NORTH-HOLLAND MATHEMATICS STUDIES 111
Notas de Matemática (101)

Editor: Leopoldo Nachbin

RECENT PROGRESS IN FOURIER ANALYSIS

Proceedings of the Seminar on Fourier Analysis held in El Escorial, Spain, June 30 - July 5, 1983

Edited by

I. PERAL and J.-L. RUBIO de FRANCIA

RECENT PROGRESS IN FOURIER ANALYSIS

Proceedings of the Seminar on Fourier Analysis held in El Escorial, Spain, June 30 - July 5, 1983

Edited by

I. PERAL and J.-L. RUBIO de FRANCIA
Universidad Autónoma de Madrid
Madrid
Spain

1985

NORTH-HOLLAND – AMSTERDAM • NEW YORK • OXFORD

ISBN: 0 444 87745 2

Publishers:
ELSEVIER SCIENCE PUBLISHERS B.V.
P.O. Box 1991
1000 BZ Amsterdam
The Netherlands

Sole distributors for the U.S.A. and Canada:
ELSEVIER SCIENCE PUBLISHING COMPANY, INC.
52 Vanderbilt Avenue
New York, N.Y. 10017
U.S.A.

Library of Congress Cataloging in Publication Data

Seminar on Fourier Analysis (1983 : Escorial) :
Recent progress in Fourier analysis.

(North-Holland mathematics studies ; 111) (Notas de matemática ; 101)
English or French.
1. Fourier analysis--Congresses. I. Peral, Ireneo. II. Rubio de Francia, J.-L., 1949- . III. Title. IV. Series. V. Series: Notas de matemática (Rio de Janeiro, Brazil) ; no. 101.
QA1.N86 no. 101a 510 s [515'.2433] 85-4531
[QA403.5]
ISBN 0-444-87745-2 (Elsevier Science Pub.)

PRINTED IN THE NETHERLANDS

RECENT PROGRESS IN FOURIER ANALYSIS

NORTH-HOLLAND MATHEMATICS STUDIES 111

Notas de Matemática (101)

Editor: Leopoldo Nachbin

Centro Brasileiro de Pesquisas Físicas
Rio de Janeiro
and University of Rochester

NORTH-HOLLAND – AMSTERDAM • NEW YORK • OXFORD

RECENT PROGRESS IN FOURIER ANALYSIS

The following contributions were presented at the Seminar on Fourier Analysis which was held in El Escorial from 30 June to 5 July 1983. This meeting was sponsored by the Asociación Matemática Española with financial support from the Comisión Asesora de Investigación Científica y Técnica (project 4192).

A decisive factor with respect to the organization was the financial help, together with the facilities, provided by the Vicerrectorado de Investigación of the Universidad Autónoma de Madrid.

The articles we present give a good idea of how work in the area has evolved and of the scientific character of the meeting. The friendly and cordial atmosphere meant that the organization, far from being a chore, became a pleasurable experience. For this we owe our sincerest thanks to all participants.

Special thanks must also go to the invited speakers for their magnificent collaboration, and to Caroline, without whose presence we hate to think what could have happened!

We should also like to express our gratitude to our colleagues in the División de Matemáticas in the Universidad Autónoma de Madrid, for their help in correcting proofs, and to Soledad, for typing the manuscript.

The Editors

CONTENTS

Recent Progress in Fourier Analysis
I. Peral and J.-L. Rubio de Francia (Editors)

FUNCTIONS OF L^p-BOUNDED PSEUDO-DIFFERENTIAL OPERATORS

Josefina Alvarez Alonso
Universidad de Buenos Aires

The aim of this paper is to construct a functional calculus over an algebra of L^p-bounded pseudo-differential operators acting on functions defined on a compact manifold without boundary.

The operators we consider here depend on amplitudes or symbols with a finite number of derivatives, without any hypothesis of homogeneity. The manifolds where the operators act are also of class C^M for a suitable M. In this way is it possible to control the number of derivatives of f that we need in order to give meaning to $f(A)$, when A is a self-adjoint operator in that algebra.

Indeed, this program was carried out in [1] and [2] when $p = 2$. In [1] an algebra of pseudo-differential operators acting on functions defined in $\mathbb{R}^n$ is constructed. The main tool to do that is the sharp L^2 estimates obtained by R. Coifman and Y. Meyer in [3]. Then, functions of those operators are defined by means of the H. Weyl formula (see [4], for example). Since it seems not to be possible to obtain directly a polynomial estimate for the exponential $\exp(-2\pi itA)$ in terms of t, a roundabout argument is employed by introducing an adapted version of the characteristic operators defined by A. P. Calderón in [5].

All this machinery is extended in [2] to non-infinitely differentiable compact manifolds without boundary.

In order to get the L^p version of these results the first thing to do is to obtain the analogous of the algebra constructed in [1]. The main point is to observe that amplitudes in a subclass of $S^0_{1,1}$ give rise to operators on which the classical theory of Calderón and Zygmund works (see [6]). Unfortunately as far as I know, it is an open question to get in the euclidean case a non trivial estimate for the exponential $\exp(-2\pi itA)$. However, when the operators act on functions defined in compact manifolds, a suitable estimate can be obtained and so, a non-infinitely differentiable functional calculus runs.

Given $0 \leq \delta < 1$, $k = 1,2,\ldots,$ let

$$N = \begin{cases} k/1-\delta & \text{if this is an integer} \\ [k/1-\delta] + 1 & \text{if not} \end{cases}$$

We will consider operators K acting on S in the following way

$$Kf = \sum_{j=0}^{N-1} \int e^{-2\pi i x\xi} \, p_j(x,\xi)\hat{f}(\xi)d\xi + Rf$$

where

i) The function p_j belongs to the class S^j; that is to say, p_j is a continuous function defined on $\mathbb{R}^n \times \mathbb{R}^n$; it has continuous derivatives in the variable ξ up to the order $n+N+2-j$ and each function $D_\xi^\gamma p_j$ has continuous derivatives in x,ξ up to the order $2[n/2]+N+k+2-j$, satisfying

$$\sup_{\substack{x,\xi \in \mathbb{R}^n \\ \alpha,\, \beta,\, \gamma}} \frac{|D_s^\alpha D_\xi^\beta D_\xi^\gamma \, p_j(x,\xi)|}{(1+|\xi|)^{-j(1-\delta)+|\alpha|\delta-|\beta+\gamma|}} < \infty$$

ii) For $1 < p_0 \leq 2$ fixed, R is a linear and continuous operator from L^p into itself for $p_0 \leq p \leq p_0'$. Moreover, R and the adjoint R^* are continuous from L^p into L_k^p, where L_k^p denotes the Sobolev space of order k and p_0' is the conjugate exponent of p_0.

Let M_k be the class of the operators K.

Now, let X be a differentiable compact manifold of dimension n and class C^M, $M = 2[n/2]+n+2N+k+5$, without boundary; X has a measure μ which in terms of any local coordinate system $x = (x_1,\ldots,x_n)$ can be express as $G(x)dx_1 \ldots dx_n$, where $G > 0$ is a function of class C^{M-1}.

We will introduce the following notation.

Let U_1, U_2 be open bounded subsets of X or $\mathbb{R}^n$ let $\phi : U_1 \to U_2$ be a diffeomorphism of class C^M; if f is a function defined on the ambient space of U_2, $\phi^*(f)$ will denote the function defined on the ambient space of U_1 which coincides with $f \circ \phi$ on U_1 and vanishes outside U_1. On the other hand, if A is an operator acting on functions defined on the ambient space of U_2, by $\phi^*(A)$ we denote the operator acting on functions defined on the ambient space of U_1 as

$$\phi^*(A)(f) = \phi^*[A(\phi^{-1*}(f))]$$

Now, we are ready to define classes of operators on X.

Given $1 < p_0 \leq 2$, R belongs to $R_k(X)$ if R is a linear continuous operator from $L^p(X)$ into itself for $p_0 \leq p \leq p_0'$ and R, R^* map continuously $L^p(X)$ into $L^p_k(X)$ for $p_0 \leq p \leq p_0'$.

$R_k(X)$ is a self-adjoint Banach algebra with the norm

$$|R|_{R_k} = |R|_{L^{p_0},L^{p_0}_k} + |R|_{L^{p_0'},L^{p_0'}_k} + |R^*|_{L^{p_0},L^{p_0}_k} + |R^*|_{L^{p_0'},L^{p_0'}_k}$$

Now, given $1 < p_0 \leq 2$, $M_k(X)$ is the class of linear continuous operators A from $L^p(X)$ into itself for $p_0 \leq p \leq p_0'$, which satisfy the following two conditions

i) Given $\phi_1, \phi_2 \in C_0^M(X)$ with disjoint supports, the operator $\phi_1 A \phi_2$ belongs to $R_k(X)$. Here ϕ_1, ϕ_2 stand for the operators of multiplication by the function ϕ_1, ϕ_2, respectively.

ii) Let $U \subset X$ be an open subset and let $\phi : U \to U_1$ be a diffeomorphism of class C^M extendable to a neighborhood of $\bar{U}$, where $U_1 \subset \mathbb{R}^n$. There exists an operator $A_1 \in M_k$ such that if $\phi_1, \phi_2 \in C_0^M(U)$,

$$\phi_1 A \phi_2 = \phi_1 \phi^*(A_1)\phi_2$$

$M_k(X)$ is a self-adjoint algebra and $R_k(X)$ is a two-sided ideal of $M_k(X)$; moreover, operators in $M_k(X)$ are continuous from $L^p_m(X)$ into itself for $p_0 \leq p \leq p_0'$, $0 \leq m \leq k$.

It is possible to endow $M_k(X)$ with a complete norm. In order to avoid technical details, we will not precise the definition. With this norm $R_k(X)$ is continuously included in $M_k(X)$ and $M_k(X)$ is continuously included in $L(L^p_m(X))$, the space of linear and continuous operators from $L^p_m(X)$ into itself, for $p_0 \leq p \leq p_0'$, $0 \leq m \leq k$.

THEOREM 1. *Let* p_0, k *and* n *be such that* $1/p_0 - k/n \leq 1/2$.

Given a self-adjoint operator $A \in M_k(X)$ *and a function* f *in the Sobolev space* L^2_s, *where* $s > 2\mu + 5/2$, $\mu = 2[n/2]+n+k+ + N(N+3)/2+4$, *the Bochner integral*

$$\int_{-\infty}^{\infty} e^{-\pi itA}\, \hat{f}(t)dt$$

belongs to $R_k(X)$ *and coincides with* $f(A)$ *calculated by means of the spectral formula in* $L(L^2(X))$.

Remarks:

a) It is possible to impose on f additional conditions under which the operator $f(A)$ belongs to $R_k(X)$.

b) When $p_0 = 2$ the above theorem remains true with $s > \mu + 3/2$.

c) The Weyl's formula also allows to define functions of a tuple of non-commuting self-adjoint operators.

We will include here the proof of the theorem 1 in a particular but significant case.

Suppose that $\delta = 0$, $k = 1$; it follows that $N = 1$. It is clear that theorem 1 can be deduced from a suitable estimate for $|\exp(-2\pi itA)|_{M_1(X)}$ in terms of $t \in \mathbb{R}$.

In order to get this estimate, some notations and results will be needed. We fix in X coordinate neighborhoods U_j, diffeomorphisms $\phi_j : U_j \to \phi_j(U_j)$ of class C^M, where $M = 2[n/2]+n+8$, functions $\theta_j \in C_0^M(U_j)$, $\theta_j \geq 0$ and a finite partition of unity $\{\eta_j\}$ of class C^M, such that $\text{supp}(\theta_j) \subset U_i$ whenever $\text{supp}(\theta_j) \cap \text{supp}(\theta_i) \neq \emptyset$; $\theta_i = 1$ in a neighborhood of $\text{supp}(\eta_j)$ if $j = i$ or if $\text{supp}(\eta_j) \cap \text{supp}(\eta_i) \neq \emptyset$.

Now, we define an space of symbols for operators in $M_1(X)$. More exactly, for each j we consider the restriction to $\phi_j(U_j)$ of a function $p^{(j)} \in S^0$. We define a norm of such a restriction as

$$|p^{(j)}| = \sup \frac{|D_x^\alpha D_\xi^\beta D_\xi^\gamma\, p^{(j)}\ (x,\xi)|}{(1 + |\xi|)^{-|\gamma+\beta|}}$$

where the supremum is taken over $x \in \phi_j(U_j)$, $\xi \in R^n$, $|\xi| \leq n+3$,

$|\alpha+\beta| \leq 2[n/2]+4$, j.

We note $\mathbb{H}_1(X)$ this space. With the pointwise multiplication

$$(p^{(j)}).(q^{(j)}) = (p^{(j)}q^{(j)})$$

as a product, $\mathbb{H}_1(X)$ becomes a commutative Banach algebra.

LEMMA. *Let $H = (p^{(j)})$ be an element in $\mathbb{H}_1(X)$; we suppose that each $p^{(j)}$ is a real function. Then, if* $t \in \mathbb{R}$,

$$|\exp(-2\pi itH)| \leq C|(1+|H|)(1+|t|)|^{\mu}$$

where $C = C(X) > 0$, $\mu = 2[n/2]+n+7$.

Proof

Since $\mathbb{H}_1(X)$ is a Banach algebra, the exponential $\exp(-2\pi itH)$ is well defined; moreover it is equal to $\exp(-2\pi itp^{(j)})_j$. According to the norm that the space $\mathbb{H}_1(X)$ has, the conclusion follows.

Now, we will introduce the space $\mathbb{M}_1(X)$ in the following way An element K of $\mathbb{M}_1(X)$ is an operator R in $R_1(X)$ and a vector $(p^{(j)})$ in $\mathbb{H}_1(X)$ subject to the condition that if $U_i \cap U_j \neq \emptyset$ and $\phi_{ij} = \phi_j \circ \phi_i^{-1}$, then

$$(p^{(i)}) = \phi_{ij}^*(p^{(j)}) \quad \text{in} \quad \phi_i(U_i \cap U_j).$$

Such an element K will be denoted as $\{(p^{(j)}),R\}$.

We define a norm in $\mathbb{M}_1(X)$ as follows

$$|K|_{\mathbb{M}_1} = |(p^{(j)})| + |R|_{R_1}$$

Given $K \in \mathbb{M}_1(X)$ we define an operator $\Lambda(K)$ in the following way

$$\Lambda(K) = \sum_j \eta_j \phi_j^*(A_j)\theta_j + R$$

where

$$(A_j f)(x) = \begin{cases} \int e^{-2\pi ix\xi}\, p^{(j)}(x,\xi)\hat{f}(\xi)d\xi & \text{if} \quad x \in \phi_j(U_j) \\ 0 & \text{if not} \end{cases}$$

It can be proved that $\Lambda(K)$ belongs to $M_1(X)$. Moreover the linear map

$$\begin{array}{ccc} \mathbb{M}_1(X) & \xrightarrow{\Lambda} & M_1(X) \\ K & \longrightarrow & \Lambda(K) \end{array}$$

is into and continuous. Furthermore, if $A \in M_1(X)$ is self-adjoint, $A = \Lambda(K)$ for some $K = \{(p^{(j)}), R\}$, with $p^{(j)}$ real for all j.

It is possible to define a product in $\mathbb{M}_1(X)$ in such a way that $\mathbb{M}_1(X)$ becomes a Banach algebra and the map Λ above is a continuous homomorphism of algebras.

Finally, let us consider the maps

$$\begin{array}{ccccc} \mathbb{M}_1(X) & \xrightarrow{\Omega} & \mathbb{N}_1(X) & \xrightarrow{\Omega_1} & \mathbb{M}_1(X) \\ \{(p^{(j)}), R\} & \longrightarrow & (p^{(j)}) & \longrightarrow & \{(p^{(j)}), 0\} \end{array}$$

Ω is a continuous homomorphism of algebras and the linear map Ω_1 is a right continuous inverse of Ω.

<u>THEOREM 2</u>. <u>*Suppose that*</u> $1/p_0 - 1/n \leq 1/2$.

<u>*Let*</u> $H = \{(p^{(j)}), R\}$ <u>*be an element of*</u> $\mathbb{M}_1(X)$ <u>*such that*</u> $\Lambda(H)$ <u>*is a self-adjoint operator and the functions*</u> $p^{(j)}$ <u>*are real for all*</u> j. <u>*Then, if*</u> $t \in \mathbb{R}$,

$$|\exp(-2\pi i t H)|_{\mathbb{M}_1} \leq C[(1 + |H|_{\mathbb{M}_1})(1 + |t|)]^{2\mu+2}$$

<u>*where*</u> $C = C(X) > 0$, $\mu = 2[n/2]+n+7$.

<u>Proof</u>

According to the notations above, we set $A = \Lambda(H)$, $K = \Omega(H) = (p^{(j)}) \in \mathbb{N}_1(X)$.

We assert that

$$e^{-2\pi i t H} - \Omega_1(e^{-2\pi i t K})$$

it an element of the form $\{(0), R(t)\}$.

In fact, since Ω is a continuous homomorphism of algebras and Ω_1 is a right inverse of Ω, we have

$$\Omega[e^{-2\pi i t H} - \Omega_1(e^{-2\pi i t K})] = e^{-2\pi i t K} - \Omega\Omega_1(e^{-2\pi i t K}) = 0.$$

On the other hand, since Ω_1 is a continuous map, according to the lemma it suffices to estimate the norm of $\{(0),R(t)\}$ in $\mathbb{M}_1(X)$, which coincides with the norm of $R(t)$ in $R_1(X)$.

We have

$$R(t) = \Lambda\left[e^{-2\pi itH} - \Omega_1(e^{-2\pi itK})\right] =$$
$$= e^{-2\pi itA} - \Lambda\Omega_1(e^{-2\pi itK}).$$

If we denote with $\dot{R}(t)$ the derivative of $R(t)$ with respect to t, we get

$$\dot{R}(t) = e^{-2\pi itA}(-2\pi iA) - \Lambda\Omega_1(e^{-2\pi itK}(-2\pi iK)) =$$
$$= \left[e^{-2\pi itA} - \Lambda\Omega_1(e^{-2\pi itK})\right](-2\pi iA) + \Lambda\Omega_1(e^{-2\pi itK})(-2\pi iA) -$$
$$- \Lambda\Omega_1(e^{-2\pi itK}(-2\pi iK)) = R(t)(-2\pi iA) + B_1(t) \qquad (1)$$

Since

$$B_1(t) = \Lambda\left[\Omega_1(e^{-2\pi itK})(-2\pi iH) - \Omega_1(e^{-2\pi itK}(-2\pi iK))\right]$$

and

$$\Omega\left[\Omega_1(e^{-2\pi itK})(-2\pi iH) - \Omega_1(e^{-2\pi itK}(-2\pi iK))\right] = 0,$$

we deduce that $B_1(t)$ belongs to $R_1(X)$ for each t.

Thus,

$$|B_1(t)|_{R_1} = |\Omega_1(e^{-2\pi itK})(-2\pi iH) - \Omega_1(e^{-2\pi itK}(-2\pi iK))|_{\mathbb{M}_1}$$
$$\leq C\left[(1 + |H|_{\mathbb{M}_1})(1 + |t|)\right]^{\mu}$$

where $C = C(X) > 0$.

Since $R(0) = 0$, from (1) it follows that

$$R(t) = \int_0^t B_1(s)e^{-2\pi i(t-s)A}\, ds \qquad (2)$$

But we can also write

$$R(t) = (-2\pi iA)e^{-2\pi itA} - \Lambda\Omega_1(e^{-2\pi itK}(-2\pi iK))$$

or

$$R(t) = (-2\pi iA)R(t) + B_2(t)$$

where $B_2(t) \in R_1(X)$ for each t and

$$|B_2(t)|_1 \leq C[(1 + |H|_{M_1})(1 + |t|)]^{\mu} , \quad c = C(X) > 0$$

So,

$$R(t) = \int_0^t e^{-2\pi i(t-s)A} B_2(s)ds \tag{3}$$

or

$$R^*(t) = \int_0^t B_2^*(s)e^{2\pi i(t-s)A} ds \tag{4}$$

where $*$ denotes the adjoint.

Now, suppose we show that

$$|e^{-2\pi i(t-s)A}|_{L^{p_0},L^p} \leq C[(1 + |H|_{M_1})(1 + |t-s|)]^{\mu+1} \tag{5}$$

We will get the same estimate for

$$|e^{-2\pi i(t-s)A}|_{L^{p_0'},L^{p_0'}}$$

Thus, according to (2) and (4), we can deduce that

$$|R(t)|_{R_1} \leq C[(1 + |H|_{M_1})(1 + |t|)]^{2\mu+2}$$

So, it remains to prove (5).

From the definition of the operator $R(t)$, it is clear that it suffices to obtain the estimate

$$|R(t)|_{L^{p_0},L^{p_0}} \leq C[(1 + |H|_{M_1})(1 + |t|)]^{\mu+1}$$

But according to the hypothesis $1/p_0 - 1/n \leq 1/2$, the Sobolev immersion theorem provides the continuous inclusion of $L_1^{p_0}(X)$ into $L^2(X)$; moreover, since $p_0 \leq 2$, we also have a continuous injection from $L^2(X)$ into $L^{p_0}(X)$. Thus, the desired estimate follows from (3).

This completes the proof of the theorem 2.

References

[1] J. Alvarez Alonso, A.P. Calderón: "Functional calculi for pseudo-differential operators, I". Proceedings of the Seminar on Fourier Analysis held in El Escorial, (1979), pp. 1-61.

[2] ________________, "Functional calculi for pseudo-differential operators, II". Proceedings of the MIT Congress in honour of I. Segal, (1979). Studies in Appl. Math., vol 8, (1983), pp. 27-72.

[3] R. Coifman, Y. Meyer: "Au delà des opérateurs pseudo-différentiels". Asterisque nº 57, (1978).

[4] M.E. Taylor: "Functions of several self-adjoint operators". Proc. Amer. Math. Soc. 19, (1968), 91-98.

[5] A.P. Calderon: "Algebras of singular integral operators", Proc. of Symp. in Pure Math., 10, (1965), 18-55.

[6] J. Alvarez Alonso: "An algebra of L^p-bounded pseudo-differential operators". Journal of Math. Analysis and Appl. 94, (1983), 268-282.