TOPOLOGY

TOPOLOGY

Murray Eisenberg
UNIVERSITY OF MASSACHUSETTS, AMHERST

HOLT, RINEHART AND WINSTON, INC.
NEW YORK CHICAGO SAN FRANCISCO ATLANTA DALLAS
MONTREAL TORONTO LONDON SYDNEY

Copyright © 1974 by Holt, Rinehart and Winston, Inc.
All rights reserved.

Eisenberg, Murray, 1939–
Topology.

Bibliography: p. 405
1. Topology.
QA611.E53 514 73-19879
ISBN 0-03-091366-7

4 5 6 7 8 038 9 8 7 6 5 4 3 2 1

Preface

This book introduces the most basic concepts, facts, and techniques of general topolog at a level appropriate to a student's first exposure to the subject It is suitable as a text for a variety of undergraduate courses of differing lengths and emphases, and for classes having varying backgrounds, some possible course outlines are suggested below It may even be used for beginning graduate students who have not previously studied topology as an introduction to one of the standard advanced texts

The only mathematical prerequisite for reading this book is calculus No knowledge of the topology of Euclidean spaces or metric spaces is assumed Some slight prior experience with "epsilontics" is desirable, but not indispensable, the greater the prior experience, the more material can be covered. Neither Zorn's lemma nor ordinal numbers are used

One of our aims has been to assist the student's mathematical maturation. Hence careful attention has been paid to motivating new notions For example, eleven pages of examples of metrics precede the actual definition of a metric, and a proof of the compactness of the unit interval (together with the corollary that continuous functions on it are bounded) precede the definition of compactness There are examples galore of everything Special pains have been taken to explain the significance of theorems and to write enough proofs in enough detail to provide models for the student's own proof making In using a preliminary version of the manuscript the author has found that students can read much of the text themselves with only minimal guidance by the instructor, so that classroom time can be devoted mainly to the exercises and discussion of more difficult points.

Chapter 0 concisely presents the necessary preliminaries on sets, maps, countability, order-completeness of the real numbers, and equivalence relations The time spent on this material will, of course, depend on the student's background, but it is suggested that all students at least read the chapter rapidly for review and to fix terminology It is a good idea to review the section on equivalence relations in conjunction with the study of quotient spaces in Chapter 3.

Chapter 1 introduces open sets, closed sets, neighborhoods, continuous maps, and convergent sequences in metric spaces The terminology used here—d-closed set, (d,d')-continuous maps, and so on—calls attention to the particular metrics involved. At the same time the entire thrust of the chapter is to justify the later definition of a topological space by demonstrating that notions of continuity and convergence remain unchanged when the metrics are replaced by equivalent ones. The section on completeness includes the Baire category theorem and its application to the existence of nowhere differentiable, continuous functions Owing to its greater technical difficulty as well as its treatment of uniform, as distinct from topological, ideas, this section may be postponed or even omitted, except for occasional mention, completeness does not appear again until Section 2 of Chapter 4, where it is used to characterize compact metric spaces and to prove the Tychonoff theorem for a sequence of compact metric spaces

The study of topology proper is begun in Chapter 2, where topologies, neighborhoods, Hausdorff spaces, bases and local bases, and countability properties are discussed. Of the whole hierarchy of separation properties, which in its entirety is liable to confuse the beginner, only the property T_2 is dealt with at length in the body of the text, the others being relegated to the exercises.

Continuity is the theme of Chapter 3 Here product and quotient spaces are constructed and their mapping properties are emphasized Here, too, the theory of convergence of nets is introduced at a level kept elementary by treating subnets only briefly and by avoiding universal nets entirely This theory deserves to be included in a first topology course It places sequential convergence in proper perspective, facilitates a later study of filter convergence, and reveals the diverse kinds of limits the student has previously encountered as instances of a single unifying concept. Nevertheless, the treatment of nets can be omitted without substantial loss the only places nets are used again are the net characterization of compactness (4 27), which can itself be omitted, and the sequence characterization of compactness of a metric space (4.36), which requires only the equivalence of sequential clustering with subsequential convergence (3.69)

The elementary facts about compactness are developed in Chapter 4. Because we avoid Zorn's lemma, the Tychonoff theorem is proved only for

此为试读,需要完整PDF请访问: www.ertongbook.com

the product of finitely many spaces and separately for the product of a sequence of metrizable spaces. Sequential compactness and other variants of compactness are not studied in their own right, only as equivalents of compactness in the metrizable case. Also considered for metric spaces is the relationship of compactness to uniform continuity and to completeness. The discussion of locally compact spaces includes the one-point compactification.

The first three sections of Chapter 5 present the standard facts about connected sets, components, locally connected spaces, and path-connected spaces. This material, which is technically if not conceptually simpler than that on compactness, can be read before Chapter 4. The final two sections, on homotopy, culminate in proofs of the Brouwer fixed-point theorem in dimension 2 and the fundamental theorem of algebra. The only thing about compactness needed in these two sections is the existence of a Lebesgue number (4.41) for an open cover of the unit interval or of the unit square. In order to keep the algebraic machinery to a minimum and make things geometrically more transparent, our treatment of homotopy avoids explicit mention of the fundamental group (except in the exercises).

The exercises, found at the end of each section, number 583 in all. Ranging in difficulty from the routine to the challenging, they are meant both to test comprehension of the ideas presented in the text and to provide applications, additional examples, and extensions of these ideas. Many call not just for proofs, but for answers to such questions as "Is it true that . . . ?" or "What can be said about . . . ?" or "Is there an analog . . . ?" Included in the exercises are a number of topics this author did not deem so essential to a first course in topology to have included them in the text proper, but which are interesting and important in their own right. These topics are as diverse as completion of a metric space, T_0 and T_1-spaces, Cartesian sum topologies, manifolds with boundary, topological groups, the closed-graph theorem, cut points, and the fundamental group.

There are surely more exercises than an instructor would want to assign to any one class. We have therefore appended a Guide to the Exercises in which we cite each exercise needed for exercises in subsequent sections.

All definitions, theorems, and examples within a single chapter are numbered consecutively, so that 3.15 refers to the fifteenth item in Chapter 3. The exercises in each chapter are separately numbered consecutively; a reference to the fifth exercise of Chapter 3 would be "Exercise 5" if made within that chapter, but "Exercise 3.5" if made in another chapter.

The Bibliography includes only those books and articles referred to in the text or suggested for further reading. References to bibliographic entries are made by numbers enclosed in brackets. Appended to the Bibliography is a list of suggested readings on special topics about which individual students might report to the class.

Suggested course outlines The list below is not meant to be exhaustive, but only suggestive of possible courses that can be based on this text. The portion of the text covered by any given class will, of course, depend on the students' preparation; it will also depend heavily on the number and difficulty of problems assigned from among the many we have provided.

A minimal course (1 quarter or 1 semester)

Chapter 0	Sections 1 through 5
Chapter 1	Sections 1 through 4
Chapter 2	omit 2.41, 2.42, and 2.56(5) and (6)
Chapter 3	Section 1 except 3.11(2); Section 2 except 3.22(3) and 3.23; Section 3 through 3.40, except 3.35(5); Section 6 of Chapter 0; Section 4 except 3.49(7) through (9); Section 5 through 3.54
Chapter 4	Section 1 except 4.27
Chapter 5	Section 1 except 5.25 and 5.26; Section 2 through 5.33 or 5.35(4); Section 3 through 5.51—optional

A second course in topology (1 quarter or 1 semester)

Chapter 1	Section 5
Chapter 2	2.41 and 2.42
Chapter 3	Section 3 from 3.41; Sections 4 and/or 5
Chapter 4	4.27 (if Section 5 of Chapter 3 is included); Sections 2 and 3, or 4.41 through 4.44 and Section 3 through 4.56
Chapter 5	Section 2 from 5.34; Sections 3 through 5

Additional readings or individual projects (see the Bibliography)

A complete course (2 semesters or 3 quarters)

Chapters 0 through 5
Additional readings or individual projects

A standard course—emphasis on geometry
(1 semester or 2 quarters)

Chapter 0	Sections 1 through 5
Chapter 1	Sections 1 through 4
Chapter 2	omit 2.41, 2.42, and 2.56(5) and (6)
Chapter 3	Section 1 except 3.11(2); Section 2 except 3.23; Sections 3 through 3.40; Section 6 of Chapter 0; Section 4; Section 5 through 3.53
Chapter 4	Section 1; Theorem 4.41
Chapter 5	omit 5.35(5); include Exercises 5.97, 5.98, 5.107

A standard course—emphasis on analysis
(1 semester or 2 quarters)

Chapter 0	Sections 1 through 5
Chapter 1	include Exercises 1.85 and 1.86
Chapter 2	
Chapter 3	Sections 1 through 3; Section 6 of Chapter 0; Section 4 except 3.49(7) through (9); Section 5
Chapter 4	
Chapter 5	Section 1; Section 2 through 5.33; Section 3 through 5.51—optional

A brief course in set theory (3 to 5 weeks)

Chapter 0

A short course on metric spaces (8 weeks)

Chapter 0	Sections 1 through 5
Chapter 1	include Exercises 1.13, 1.14, 1.68, 1.85 through 1.89

A course in special topics (variable time)

Chapter 1	1.71 through 1.73; 1.69, 1.70, and Exercises 1.85 and 1.86
Chapter 3	3.23 (include Example 1.9 and Exercise 1.23)
Chapter 4	4.28 and 4.29; 4.41
Chapter 5	Examples 5.35(5) and/or 5.52; Sections 3 through 5

Acknowledgments My students at the University of Massachusetts spotted numerous errors and detected defects in exposition in a preliminary version of this text. Professor Victor Klee and several anonymous reviewers corrected some infelicities of style and suggested many improvements. Miss Margo Vidrine and Mrs. Rita Warner speedily and accurately typed the manuscript. To all these people I express my gratitude for making this book possible.

MURRAY EISENBERG
Amherst, Massachusetts

Contents

CHAPTER

0

Sets and Maps

In this preliminary chapter we collect some essential facts about sets and maps used throughout the text. Much of this material will doubtless not be new to the reader and is therefore covered rapidly, with few examples or proofs, to remind him of what he already knows and to fix the particular terminology and notation adopted here. Those topics that are likely to be less familiar—countability, order-completeness of the real numbers, and equivalence relations—are covered in somewhat greater detail. For a fuller treatment of these preliminaries at an elementary level see Fairchild and Ionescu Tulcea [10] or Foulis [12]; for an advanced, axiomatic treatment see Eisenberg [9].

1. SETS

Two logical connectives will be used frequently:

$\Rightarrow$ means *implies* or *if . . . then,*

$\Leftrightarrow$ means *if and only if.*

A *set* is a collection of mathematical objects. If x is one of the objects comprising a set X, we write

$$x \in X$$

and say that x is an *element, member,* or *point of* X and that x *belongs to* X; in the contrary case we write

$$x \notin X.$$

此为试读，需要完整PDF请访问：www.ertongbook.com

Two sets X and Y are *equal* to one another, in symbols

$$X = Y,$$

precisely when they have the same elements, that is, when

$$x \in X \iff x \in Y.$$

When it is not the case that $X = Y$, we write

$$X \neq Y.$$

Similar use of a slash mark to negate a statement will be made in the future without further explanation.

Two notational devices are used to specify particular sets. The first simply lists or indicates the elements of the set between braces. For example,

$$\{-1, 1\}$$

is the set having the two elements -1 and 1, and

$$\{2, 4, 6, \ldots\}$$

is the set of all even positive integers (since the latter set is infinite, its elements cannot all be listed explicitly, but their identity is supposed to be implicit in the few actually listed in conjunction with the context).

The second device uses the notation

$$\{x \mid P\}$$

to specify the set consisting of those objects x having a given property P. For example, if R denotes the set of all real numbers, then

$$\{x \mid x \in \mathsf{R},\, x^2 = 1\} = \{-1, 1\};$$

this set, which consists of those elements of the set R satisfying a certain condition, may also be specified by the modified notation

$$\{x \in \mathsf{R} \mid x^2 = 1\}.$$

Because the vertical bar $|$ is also used as part of other notations (for example, for absolute value), a colon is sometimes used in place of the vertical bar in $\{x \mid P\}$ to avoid confusion. Thus

$$\{x \colon x \in \mathsf{R},\, |x| = 1\} = \{-1, 1\} = \{x \in \mathsf{R} \colon |x| = 1\}.$$

0.1 Special sets. If x is an object, then the *singleton*

$$\{x\}$$

is the set having the lone member x; the set $\{x\}$ is just as different from the object x as a caged lion is from a loose lion. The *empty set* is the set $\varnothing$ having no members at all. Thus

$$\{x \mid x \neq x\} = \varnothing = \{x \in \mathsf{R} \mid x < x\}.$$

Any other set is *nonempty.*

Some sets of numbers for which we reserve special notation are.

N = the set of all natural numbers = $\{0, 1, 2, \ldots\}$
Z = the set of all integers = $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$
Q = the set of all rational numbers
R = the set of all real numbers
C = the set of all complex numbers
I = $\{x \in \mathsf{R} \mid 0 \leq x \leq 1\}$

0.2 Subsets. We say that a set X is *contained in* a set Y, call X a *subset of* Y, and write

$$X \subset Y$$

to mean each element of X is an element of Y, that is,

$$x \in X \Rightarrow x \in Y.$$

We also write

$$Y \supset X$$

to mean the same thing and then say that Y *contains* X. For example,

$$\mathsf{N} \subset \mathsf{Z} \subset \mathsf{Q} \subset \mathsf{R} \subset \mathsf{C}, \qquad \mathsf{R} \supset \mathsf{I},$$

but

$$\mathsf{I} \not\subset \mathsf{Q}.$$

If x is an object, then

$$\{x\} \subset X \Leftrightarrow x \in X.$$

The empty set is a subset of every set X:

$$\varnothing \subset X.$$

(*Proof:* Since $\varnothing$ has no elements at all, it does not have any element that fails to be an element of X.)

Evidently

$$X = Y \Leftrightarrow X \subset Y \text{ and } Y \subset X.$$

Thus the inclusion $X \subset Y$ does not preclude the possibility that $X = Y$. When $X \subset Y$ but $X \neq Y$, we call X a *proper* subset of Y.

The language of subsets may be used to state the *principle of mathematical induction:*

> Let $E \subset \mathsf{N}$. Suppose $0 \in E$ and suppose $n + 1 \in E$ whenever $n \in E$ Then $E = \mathsf{N}$.

This principle, which we accept as a fundamental property of the natural numbers, is the basis for "proof by induction".

To illustrate proof by induction, let us prove

$$(*) \qquad 2^n > n \qquad (n \in \mathbf{N}).$$

First, $2^0 = 1 > 0$. Next, suppose $n \in \mathbf{N}$ and

$$2^n > n.$$

If $n > 0$, then

$$\begin{aligned} 2^{n+1} &= 2 \cdot 2^n \\ &> 2 \cdot n \qquad \text{(by the assumption } 2^n > n) \\ &= n + n \geq n + 1; \end{aligned}$$

if $n = 0$, then $2^{n+1} = 2 > 1 = n + 1$. Thus $2^{n+1} > n + 1$ whenever $2^n > n$. This proves (*), for if we let

$$E = \{n \in \mathbf{N} \mid 2^n > n\},$$

then $E \subset \mathbf{N}$, and we have shown that $0 \in E$ and that $n + 1 \in E$ whenever $n \in E$; hence from the principle of mathematical induction we can conclude that $E = \mathbf{N}$.

The *power set* of a given set X is the collection

$$\mathcal{P}(X) = \{A \mid A \subset X\}$$

consisting of all the subsets of X. For example,

$$\mathcal{P}(\{0, 1\}) = \{\varnothing, \{0\}, \{1\}, \{0, 1\}\}.$$

In general,

$$\varnothing \in \mathcal{P}(X), \qquad X \in \mathcal{P}(X)$$

for any set X.

0.3 Union and intersection of two sets. Let A and B be sets. The *union of A and B* is the set

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

of all those objects that belong to at least one of the sets A and B. The *intersection of A and B* is the set

$$A \cap B = \{x \mid x \in A, x \in B\}$$

of all those objects that belong to both of the sets A and B. The set A is *disjoint from B* when

$$A \cap B = \varnothing,$$

that is, when A and B have no elements in common; A *intersects* B in the contrary case.

Some handy formulas concerning union and intersection are:

$$A \cap B \subset A \subset A \cup B$$

$$A \cup A = A = A \cap A$$

$$A \cup \varnothing = A \qquad A \cap \varnothing = \varnothing$$

$$A \cup B = B \cup A \qquad A \cap B = B \cap A$$

$$A \cup (B \cup C) = (A \cup B) \cup C \qquad A \cap (B \cap C) = (A \cap B) \cap C$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \qquad A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup B = B \iff A \subset B \iff A \cap B = A$$

0.4 Complements. For sets A and X, the *complement of* A *in* X is the set

$$X \backslash A = \{x \in X \mid x \notin A\}$$

of those elements of X that do not belong to A. For any sets A and X:

$$X \backslash \varnothing = X \qquad X \backslash X = \varnothing$$

$$A \cup (X \backslash A) = X \qquad A \cap (X \backslash A) = \varnothing$$

$$X \backslash (X \backslash A) = A$$

If A and B are subsets of X, then

$$A \subset B \iff X \backslash B \subset X \backslash A.$$

For any sets X, A, and B,

$$X \backslash (A \cup B) = (X \backslash A) \cap (X \backslash B),$$

$$X \backslash (A \cap B) = (X \backslash A) \cup (X \backslash B).$$

These two *De Morgan's laws* will be generalized in Section 3.

0.5 Ordered pairs and products. The *ordered pair* (x, y) formed from objects x and y is a new object in which x is the *first coordinate* and y is the *second coordinate*. Equality of ordered pairs is governed by the rule

$$(x, y) = (a, b) \iff x = a \text{ and } y = b.$$

[It is interesting, but unnecessary for our needs, to know that (x, y) may be defined as $\{\{x\}, \{x, y\}\}$; then the preceding rule may be deduced from this definition.] Observe that (x, y) is not the same thing as $\{x, y\}$: although

$$\{x, y\} = \{y, x\}$$

for any x and y,

$$x \neq y \implies (x, y) \neq (y, x).$$

The *product of* two sets X and Y is the set

$$X \times Y = \{(x, y) \mid x \in X, y \in Y\}$$

of all ordered pairs whose first coordinate belongs to X and whose second coordinate belongs to Y. Clearly

$$X \times Y \neq \varnothing \quad \Leftrightarrow \quad X \neq \varnothing \quad \text{and} \quad Y \neq \varnothing.$$

A subset R of $X \times Y$ is called a *relation in* X *to* Y; for $x \in X$ and $y \in Y$ we write

$$xRy$$

to mean

$$(x, y) \in R$$

and interpret this statement to say that R "relates y to x". For example, the *usual ordering of* R is the relation

$$R = \{(x, y) \in \mathsf{R} \times \mathsf{R} \mid x \leq y\}$$

which satisfies

$$xRy \quad \Leftrightarrow \quad x \leq y \qquad (x, y \in \mathsf{R}).$$

(Here the parenthetical expression on the right means "for every $x \in \mathsf{R}$ and for every $y \in \mathsf{R}$" and qualifies the statement to its left.)

EXERCISES

1. (**a**) Do there exist two sets each of which is a proper subset of the other?
(**b**) If X is a proper subset of Y and Y is a proper subset of Z, must X be a proper subset of Z?

2. Given a real number $\epsilon > 0$, find a real number $\delta > 0$ such that

$$\{x: |x - 1| < \delta\} \subset \{x: |(3x - 1) - 2| < \epsilon\}.$$

3. Let

$$A = \{\varnothing\}, \qquad B = \{\varnothing, A\}, \qquad C = \{\varnothing, A, B\}.$$

(**a**) Compute the union and intersection of each pair of these three sets.
(**b**) Compute $\mathcal{P}(A)$, $\mathcal{P}(B)$, and $\mathcal{P}(C)$.
(**c**) Considering A, B, and C together with all the sets you computed in (a) and (b), determine which are elements of others, which are subsets of others, and which are equal to others.

4. Let

$$A = \{x \in \mathsf{R} \mid x^2 < 2\}, \qquad B = \{x \in \mathsf{R} \mid x^2 = 2\}.$$

(**a**) Compute and draw pictures of the sets $A \cup B$, $A \cap B$, $\mathsf{R} \setminus A$, and $\mathsf{R} \setminus B$.
(**b**) Compute $B \cap \mathbf{Q}$ and $A \cap \mathbf{Z}$.

5. Establish the *absorption laws*

$$(A \cup B) \cap B = B, \qquad (A \cap B) \cup B = B.$$

6. Exhibit subsets A and B of R for which:

(**a**) $\mathsf{R}\setminus(A \cup B) \neq (\mathsf{R}\setminus A) \cup (\mathsf{R}\setminus B)$.

(**b**) $\mathsf{R}\setminus(A \cap B) \neq (\mathsf{R}\setminus A) \cap (\mathsf{R}\setminus B)$.

7. Let A and B be subsets of a set X. Prove:

(**a**) $A = B \quad \Leftrightarrow \quad X\setminus A = X\setminus B$.

(**b**) $A \subset B \quad \Leftrightarrow \quad A \cap (X\setminus B) = \varnothing$.

8. (**a**) Express the intersection $(A \times B) \cap (C \times D)$ of products as the product of two sets.

(**b**) Show by example that a union $(A \times B) \cup (C \times D)$ of products is not necessarily a product of two sets.

9. For which sets X and Y does $X \times Y = Y \times X$?

10. Determine all relations in $\{0, 1\}$ to $\{0, 1\}$.

11. Let $X = \{x \mid x \notin x\}$. Is $X \in X$? If not, is $X \notin X$?

2. MAPS

A *map* (or *function*)

$$f: X \to Y$$

from (or *on*) X *to* (or *into*) Y consists of sets X and Y together with a rule f which assigns to each $x \in X$ a unique element $f(x) \in Y$ called the *value of f at x*. The set X is the *domain*, the set Y is the *codomain*, and the rule f is the *graph* of the map. [It is unnecessary for our purposes to know that such a "rule" is actually a relation $f \subset X \times Y$ such that for each $x \in X$ there is exactly one $y \in Y$ with $(x, y) \in f$, and then $y = f(x)$.]

Let $f: X \to Y$ be a map. If $x \in X$ and $y = f(x)$, we write

$$x \mapsto y$$

and say that f *sends* or *maps x to y*. When $f(x)$ is specified by a single formula involving x for arbitrary $x \in X$, we write

$$f: X \to Y$$
$$x \mapsto f(x).$$

For example,

$$f: \mathsf{R} \to \mathsf{R}$$
$$x \mapsto x^2$$

is the map from R to R such that

$$f(x) = x^2 \qquad (x \in \mathsf{R}).$$

Sometimes several formulas specifying $f(x)$ are needed for various parts of the domain to which x might belong. For example, if $A \subset X$, then the

此为试读，需要完整PDF请访问：www.ertongbook.com