}\right)\right) + \left(Y_p^2 + 1\right) e^{\lambda p t}(\varepsilon\delta)^{-p} m_p \right\}.$$

Of course, all constants are independent of p.

3.4. *Estimates for the derivatives of the pressure.* We have

$$\Delta p = \sum \frac{\partial u_j}{\partial x_k}\frac{\partial u_k}{\partial x_j} - \operatorname{div} f \equiv A,$$

so

$$g_2 = \partial_x^{\alpha+\beta} \nabla p = \Delta^{-1}\partial_{x_j}\nabla\partial_x^{\alpha+\beta-1_j} A = K\partial_x^{\alpha+\beta-1_j} A,$$

where Δ^{-1} means the usual convolution, and $K = \Delta^{-1}\partial_j\nabla$ is an analytic pseudodifferential operator of order 0.

The remarkable fact here is that the type of control of the derivatives of u expressed by the inequalities $Y_p \leqslant H$ ($p \leqslant N$) is preserved by the action of K (even the constants ε, δ, λ are preserved).

This is precisely stated in Lemma 3.5, and it allows us to prove for g_2 the inequality (3.3).

3.5 *The crucial (pseudolocal) lemma.*

LEMMA 3.5. *Let $w \in H^\mu(\mathbb{R}^n_x) \cap C^\infty(\bar\Omega)$, and let K be an analytic pseudodifferential operator of order 0.*

Set $|\varphi|_{p,\delta} = \sup_{|\alpha|=p}\|\partial_x^\alpha\varphi\|_{H^\mu(\Omega_\delta)}$. Then there exist C, $\varepsilon_0 > 0$, $\delta_0 > 0$, s.t. for $p \geqslant 1$, $0 < \varepsilon \leqslant \varepsilon_0$, $0 < \delta \leqslant \delta_0$, we have

$$|Kw|_{p,\delta} \leqslant C\left\{ |w|_{p,\delta(1-1/(p+1))} + (\varepsilon\delta)^{-p} m_p\left([w]_{p-1} + \|w\|_{H^\mu(\mathbb{R}^n)}\right)\right\},$$

where

$$[w]_p = \sup_{\substack{0 \leq q \leq p \\ \delta \in]0, \delta_0]}} \frac{(\varepsilon\delta)^q |w|_{q,\delta}}{m_q}. \qquad \blacksquare$$

When this lemma is applied to the situation of 3.4, t is to be considered as a parameter.

A very similar result has been obtained by Baouendi–Goulaouic [6], who deal with functions holomorphic in certain domains of the complex space.

This lemma is proved by cutting the kernel k of K in p zones concentric about the origin.

3.6. *End of the proof*. Inequality (3.3) for g and the energy estimate 3.2 imply easily (taking into account the analyticity of u_0)

$$Y_{p+1} \leq \sup\left\{ Y_p, H_1 + C_1\left(Y_p^2 + 1\right)/(\lambda - \lambda_1)\right\}$$

for some constants C_1, H_1 and all ε, $1/\lambda$ small enough. Appropriate choices of H, ε, λ then give $Y_p \leq H$.

REFERENCES

1. S. Alinhac and G. Metivier, *Propagation de l'analyticité des solutions de systèmes hyperboliques non-linéaires*, Invent. Math. **75** (1984), 189–204.

2. ______, *Propagation de l'analyticité des solutions d'équations non-linéaires de type principal*, Comm. Partial Differential Equations 9 (1984), 523–537.

3. ______, *Propagation de l'analyticité locale pour les solutions de l'équation d'Euler*, Arch. Rational Mech. Anal. (to appear).

4. M. S. Baouendi and C. Goulaouic, *Problèmes de Cauchy pseudo-différentiels analytiques non-linéaires*, Séminaire Goulaouic-Schwartz 75–76 (No. 13), École Polytechnique, Paris.

5. ______, *Solutions analytiques de l'équation d'Euler d'un fluide incompressible*, Séminaire Goulaouic-Schwartz 76–77(No. 22), École Polytechnique, Paris.

6. ______, *Sharp estimates for analytic pseudo-differential operators and application to Cauchy problems*, J. Differential Equations **48** (1983), 241–268.

7. C. Bardos, *Analyticité de la solution de l'équation d'Euler dans un ouvert de* $\mathbf{R}^n$, C.R. Acad. Sci. Paris **283** (1976), 255–258.

8. C. Bardos, S. Benachour and M. Zerner, *Analyticité des solutions périodiques de l'équation d'Euler en dimension deux*, C.R. Acad. Sci. Paris **282** (1976), 995–998.

9. S. Benachour, *Analyticité des solutions périodiques de l'équation d'Euler en dimension trois*, C. R. Acad. Sci. Paris **283** (1976), 107–110.

10. J. M. Delort, *Estimations fines pour des opérateurs pseudo-différentiels analytiques sur un ouvert à bord de* $\mathbf{R}^n$. *Application aux équations d'Euler*, Comm. Partial Differential Equations (to appear).

11. D. G. Ebin and J. Marsden, *Groups of diffeomorphisms and the motion of an incompressible fluid*, Ann. of Math. (2) **92** (1970), 102–163.

12. T. Kato, *Nonstationary flows of viscous and ideal fluids in* $\mathbf{R}^3$, J. Funct. Anal. **9** (1972), 296–305.

13. Le Bail, Univ. de Rennes, preprint.

14. G. Metivier, *Un théorème de Cauchy-Kowalevski pseudo-différentiel local*, Séminaire Goulaouic-Schwartz 83–84, No. 16, École Polytechnique, Paris.

15. R. Temam, *On the Euler equtions of incompressible perfect fluids*, J. Funct. Anal. **20** (1975), 32–43.

UNIVERSITÉ PARIS-SUD, FRANCE

UNIVERSITÉ DE RENNES 1, FRANCE

Proceedings of Symposia in Pure Mathematics
Volume 43 (1985)

A Functional Calculus for a Class of Pseudodifferential Operators with Singular Symbols

JOSÉ L. ANTONIANO AND GUNTHER A. UHLMANN[1]

0. Introduction and statement of results. Distributions whose wave front set is contained in several intersecting Lagrangian manifolds appear naturally in many situations. For instance, the Schwartz kernel of the parametrices constructed by Duistermaat and Hörmander (see [DH]) for pseudodifferential operators P of real principal type have wave front set contained in the diagonal and in the flow-out, from the diagonal intersected with $p = 0$, by the integral curves of H_p. Also in the case of operators with double characteristics, several intersecting Lagrangian manifolds appear due to the interaction of the different flows (see [GU, MU, M, U]).

In this paper we extend the symbol calculus developed in Guillemin–Uhlmann [GU] for the case of two Lagrangian manifolds intersecting cleanly to a functional calculus under certain restrictions. Of particular importance is the case of a functional calculus for pseudodifferential operators with singular symbols. We now describe our results more precisely.

Let X be a C^∞ manifold of dimension n, $\Delta = \Lambda_0$ the diagonal in $T^*(X) \times T^*(X)$, p a symbol of real principal type (i.e., p is a real-valued canonical C^∞ function homogeneous of degree 1 on $T^*X \setminus 0$), $dp \neq 0$ on $p = 0$, dp and the one-form are linearly independent on $p = 0$. Let Λ_1 be the Lagrangian manifold obtained as the flow-out from $\Delta \cap \{p = 0\}$ by the integral curves of the Hamiltonian vector field associated to p. In this case the compositions of the different Lagrangians do not generate new manifolds since $\Lambda_0 \circ \Lambda_0 = \Lambda_0$, $\Lambda_0 \circ \Lambda_1 = \Lambda_1$, $\Lambda_1 \circ \Lambda_0 = \Lambda_1$, $\Lambda_1 \circ \Lambda_1 = \Lambda_1$. Notice also that the intersection of Λ_1 with itself is clean. Thus, the question naturally arises of whether the

1980 *Mathematics Subject Classification*. Primary 35S99; Secondary 58G15.

[1] The author was partially supported by NSF grant DMS 8402581. The author is an Alfred P. Sloan Research Fellow.

composition of two operators in $I^{\cdot}(X \times X; \Lambda_0, \Lambda_1)$, as defined in [GU], is in the same class. We prove

THEOREM 0.1. *Let* $A \in I^{p,l}(X \times X; \Lambda_0, \Lambda_1)$, $B \in I^{r,s}(X \times X; \Lambda_0, \Lambda_1)$ *be properly supported. Then* $A \circ B \in I^{\tilde{p},\tilde{l}}(X \times X; \Lambda_0, \Lambda_1)$, *with* $\tilde{p} = p + r - n/4$, $\tilde{l} = l + s - 1/2$.

Theorem 0.1 is proved in §2. In §3 we compute the symbol of the composition on Λ_0, Λ_1 away from the intersection $\Sigma = \Lambda_0 \cap \Lambda_1$. $A \circ B$ is a pseudodifferential operator on $\Lambda_0 - \Sigma$. We have

$$\sigma(A \circ B)|_{\Lambda_0 - \Sigma} = \sigma(A)|_{\Lambda_0 - \Sigma}\, \sigma(B)|_{\Lambda_0 - \Sigma}. \tag{0.1}$$

Also $A \circ B$ is a Fourier integral operator on $\Lambda_1 - \Sigma$. Let $(x, \xi, z, \zeta) \in \Lambda_1 - \Sigma$. Then, microlocally,

$$\begin{aligned} &\sigma(A \circ B)|_{\Lambda_1 - \Sigma}(x, \xi, z, \zeta) \\ &\quad = \int \sigma(A)|_{\Lambda_1}(x, \xi, y(t), \eta(t))\, \sigma(B)|_{\Lambda_1}(y(t), \eta(t), z, \zeta)\, dt, \end{aligned} \tag{0.2}$$

where $(y(t), \eta(t))$ denotes the (maximally extended) bicharacteristic curve joining (x, ξ) and (z, ζ). (0.2) does not take into account Maslov contributions and half-densities. Also, (0.2) is, at the moment, a formal expression because of the singularities of $\sigma(A)$ and $\sigma(B)$ on Σ. However, as we shall see in §3, the singularities of $\sigma(A)|_{\Lambda_1}$ and $\sigma(B)|_{\Lambda_1}$ in (0.2) occur at different t's, and one can make sense of (0.2) in the sense of distributions. In the case

$$A \in I^{p-n/4}(X \times X; \Lambda_1) \subseteq I^{p,l}(X \times X; \Delta, \Lambda_1),$$
$$B \in I^{r-n/4}(X \times X; \Lambda_1) \subseteq I^{r,s}(X \times X; \Delta, \Lambda_1),$$

formula (0.2) was obtained by Duistermaat and Guillemin (see [DG]). In §1 we briefly review the symbol calculus of [GU]. In §4 we give an application. We prove

THEOREM 0.2. *There exist elliptic* $\mathcal{U}, \tilde{\mathcal{U}} \in I^{\cdot}(\mathbb{R}^n \times \mathbb{R}^n; \Delta, \Lambda_1)$ *(in the sense described in* §1) *such that*

$$\left(\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2} + A\right)\mathcal{U} = \tilde{\mathcal{U}}\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2} \quad \textit{microlocally near } \Delta \cap \{\xi_1 = 0\}.$$

Here A *is a classical pseudodifferential operator of order* 0 *in* $\mathbb{R}^n$, $n > 2$, *and* $p = \xi_1$ *in this case.*

We recall that operators of the form $(\partial/\partial x_1)(\partial/\partial x_2) + A$ are a microlocal model for operators with double involutive characteristics of product type satisfying the Levi condition [U]. In §4 we also give further examples of operators in $I^{\cdot}(X \times X; \Delta, \Lambda_1)$, namely, pseudodifferential powers of operators with simple, real characteristics.

Theorem 0.1 was obtained independently by Jiang and Melrose (see [JM]).

1. Symbol calculus (see [GU] for details). Let X be a smooth manifold of dimension $n > 2$, Λ_0, Λ_1 two conic Lagrangian submanifolds of $T^*X \setminus 0$ intersecting cleanly in a submanifold of codimension 1 on each Λ_i, $i = 0, 1$, and

$$T_\lambda(\Lambda_0) \cap T_\lambda(\Lambda_1) = T_\lambda(\Lambda_0 \cap \Lambda_1) \quad \text{for all } \lambda \in \Lambda_0 \cap \Lambda_1. \tag{1.1}$$

A basic example of such an intersecting pair in $T^*(\mathbf{R}^n)$ is given by

$$\begin{aligned} \tilde{\Lambda}_0 &= \{(0, \xi) \in T^*\mathbf{R}^n \setminus 0\}, \\ \tilde{\Lambda}_1 &= \{(x, \xi) \in T^*\mathbf{R}^n \setminus 0 \mid x' = 0, \xi_1 = 0\}. \end{aligned} \tag{1.2}$$

Here (x, ξ) are the standard coordinates in $T^*(\mathbf{R}^n)$, $x = (x_1, x') \in \mathbf{R} \times \mathbf{R}^{n-1}$, and $\xi = (\xi_1, \xi') \in \mathbf{R} \times \mathbf{R}^{n-1}$. Let $\mathbf{R}^m = \mathbf{R}^n \times \mathbf{R}$. Let z, x, s be coordinates on $\mathbf{R}^m$, $\mathbf{R}^n$, and $\mathbf{R}$, respectively, and let ξ, σ be the dual variables of x, s.

$S^{p,l}(m, n, 1)$ denotes the space of all C^∞ functions in (z, ξ, σ) compactly supported in z and satisfying

$$\left| \left(\frac{\partial}{\partial z}\right)^\alpha \left(\frac{\partial}{\partial \xi}\right)^\beta \left(\frac{\partial}{\partial \sigma}\right)^\gamma a \right| \leqslant C_{\alpha,\beta,\gamma}(1 + |\xi|)^{p-|\beta|}(1 + |\sigma|)^{l-|\gamma|} \tag{1.3}$$

uniformly in (z, ξ, σ).

Let $a_r(z, \xi, \sigma)$ be a smooth function on the set $\xi \neq 0$, homogeneous of degree r in ξ, and let $a_{r,s}(z, \xi, \sigma)$ be a smooth function on the set $\xi \neq 0$, $\sigma \neq 0$, bihomogeneous of degree (r, s) in (ξ, σ). We shall say

$$a_r \sim \sum_{s=l}^{-\infty} a_{r,s} \quad \text{if } \rho(\xi, \sigma)\left(a_r - \sum_{s=l}^{-N} a_{r,s} \right) \in S^{r,-(N+1)}, \tag{1.4}$$

where ρ is a smooth function that is zero near $\xi = 0$ and $\sigma = 0$ and one outside a compact set. $S^{p,l}_{cl}(m, n, 1)$ denotes the subspace of $S^{p,l}(m, n, 1)$ consisting of all symbols that admit an asymptotic expansion of the form

$$a \sim \sum_{r=p}^{-\infty} a_r, \tag{1.5}$$

with the a_r's as in (1.4). Here $\sim$ means

$$\rho(\xi)\left(a - \sum_{r=p}^{-N} a_r \right) \in S^{-(N+1),l}(m, n, 1),$$

where ρ is a smooth function that is zero near $\xi = 0$ and one outside a compact set.

$I^{p,l}(X; \Lambda_0, \Lambda_1)$ denotes the space of oscillatory integrals with singular symbols as defined in [GU]. A microlocal model is $I^{p,l}(\mathbf{R}^n; \tilde{\Lambda}_0, \tilde{\Lambda}_1)$. We say $\mu \in I^{p,l}(\mathbf{R}^n; \tilde{\Lambda}_0, \tilde{\Lambda}_1)$ if $\mu = \mu_1 + \mu_2$, with $\mu_1 \in C_0^\infty(\mathbf{R}^n)$, and μ_2 can be represented by means of the oscillatory integral

$$\mu_2 = \int e^{i[(x_1 - s)\xi_1 + x'\xi' + s\sigma]} a(s, x, \xi, \sigma)\, ds\, d\xi\, d\sigma, \tag{1.6}$$

with $a \in S^{p',l'}_{cl}(m, n, 1)$, $p' = p - n/4$, $l' = l - 1/2$.

We have the following facts from [GU].

PROPOSITION 1.1. *Let* $\tilde{\Lambda}_0$ *and* $\tilde{\Lambda}_1$ *be as in* (1.2), *and let* $\tilde{\Sigma} = \tilde{\Lambda}_0 \cap \tilde{\Lambda}_1$. *Then* $\mathrm{WF}(\mu_2) \subseteq \tilde{\Lambda}_0 \cup \tilde{\Lambda}_1$. *Moreover, near* $\tilde{\Lambda}_0 - \tilde{\Sigma}$, μ_2 *is microlocally in the space* $I^{p+l+n/4}(\mathbf{R}^n; \tilde{\Lambda}_0)$, *and its leading symbol is* $a_{p,l}(z,\xi,\sigma)|_{z=0,\sigma=\xi_1}$, *omitting half-densities and Maslov contributions. Near* $\tilde{\Lambda}_1 - \tilde{\Sigma}$, μ_2 *is microlocally in the space* $I^{p+n/4}(\mathbf{R}^n; \tilde{\Lambda}_1)$, *and its leading symbol is*

$$\left(\int a_p(z,\xi,\sigma)e^{is\sigma}\,d\sigma\right)\Bigg|_{\substack{x'=0\\ \xi_1=0\\ s=x_1}}, \tag{1.7}$$

omitting half-densities and Maslov contributions.

PROPOSITION 1.2. *With* l *fixed*, $\bigcap_p I^{p,l} = C_0^\infty(\mathbf{R}^n)$. *With* p *fixed*, $\bigcap_l I^{p,l} = I^p(\mathbf{R}^n; \tilde{\Lambda}_1)$.

DEFINITION 1.1. Given $u \in I^{p,l}(X; \Lambda_0, \Lambda_1)$, let $\sigma_0(u) = \sigma(u)|_{\Lambda_0 - \Sigma}$ and $\sigma_1(u) = \sigma(u)|_{\Lambda_1 - \Sigma}$. $\sigma_0(u)$ is called the principal symbol of u.

The main result is

PROPOSITION 1.3. *The following sequence*

$$0 \to I^{p,l-1}(X;\Lambda_0,\Lambda_1) + I^{p-1,l}(X;\Lambda_0,\Lambda_1) \to I^{p,l}(X;\Lambda_0,\Lambda_1)$$
$$\to S^{p,l}(X;\Lambda_0,\Sigma) \to 0$$

is exact, where $S^{p,l}(X;\Lambda_0,\Sigma)$ *is the subspace of* $R^{l-1/2}(\Omega_0 \otimes L_0; \Lambda_0, \Sigma)$ (*as defined in* [GU]). $R^{l-1/2}$ *is, intuitively, "the space of smooth functions on* $\Lambda_0 - \Sigma$ *that have a singularity of order* l *at* Σ", *and consist of elements that are homogeneous of degree* $p + l + n/4$.

Let $X = \mathbf{R}^n \times \mathbf{R}^n$, let $\Lambda_0 = \Delta$ be the diagonal, and let Λ_1 be the flow-out from $\Lambda \cap \{\xi_1 = 0\}$ by the integral curves of $\partial/\partial x_1$. It is easy to check

PROPOSITION 1.4. *If* $A \in I^{p,l}(X;\Delta,\Lambda_1)$ *then* $A^t \in I^{p,l}(X;\Delta,\Lambda_1)$.

DEFINITION 1.2. $A \in I^{p,l}(X;\Delta,\Lambda_1)$ is elliptic if $\sigma_0(A) \neq 0$ on $\Delta - \Sigma$.

2. Proof of Theorem 0.1. Using the invariance of the distributions in $I^{p,l}(X;\Lambda_0,\Lambda_1)$ under conjugation by Fourier integral operators (see [GU]), we can assume $X = \mathbf{R}^n \times \mathbf{R}^n$, Λ_0 is the diagonal in $T^*(\mathbf{R}^n) \times T^*(\mathbf{R}^n)$, and Λ_1 is the flow-out from $\Lambda_0 \cap \{\xi_1 = 0\}$ by the integral curves of $\partial/\partial x_1$; i.e.,

$$\Lambda_0 = \{((x,\xi),(x,\xi)) \in T^*(\mathbf{R}^n \times \mathbf{R}^n)\setminus 0\},$$
$$\Lambda_1 = \{((x_1,x',0,\xi'),(y_1,x',0,\xi')) \in T^*(\mathbf{R}^n \times \mathbf{R}^n)\setminus 0\},$$

where $x = (x_1,x') \in \mathbf{R} \times \mathbf{R}^{n-1}$, $\xi = (\xi_1,\xi') \in \mathbf{R} \times \mathbf{R}^{n-1}$. We take the Schwartz kernel of A and B to be

$$k_A = \int e^{i[(x_1-y_1-s)\xi_1 + (x'-y')\xi' + s\sigma]} a(s,x,y,\xi,\sigma)\,ds\,d\xi\,d\sigma, \tag{2.1}$$

with $a \in S_{cl}^{p',l'}(m, n, 1)$ a product type symbol in ξ, σ; $m = 2n + 1, p' = p - n/4$, $l' = l - 1/2, s, \sigma \in \mathbf{R}$ and

$$k_B = \int e^{i[(y_1 - z_1 - t)\eta_1 + (y' - z')\eta' + t\tau]} b(t, y, z, \eta, \tau)\, dt\, d\eta\, d\tau, \tag{2.2}$$

with $b \in S_{cl}^{r',s'}(m, n, 1)$ a product type symbol in (η, τ), m as above, $r' = r - n/4$, $s' = s - n/2$, $t, \tau \in \mathbf{R}$. We first cut off a near $\xi = 0$, $\sigma = 0$ and b near $\eta = 0$, $\tau = 0$. Let $\chi \in C^\infty(\mathbf{R}^n)$ be a homogeneous function of degree 0 for $|\xi| \geqslant 1$ such that $\chi = 0$ near $\xi = 0$, $\chi = 1$ for $|\xi| \geqslant 1$, and let $\tilde{\chi} \in C^\infty(\mathbf{R})$ be a homogeneous function of degree 0 for $|\sigma| \geqslant 1$ s.t. $\tilde{\chi} = 0$ near $\sigma = 0$, $\tilde{\chi} = 1$ for $|\sigma| \geqslant 1$. Then

$$k_A = \int e^{i[(x_1 - y_1 - s)\xi_1 + (x' - y')\xi' + s\sigma]} \chi(\xi)\tilde{\chi}(\sigma) a(s, x, y, \xi, \sigma)\, ds\, d\xi\, d\sigma + \tilde{A} \quad \text{with } \tilde{A} \in I^{p-n/4}(\mathbf{R}^n \times \mathbf{R}^n; \Lambda_1) \tag{2.3}$$

and

$$k_B = \int e^{i[(y_1 - z_1 - t)\eta_1 + (y' - z')\eta' + t\tau]} \chi(\eta)\tilde{\chi}(\tau) b(t, y, z, \eta, \tau)\, dt\, d\eta\, d\tau + \tilde{B} \quad \text{with } \tilde{B} \in I^{r-n/4}(\mathbf{R}^n \times \mathbf{R}^n; \Lambda_1). \tag{2.4}$$

Thus we may assume that a (resp. b) in (2.1) (resp. (2.2)) vanishes near $\xi = 0$, $\sigma = 0$ (resp. $\eta = 0$, $\tau = 0$), since the terms $A \circ \tilde{B}$, $\tilde{A} \circ B$ can be shown to be in the appropriate class by a similar argument to the one used below. We have

$$k_{A \circ B} = \int e^{i(\phi_1 + \phi_2)} a \cdot b\, dy\, ds\, d\xi\, d\sigma\, dt\, d\eta\, d\tau, \tag{2.5}$$

where

$$\phi_1 = (x_1 - y_1 - s)\xi_1 + (x' - y')\xi' + s\sigma,$$
$$\phi_2 = (y_1 - z_1 - t)y_1 + (y' - z')\xi' + t\tau.$$

Let $0 < \varepsilon < 1$, and let χ_1 (resp. χ_2) be a C^∞ homogeneous function of degree 0 in (ξ, η) such that $\chi_1 = 1$ for $|\eta| \leqslant \frac{1}{2}\varepsilon|\xi|$, $\chi_1 = 0$ for $|\eta| \geqslant \varepsilon|\xi|$ if $|(\xi, \eta)| = 1$ (resp. $\chi_2 = 1$ for $|\xi| \leqslant \frac{1}{2}\varepsilon|\eta|$, $\chi_2 = 0$ for $|\xi| \geqslant \varepsilon|\eta|$ for $|(\xi, \eta)| = 1$). Integrating by parts in the y-variable, we can easily show that the term

$$\int e^{i(\phi_1 + \phi_2)} \chi_i a \cdot b\, dy\, ds\, d\xi\, d\sigma\, dt\, d\eta\, d\tau$$

is smoothing, $i = 1, 2$. We then assume that the amplitude in (2.5) satisfies

$$\tfrac{1}{2}\varepsilon \cdot |\eta| \leqslant |\xi| \leqslant 2|\eta|/\varepsilon \quad \text{on } \operatorname{supp}(a \cdot b). \tag{2.6}$$

Now making the change of variables $s = u + r$, $t = -u + r$, and $y_1 = -u + v$, we obtain

$$k_{A \circ B} = 2\int e^{i[(u+r)\sigma + (-u+r)\tau]} D\, du\, dr\, d\xi\, d\sigma\, d\tau, \tag{2.7}$$

where

$$D = \int e^{i[(x_1 - v - r)\xi_1 + (x' - y')\xi' + (v - z_1 - r)\eta_1 + (y' - z')\eta']} a \cdot b\, dv\, dy'\, d\eta_1\, d\eta'. \tag{2.8}$$

In order to be able to apply the stationary phase method, we introduce polar coordinates in ξ. Writing $\omega = \xi/|\xi| \in S^{n-1}$, $\xi = |\xi|\omega$, and making the change of variables $\eta = |\xi|\bar{\eta}$ in (2.8), we get

$$(2.9)\ D = \int e^{i|\xi|[(x_1 - v - r)\omega_1 + (z' - y')\omega' + (v - z_1 - r)\bar{\eta}_1 + (y' - z')\bar{\eta}']}$$

$$\cdot a \cdot b(u + r, -u + r, x, (-u + v, y'), z, |\xi|\omega, |\xi|\bar{\eta}, \sigma, \tau)\, dv\, dy'\, d\eta.$$

Observe that $\bar{\eta} \leqslant 2/\varepsilon$ on supp $a \cdot b$ ((2.6)). We are now in position to apply stationary phase in $(v, y', \bar{\eta})$. We obtain

$$(2.10)\qquad D \sim (2\pi/|\xi|)^n e^{i[(x_1 - z_1 - 2r)\xi_1 + (x' + z')\xi']}$$

$$\cdot d(u + r, -u + r, x_1, (-u + v, y'), z, \xi, \sigma, \tau),$$

where $d \in S_{cl}^{p'+r', l', s'}(n, 1, 1)$ is a product type symbol of order $p' + r'$, l', s' in ξ, σ, τ, respectively; $\sim$ in (2.10) means that

$$\left(D - \sum_{j=1}^{N} d_j\right) \in S^{p'+r'-(N+1), l', s'}(n, 1, 1),$$

where $d \sim \sum d_j$ as in §1.

Let us now consider

$$(2.11)\qquad F = \int e^{i[(u+r)\sigma + (-u+r)\tau]} d\, du\, d\tau.$$

Let $\chi_1(\sigma, \tau)$, $\chi_2(\sigma, \tau)$ be defined in a similar way as above. Then

$$\int e^{i[(u+r)\sigma + (-u+r)\tau + (x_1 - z_1 - 2r)\xi_1 + (x' - z')\xi']} \chi_i\, d\, du\, d\tau$$

is an element of $I^{p+l}(\mathbf{R}^n \times \mathbf{R}^n \Lambda_1)$. Therefore, we may assume

$$(2.12)\qquad \tfrac{1}{2}\varepsilon|\tau| \leqslant |\sigma| \leqslant 2|\tau|/\varepsilon \quad \text{on supp } d.$$

Then we apply the stationary phase Lemma to F in u, $\bar{\tau}$, with $\bar{\tau} = \tau/|\sigma|$. Putting everything together we find that

$$(2.13)\quad k_{A \circ B} = \int e^{i[(x_1 - z_1 - 2r)\xi_1 + (x' - y')\xi' + 2r\sigma]} c(r, x, z, \xi, \sigma)\, d\xi\, d\sigma\, dr,$$

where $c \in S_{cl}^{p', r'}$ is a classical product type symbol in the (ξ, σ) variables ending the proof of the theorem.

3. Computation of the principal symbol of $A \circ B$. First we compute the principal symbol of $A \circ B$ on $\Delta - \Sigma$, where A, B, $A \circ B$ are pseudodifferential operators. We have

$$\sigma(A)|_{\Delta - \Sigma} = a_{p', l'}(0, x, x, \xi, \xi_1)|_{\xi_1} \neq 0,$$

$$\sigma(B)|_{\Delta - \Sigma} = b_{r', s'}(0, z, z, \zeta, \zeta_1)|_{\zeta_1} \neq 0$$

(see §1). From the proof in §2 it is easy to see that the highest order of homogeneity of c (as in (2.13)) in ξ is $a_{p', l'} \cdot b_{r', s'}$. Therefore, we conclude that

$$(3.1)\qquad \sigma(A \circ B)|_{\Delta - \Sigma} = \sigma(A)|_{\Delta - \Sigma} \cdot \sigma(B)|_{\Delta - \Sigma}.$$

We next compute the principal symbol of $A \circ B$ on $\Lambda_1 - \Sigma$. We first show that we can extend the C^∞ function on $\Lambda_1 - \Sigma$, $\sigma(A)|_{\Lambda_1 - \Sigma}$, to a symbol-valued conormal distribution associated to $\Sigma \subseteq \Lambda_1$. Let

$$f_{p'}^{A}(s, x, y, \xi) = \int e^{is\sigma} a_{p'}(s, x, y, \xi, \sigma)\, d\sigma,$$

where $a \sim \sum_{j \leq p'} a_j$ as in §1. Then $f_{p'}^{A} \in C^\infty(\mathbf{R}_x^n \times \mathbf{R}_y^n \times \mathbf{R}_\xi^n; \mathscr{S}'(\mathbf{R}_s))$. In fact, $f_{p'}^{A}$ is a conormal distribution associated with $s = 0$. We also have (see §1)

$$\sigma(A)|_{\Lambda_1 - \Sigma} = f_{p'}^{A}\,(x_1 - y_1, (x_1, x'), (y_1, x'), (0, \xi'))|_{x_1 \neq y_1}.$$

Then we extend $\sigma(A)|_{\Lambda_1 - \Sigma}$ to Λ_1 as an element of

$$C^\infty\Big(\mathbf{R}_{x'}^{n-1} \times \mathbf{R}_{\xi'}^{n-1}; \mathscr{D}'\Big(\mathbf{R}^2_{(x_1, y_1)}\Big)\Big)$$

—in fact, as a conormal distribution associated with $x_1 = y_1$. Clearly, $(\sigma(A)|_{\Lambda_1}, \psi(x_1, y_1))$ is a homogeneous function of degree p' in ξ' depending smoothly on x', y' $\forall \psi \in C_0^\infty(\mathbf{R}^2)$. Let

$$(3.2) \qquad f_A = \int e^{is\sigma} a(s, x, y, \xi, \eta)\, d\sigma, \quad f_B = \int e^{it\tau} b(t, y, z, \eta, \tau)\, d\tau.$$

We now write

$$(3.3) \qquad k_{A \circ B} = \pi_* \tilde{K}$$

where π: $\mathbf{R}_x^n \times \mathbf{R}_z^n \times \mathbf{R}_s \times \mathbf{R}_t \to \mathbf{R}_x^n \times \mathbf{R}_z^n$ is the projection. $\tilde{K}$ is formally given by

$$(3.4) \qquad (\tilde{K}, \phi) = \int e^{i(\psi_1 + \psi_2)}(f_A f_B, \phi)\, dy\, dx\, dz\, d\xi\, d\eta$$

$\forall \phi \in C_0^\infty(\mathbf{R}_x^n \times \mathbf{R}_z^n \times \mathbf{R}_s \times \mathbf{R}_t)$, where

$$\psi_1 = (x_1 - y_1 - s)\xi_1 + (x' - y')\xi', \quad \psi_2 = (y_1 - z_1 - t)\eta_1 + (y' - z')\eta'.$$

Formally,

$$(3.5) \qquad (f_A f_B, \phi) = \int f_A f_B \phi\, ds\, dt \quad \forall \phi$$

as above.

PROPOSITION 3.1. *$(f_A f_B, \phi)$, as in (3.5), is a symbol-valued symbol in ξ, η; i.e.,*

$$\left|D_x^\alpha D_z^\beta D_\xi^\gamma D_\eta^\delta (f_A f_B, \phi)\right| \leq C_{\alpha,\beta,\gamma,\delta}(1 + |\xi|)^{p' - |\gamma|}(1 + |\eta|)^{r' - |\delta|}$$

uniformly on compact subsets in (x, z).

PROOF. We write

$$(f_A f_B, \phi) = \int e^{i(s\sigma + t\tau)} a \cdot b\, \phi\, ds\, dt.$$

We develop a (resp. b) in a Taylor series around $s = 0$ (resp. $t = 0$) and obtain

$$(3.6)\qquad (f_A f_B, \phi) = \int \sum_{j=0}^{N} \sum_{k=0}^{N} \frac{d^k}{ds^k} a \Big|_{s=0} \frac{d^j}{dt^j} b \Big|_{t=0} \widehat{D_s^k D_t^j \phi}\, d\sigma\, d\tau + (R_N, \phi),$$

where the Fourier transform is taken in the s, t variables. The first term in (3.6) satisfies the estimate in the proposition, since now the amplitude is rapidly decreasing in σ, τ, and we use the estimates for a and b as product type symbols. (R_N, ϕ) contains terms of the form

$$(3.7)\qquad \begin{aligned} &\int e^{i(s\sigma + t\tau)} \frac{s^{(N+1)}}{(N+1)!} h_N \frac{d^j}{dt^j} b \Big|_{t=0} \phi\, d\tau\, d\sigma\, dt\, ds \\ &= \int \frac{D_\sigma^{N+1}(e^{is\sigma})}{(N+1)!} h_N \frac{d^j}{dt^j} b \Big|_{t=0} \hat{\phi}\, d\tau\, d\sigma\, ds, \end{aligned}$$

where the Fourier transform is in the t variable. h_N is the remainder term of order $N + 1$ in the Taylor series of a. Integrating by parts in (3.7) in the σ variable, we obtain that, for N large, the integrand in (3.7) is absolutely integrable in σ, τ, and, therefore, the estimate of Proposition 3.1 is solid for terms of the form (3.4). The other terms in (R_N, ϕ) are estimated similarly.

PROPOSITION 3.2. $\tilde{K} \in \mathscr{D}'(\mathbf{R}^n_x \times \mathbf{R}^n_z \times \mathbf{R}_s \times \mathbf{R}_t)$.

PROOF. We may assume $\xi, \eta \neq 0$. Then we can write (3.4) in the form

$$(3.8)\qquad (\tilde{K}, \phi) = \int \Delta_x \Delta_z e^{i(\psi_1 + \psi_2)} (f_A f_B, \phi) \frac{1}{|\xi|^2} \frac{1}{|\eta|^2}\, dy\, dx\, dz\, d\xi\, d\eta,$$

where Δ_x, Δ_z denotes the Laplacian in the x and z variables, respectively. Integrating by parts in the x, z variables, we obtain the result. Thus

$$k_{A \circ B} = \int e^{i(\psi_1 + \psi_2)} f_A f_B\, dy\, ds\, d\xi\, dt\, d\eta$$

is a well-defined distribution in $\mathscr{D}'(\mathbf{R}^n \times \mathbf{R}^n)$. Now in order to prove (0.2) we make the change of variables $s = u + r$, $t = -u + r$, $y_1 = -u + v$. As in §2 we apply the stationary phase lemma in v, y', η. We obtain

$$(3.9)\qquad \begin{aligned} k_{A \circ B} = \int & e^{i[(x_1 - z_1 - 2r)\xi_1 + (x' - z')\xi']} \\ & \cdot f_C(u + r, -u + r, x, (z_1 + r - u, z'), z, \xi)\, du\, dr\, d\xi, \end{aligned}$$

where

$$f_C \sim \left(\frac{2\pi}{|\xi|}\right)^n \sum_{j=0}^{\infty} \frac{(i)^j}{j!} \left(\frac{\partial}{\partial v} \frac{\partial}{\partial y'} \frac{\partial}{\partial \eta}\right)^j (f_A f_B) \Bigg|_{\substack{\eta = \xi_1 \\ v = z_1 + r \\ y' = z'}} |\xi|^{-j}.$$