Encyclopaedia of Mathematical Sciences

Volume 26

S.M. Nikol'skiĭ (Ed.)

Analysis III

Springer-Verlag

世界图书出版公司

S.M. Nikol'skiĭ (Ed.)

Analysis III

Spaces of Differentiable Functions

With 22 Figures

Springer-Verlag
Berlin Heidelberg New York
London Paris Tokyo
Hong Kong Barcelona
世界图书出版公司
北京·广州·上海·西安

Title of the Russian edition:
Itogi nauki i tekhniki, Sovremennye problemy matematiki,
Fundamental'nye napravleniya, Vol. 26, Analiz 3
Publisher VINITI, Moscow 1988

Reprinted by World Publishing Corporation, Beijing, 1993
for distribution and sale in The People's Republic of China only
ISBN 7-5062-1429-6

Mathematics Subject Classification (1980): 46Exx

ISBN 3-540-51866-5 Springer-Verlag Berlin Heidelberg New York
ISBN 0-387-51866-5 Springer-Verlag New York Berlin Heidelberg

Library of Congress Cataloging-in-Publication Data
Analysis integral representations and asymptotic methods.
(Encyclopaedia of mathematical sciences; v. 13)
Translation of Analiz, issued as part of the serial: Itogi nauki i tekhniki.
Seriia Sovremennye problemy matematiki. Fundamental'nye napravleniia.
Includes bibliographical references.
1 Operational calculus 2 Integral representations.
3 Integral transforms 4 Asymptotic expansions.
I. Gamkrelidze, R V II Series III Series Encyclopaedia of mathematical sciences; v. 13, etc.
QA432 A6213 1991 515'.72 89-6163
ISBN 0-387-17008-1 (U.S.: v. 1: alk. paper)

List of Editors, Contributors and Translators

Editor-in-Chief

R.V. Gamkrelidze, Academy of Sciences of the USSR, Steklov Mathematical Institute, ul. Vavilova 42, 117966 Moscow, Institute for Scientific Information (VINITI), Baltiiskaya ul. 14, 125219 Moscow, USSR

Consulting Editor

S.M. Nikol'skiĭ, Steklov Mathematical Institute, ul. Vavilova 42, 117333 Moscow, USSR

Contributors

L.D. Kudryavtsev, Steklov Mathematical Institute, ul. Vavilova 42, 117333 Moscow, USSR

V.G. Maz'ya, Institute for Research on Machines, Leningrad Department of the Academy of Sciences of the USSR, Bolshoĭ prospekt V.O. 61, 199179 Leningrad, USSR

S.M. Nikol'skiĭ, Steklov Mathematical Institute, ul. Vavilova 42, 117333 Moscow, USSR

Translator

J. Peetre, University of Stockholm, Department of Mathematics, Box 6701, S-113 85 Stockholm, Sweden

Contents

Introduction

In the Part at hand the authors undertake to give a presentation of the historical development of the theory of imbedding of function spaces, of the internal as well as the externals motives which have stimulated it, and of the current state of art in the field, in particular, what regards the methods employed today.

The impossibility to cover all the enormous material connected with these questions inevitably forced on us the necessity to restrict ourselves to a limited circle of ideas which are both fundamental and of principal interest. Of course, such a choice had to some extent have a subjective character, being in the first place dictated by the personal interests of the authors. Thus, the Part does not constitute a survey of all contemporary questions in the theory of imbedding of function spaces. Therefore also the bibliographical references given do not pretend to be exhaustive; we only list works mentioned in the text, and a more complete bibliography can be found in appropriate other monographs.

O.V. Besov, V.I. Burenkov, P.I. Lizorkin and V.G. Maz'ya have graciously read the Part in manuscript form. All their critical remarks, for which the authors hereby express their sincere thanks, were taken account of in the final editing of the manuscript.

Chapter 1

Function Spaces

§ 1. The Concept of Space

Around the turn of the century mathematicians formulated the notion of *function space*. By this is meant a set of functions of similar type in a suitable sense, equipped with additional structure: algebraic, topological or metric. Already in the 17th and 18th centuries, in the very period of birth of mathematical analysis and the creation of its methods, much use was made of the linearity of the operations of differentiation and integration. In the sequel it was realized that many important classes of functions, analytic, differentiable or integrable, in some sense or other, enjoy the property of being closed under linear operations: if we take any pair of functions of the class in question then any linear combination of them belongs to the same class. In the second half of the 19th century, after one had rigorously defined the concepts of continuity and continuous differentiability and begun to use these notions more widely, one also began to consider classes of continuous, respectively continuously differentiable functions. In this way ground was prepared for the appearance of the notion of linear (vector) space or, as one often says, linear manifold. However, in the beginning it was not necessary to consider such classes as a kind of spaces, that is, to consider them as in a certain sense homogeneous sets, that is, sets consisting of elements all regarded as equivalent, otherwise put, to substitute points for functions, because in this epoch the individual functions were the central objects of study. But already isolated results of this period, established essentially for function classes, (as, for instance, the Cauchy-Bunyakovskiĭ(-Schwarz) inequality (cf. §6 of Chap. 1) pointed to an analogy between the class of square integrable functions and the geometry of an infinite dimensional vector space) inexorably foreshadowed a time when the notion of function space unfolded itself in a natural way.

An important step in the development of function theory, accelerating the clarification of this notion, was the proof of various limit theorems, asserting the validity of a property of the limit function if this property holds true for all the functions whose limit we consider. Thus one obtained classes which are closed not only under linear combinations but also under a passage to the limit. As an example of such a set we quote the family of continous functions under passage to the limit in the sense of uniform convergence. From this one might infer that only a small step remained for the introduction of the notion of function space. The difficulty was connected with the fact that in the solution of many problems with the aid of a suitable passage to the limit within a given class (for instance, the search of an exact solution of a differential equation as a limit of an approximate solution) the required smoothness properties are not always preserved. For instance, one cannot find a continuously differentiable curve of minimal length passing through three given points, not all three lying on a line; the minimal curve in the case at hand is a two linked broken line consisting of two links passing through the given points. In view of this, in the process of solving a problem one can sometimes recover the function from the set over which the limit is taken but in other cases one has to modify the sense in which the limit is taken (in such a case one uses essentially methods which regularize illposed problems). All these circumstances hindered the clarification of the notion of function space in mathematics. Before one could arrive at the idea of the soundness of such a concept, one had to realize the value and the sense of various modes of transition to the limit in the function classes under consideration. Thus, for instance, uniform convergence or convergence in an integral metric lead subsequently to the notion of metric and, in particular, normed space, while pointwise convergence required the more general notion of topological space.

As an example we may refer to Riemann's solution of the first boundary problem with the aid of the so-called *Dirichlet's principle*, which amounts to minimizing the energy integral, which at the case at hand is the *Dirichlet integral*, i.e. the integral of the squares of all first order partial derivatives extended over the domain. Riemann started out with a pointwise converging sequence of piecewise differentiable functions, the limit of which, generally speaking, is not a piecewise differentiable, which is required in the statement of the problem. In order to maintain the required properties of the limit function Riemann tried to use a kind of regularization process but he never succeeded to carry out this idea to the end. Subsequently, passing to an integral metric, Hilbert gave a foundation of Dirichlet's principle in the assumption of the existence of at least one admissible function, i.e. a function which takes given boundary values and has a finite Dirichlet integral in the domain.

Similar circumstances made mathematicians realize that the problems of mathematical physics may be illposed in some spaces (for instance, in function spaces with pointwise or uniform convergence) and wellposed in others (for instance, in spaces with an appropriate integral metric).

Essentially the first function space, to occupy a more permanent position in mathematics, was the space of square integrable functions. The notion of this space arose in the first place in the work of Hilbert and Schmidt, and later the study of its properties was pursued in investigations by Fischer, F. Riesz (let us for instance recall the famous Riesz-Fischer theorem on the completeness of the space of square integrable functions), J. v. Neumann and others. We underline that to great extent the completion of the general theory of the space of square integrable functions became possible only after the introduction of the notion of Lebesgue integral.

In the 30's of this century one began a systematic study of function spaces, both in their own right (more exactly in connection with the internal needs of function theory) and in connection with the demands of the theory of differential equations and of probability theory. Before we pass to an account of the current state of the theory of function spaces let us briefly recall the basic types of abstract spaces (i.e. spaces consisting of elements of an arbitrary nature) to which belong the function spaces that one encounters most frequently in modern mathematics. More specifically, we will recall the notion of linear space, of topological space, of metric space, of normed space, of inner product space. As for examples of such spaces we restrict our attention to function spaces only.

§ 2. Linear Spaces

A set $X = \{x, y, z, \dots\}$ is called a *real (complex) linear space* or a *vector space* over the field of real (complex) numbers if

1) it is an Abelian group (writing the group operation additively);

2) to each element $x \in X$ and each real (complex) number λ there corresponds a unique element of X, called the product of λ and x and denoted by λx, assuming that

a) $1x = x \quad \forall x \in X$;
b) $\lambda(\mu x) = (\lambda\mu)x \quad \forall x \in X, \forall \lambda \in \mathbb{R}, \forall \mu \in \mathbb{R}$ (respectively $\forall \lambda \in \mathbb{C}, \forall \mu \in \mathbb{C}$);
c) $\lambda(x + y) = \lambda x + \lambda y \quad \forall x \in X, \forall y \in X, \forall \lambda \in \mathbb{R}$ (respectively $\forall \lambda \in \mathbb{C}$).

This definition extends in a natural way to the definition of linear space over an arbitrary field.

A subset Y of a linear space X is a *subspace* of X if for any elements $x \in Y, y \in Y$ and any numbers λ and μ (respectively real or complex) the element $\lambda x + \mu y$ also belongs to Y. It is clear that a subspace of a linear space itself is a linear space.

If we have in a linear space X a finite number of vectors such that each vector in X is a linear combination of these vectors, we say that *the space X is finite dimensional*; the minimal number of vectors whose linear combinations give the entire space is called the *dimension* of the space. If X does not contain any finite system of vectors with this property then we say that X is *infinite dimensional.*

If E is any set, then the family $\mathbb{R}(E)$ (respectively $\mathbb{C}(E)$) of real- (complex-) valued functions on E forms a linear space. Then set $B(E)$ of all bounded (on E) functions $f \in \mathbb{R}(E)$ ($f \in \mathbb{C}(E)$) is a subspace of the linear space $\mathbb{R}(E)$ (respectively $\mathbb{C}(E)$). If E is infinite then the spaces $\mathbb{R}(E), \mathbb{C}(E), B(E)$ (of real- or complex-valued functions) are infinite dimensional.

Let $(E, \mathscr{F}, \mu)$ be a space E with a σ-algebra $\mathscr{F}$ of measurable subsets and a σ-finite measure μ. (Here and in the sequel we always assume that the measure is *complete*, i.e. if $E_1 \in \mathscr{F}, \mu(E) = 0$ and $E_2 \subset E_1$ then $\mu(E_2) = 0$.) Then the set $M(E, \mu)$ of all measurable functions, the set $BM(E, \mu)$ of all bounded measurable functions, the set $L_\infty(E, \mu)$ of all *essentially bounded* functions on E, i.e. functions f such that

$$\operatorname*{vrai\,sup}_{x \in E} |f(x)| = \min\{y : \mu\{x : |f(x)| > y\} = 0\} =$$

$$= \sup\{y : \mu\{x : |f(x)| > y\} > 0\} = \inf_{\{X : \mu X = 0\}} \sup_{x \in E \setminus X} |f(x)| < +\infty,$$

the set $L_p(E, \mu)$ of all measurable functions whose p-th powers, $1 \leq p < +\infty$, are integrable over E are all linear subspaces of $\mathbb{R}(E)$ (respectively $\mathbb{C}(E)$). The spaces $L_p(E, \mu)$, $1 \leq p \leq +\infty$, are called *Lebesgue spaces*. If there is no ambiguity about which measure we have in mind, for instance, if E is a measurable subset with respect to n-dimensional Lebesgue measure of the n-dimensional space and μ denotes this measure then we usually write $M(E), BM(E), L_p(E)$ instead of $M(E, \mu), BM(E, \mu), L_p(E, \mu)$. In place of $L_1(E)$ we often write simply $L(E)$.

A function $f : X \to \mathbb{R}$ (or $f : X \to \mathbb{C}$), given on a Lebesgue measurable subset X of $\mathbb{R}^n$, is said to belong locally to the space L_p and we write then $f \in L_p(X, \text{loc})$ if for each compact subset $E \subset X$ the restriction of f to E belongs to the Lebesgue space $L_p(E)$, $1 \leq p \leq +\infty$. A function f belonging to $L_1(X, \text{loc})$ is also termed *locally integrable* on X.

The set of all functions in $L_p(X, \text{loc})$ is a linear space.

If E is a topological space (cf. § 3) then the set $C(E)$ of all continuous functions on E and likewise the set $BC(E)$ of all bounded and continuous functions on E are linear subspaces of $\mathbb{R}(E)$ ($\mathbb{C}(E)$).

Finally, if E is a domain in n-dimensional Euclidean space, then the set $\mathscr{D}^{(l)}(E)$ of all l times differentiable functions on E, the set $C\mathscr{D}^{(l)}(E)$ of all l times continuously differentiable functions on E, the set $W_p^{(l)}(E)$ of l times differentiable functions in a generalized sense whose partial derivatives (generalized) of orders up to l ($l \geq 0$) belong to $L_p(E)$ all are subspaces of $\mathbb{R}(E)$ ($\mathbb{C}(E)$); it is clear that $W_p^{(0)}(E) = L_p(E)$. In the following sections we will encounter also other examples of linear function spaces.

In the sequel we will only deal with numerical functions, generally speaking, complex valued. All sets of functions to be considered below will always be assumed to consist either of functions, defined by suitable properties, which take only real values (i.e. they are subsets of $\mathbb{R}(E)$), or else they may take arbitrary complex values (i.e. we have a subset of $\mathbb{C}(E)$). When the notions considered make sense in the first as well as in the second case (for example, boundedness, measurability, integrability, continuity, differentiability etc.), we will speak only of functions, with the tacit understanding that we in reality can only deal with each of the two cases at a time.

If the set E, on which the functions are given, is a segment of the real axis, for example the open interval (a, b) or the closed interval $[a, b]$, then we write $\mathbb{R}[a, b], \mathbb{C}(a, b)$ etc. in place of $\mathbb{R}([a, b]), \mathbb{C}((a, b))$ etc.

§ 3. Topological Spaces

A set X consisting of any kind of elements is called a *topological space* if on it is given a system $\Omega = \{G\}$ of subsets satisfying the following conditions:

1°. The intersection of finitely many sets in Ω belongs to the same system;

2°. The union of arbitrary many sets in Ω is in the system;

3°. $X \in \Omega, \emptyset \in \Omega$.

The sets in Ω will be called *open* sets of the topological space and the system Ω itself the *topology of the space* X.

For each point $x \in X$, any set $G \in \Omega$ such that $x \in G$, i.e. any open se containing x, will be termed a *neighborhood* of this *point*.

If any two points of a topological space have nonintersecting neighborhoods, then the space is said to be *Hausdorff*.

Every metric space is an example of a Hausdorff topological space (cf. § 4 of Chap. 1).

A subsystem B of Ω of open subsets of a topological space X is called a *base for the topology* of the space, if each nonempty open set is a union of a suitable family in B.

A system $B(x)$ of neighborhoods of a point x of a topological space X is called a *local base for the topology* at the point, if for each neighborhood U of x in X there exists a neighborhood $V \in B(x)$ such that $V \subset U$.

The union of local bases of the topology at all points forms a base of the topology of the entire space, as every nonempty open set can be written as a union of neighborhoods of its points contained in it, where the neighborhoods in question can be taken from the given local basis of the topology. By the same token, one can define the topology of a set by first writing down the bases of the topology at all of its points.

In terms of topological spaces one can describe the *pointwise convergence of sequences of functions*. Let X be a family of functions defined on some set E and let $f_0 \in X$. A local basis $B(f_0)$ of the topology at the point f_0 is defined by the system of all possible sets of the form

$$U(f_0) = \{f \in X : |f(a_i) - f_0(a_i)| < \varepsilon, i = 1, 2, \dots, k\},$$

where $a_i \in E$, $i = 1, 2, \dots, k$, is any finite family of points in E and $\varepsilon > 0$ an arbitrary fixed number. If now $f_n \in X$, $n = 1, 2, \dots$, and for each point $a \in E$ holds the identity $\lim_{n\to\infty} f_n(a) = f_0(a)$ then this is equivalent to the statement that for each neighborhood $U(f_0)$ of the function f_0 there exists a number n_0 such that for each index $n > n_0$ there holds the inclusion $f_n \in U(f_0)$.

§ 4. Metric Spaces

A set X is said to be a *metric space* if on the set of order pairs (x, y) in X there is given a nonnegative function, called the *distance* or the *metric* and written $\varrho(x, y)$, satisfying the following conditions:

1) $\varrho(x, y) = 0$ iff $x = y$, $\forall x \in X, y \in X$;
2) $\varrho(x, y) = \varrho(y, x) \quad \forall x \in X, y \in X$;
3) $\varrho(x, y) \le \varrho(x, z) + \varrho(z, y) \quad \forall x \in X, y \in X, z \in X$.

These conditions are called the *distance axioms*.

Each subset of a metric space is a metric space in its own right.

If $B(E)$ is the family of bounded function on a set E (cf. § 2 of Chap. 1) then the function

$$\varrho(f, g) = \sup_{x\in E} |f(x) - g(x)|, \quad \forall f \in B(E), g \in B(E), \tag{1.1}$$

is a metric and, consequently, $B(E)$ can be viewed as a metric space.

An important example of a metric in function spaces are *integral metrics*. Let E be a space equipped with a complete measure μ. Then

$$\varrho(f, g) = \left(\int_E |f(x) - g(x)|^p dx\right)^{\frac{1}{p}}, \quad \forall f \in L_p(E), g \in L_p(E), \tag{1.2}$$

is a metric in the space $L_p(E), 1 \leq p < +\infty$. Note that the metric in $L_\infty(E)$ (cf. §2 of Chap. 1) is given by

$$\varrho(f, g) \overset{\text{def}}{=} \operatorname{vrai} \sup_{x \in E} |f(x) - g(x)|. \tag{1.3}$$

Two metric spaces are said to be *isometric* if there exists a bijective correspondence which preserves the distance.

A *sequence* of points $x_n \in X$, $n = 1, 2 \ldots$, of a metric space X is said to *converge* to the point $x \in X$ if

$$\lim_{n \to \infty} \varrho(x_n, x) = 0.$$

In this case x is said to be the *limit of the sequence* $\{x_n\}$ and one writes $\lim_{n \to \infty} x_n = x$ or $x_n \to x$. Convergence in the metric of the space $B(E)$ is just uniform convergence.

A basic notion is the notion of fundamental sequence. A sequence $\{x_n\}$ of points of a metric space is termed *fundamental* if

$$\lim_{n,m \to \infty} \varrho(x_n, x_m) = 0.$$

It is clear that if a sequence converges then it is fundamental; the converse is not true. If every fundamental sequence in a metric space converges to an element of the space, then the space is said to be *complete.*

As examples of complete metric spaces we mention the spaces $L_p(E)$, $1 \leq p \leq +\infty$. If $1 \leq p < +\infty$ the completeness follows from the properties of the Lebesgue integral (this circumstance, together with the fact that with the aid of the Lebesgue integral one can obtain the primitives for functions in a sufficiently wide class, also explains the great usefulness of the Lebesgue integral in various branches of mathematics).

Another important example of a complete metric space is the space $BC(E)$ of bounded continuous functions on a topological space E, with the metric (1.1) (that is, $BC(E)$ is considered as a subspace of $B(E)$). In this case the completeness follows from the observation that convergence in the metric (1.1) is the same as uniform convergence, that the limit of a uniformly converging sequence of continuous functions is again a continuous function and that for a sequence in $C(E)$ to be fundamental means that Cauchy's condition for uniform convergence holds for the sequence in question.

For a point x in a metric space one defines in a natural way the notion of an *ε-neighborhood* $U(x, \varepsilon)$ (here $\varepsilon > 0$ is an arbitrary positive number) as the set of all points y in this space such that $\rho(y, x) < \varepsilon$.

If X is a metric space and E a subset of X, we say that a point $x \in E$ is an *interior* point of E if there exists an $\varepsilon > 0$ such that the ε-neighborhood $U(x, \varepsilon)$ of x is contained in E.

A set $E \subset X$ is said to be open if all points of E are interior. The system of all open sets is a topology in the sense of the definition in §3 of Chap. 1.

Thus, each metric space is topological. On the other hand, the space of all functions with pointwise convergence is a topological space where it is not possible to introduce a metric which generates there the topology of pointwise convergence, this provided the underlying set is uncountable.

A point x of a metric space X is called a *limit* point for the set $E \subset X$ if each neighborhood of x contains points in E different from x itself.

A set which contains all its limit points is called *closed*. In a metric space X the complement $X \backslash E$ of any open set E is closed and, conversely, the complement of a closed set is open.

The minimal closed set containing a given set $E \subset X$ is called the *closure* of E and is written $\bar{E}$ (the closure $\bar{E}$ of E is also the intersection of all closed sets containing E).

A set $E \subset X$ is called *dense* in X if the closure of E coincides with X, i.e. $\bar{E} = X$. Every metric space is contained as a dense subset in a suitable complete metric space, called the *completion* of the given one.

The construction of the completion of a given metric space X can be effectuated in the following manner. Two fundamental sequences $\{x_n\}$ and $\{y_n\}$ are termed equivalent if $\lim_{n\to\infty} \rho(x_n, y_n) = 0$. Let X^* be the set of all equivalence classes of fundamental sequences in X. If $x^* \in X^*$, $y^* \in X^*$ and $\{x_n\} \in x^*$, $\{y_n\} \in y^*$, then the limit

$$\rho^*(x^*, y^*) \stackrel{\text{def}}{=} \lim_{n\to\infty} \rho(x_n, y_n)$$

exists, does not depend on the choice of the sequences $\{x_n\}$, $\{y_n\}$ and defines a metric in X^*. The given space X is isometric to the subset of all classes which contain a stationary sequence $\{x, x, \ldots, x, \ldots\}$, $x \in X$, and so can be identified with this subset. Thus X can be viewed as a subset of X^*. It turns out that X is dense in X^* and that X^* is complete, that is, X^* is the completion of X.

If a metric space contains a countable dense set then we say that it is *separable*.

An important notion in the theory of metric spaces is compactness.

A metric space is called *compact* if each sequences of points contains a subsequence which converges to a point of the space.

In order to formulate *compactness criterion for a space* we first formulate the notions of bounded and totally bounded metric spaces.

The number

$$\operatorname{diam} X = \sup_{x\in X, y\in X} \rho(x, y)$$

is called the *diameter* of the space X. If the diameter of X is finite then X is termed *bounded*.

Let A and B be given subsets of a metric space X and let ε be a number > 0. We say that B is an *ε-net* for A if for each $x \in A$ there exists a point $y \in B$ such that $\rho(x, y) \le \varepsilon$.

A set $A \subset X$ is called *totally bounded* if for each $\varepsilon > 0$ there exists a finite ε-net for it.

In n-dimensional Euclidean space $\mathbb{R}^n$ the notions of bounded and totally bounded sets are *equivalent*. In an infinite dimensional space each totally bounded set is clearly bounded; the converse does not hold true. Let us further mention that if a metric space is totally bounded then it is separable.

A necessary and sufficient condition for a metric space to be compact is that it is bounded and complete.

From this it follows that a subset of a complete metric space is compact iff it is totally bounded and complete. In particular, a set in finite dimensional Euclidean space is compact iff it is bounded and closed.

Let us remark that by what was just said it follows from our compactness criterion that each compact metric space is separable.

Another compactness criterion is connected with the notion of covering of a set. Let $E \subset X$. A family $\Omega = \{E_\alpha\}$, $\alpha \in \mathscr{A}$ of sets $E_\alpha \subset X$ ($\mathscr{A} = \{\alpha\}$ is a suitable set of indices α) is called a *covering* of E if

$$E \subset \bigcup_{\alpha \in \mathscr{A}} E_\alpha.$$

A covering consisting of finitely many sets E_α is called *finite*.

It turns out that a metric space is compact iff each open covering of it contains a finite subcovering.

In the case of finite dimensional Euclidean space this entails the well-known *Borel-Lebesgue lemma*: every covering of a bounded closed subset of $\mathbb{R}$ contains a finite covering.

Concluding our discussion of compactness, let us mention that that a subset A of a metric space X is called *pre-compact* if its closure $\bar{A}$ in X is compact.

The notions of closed set, closure, density of a set and compactness make sense also for topological spaces but in the sequel we will only encounter them in the context of metric spaces. Therefore we content ourselves with what has been said.

§ 5. Normed and Seminormed Spaces. Banach Spaces

A linear space (real or complex) is said to be a *normed space* if on the set of its points x there is defined a nonnegative function called, the *norm*, written $\|x\|_X$ or simply $\|x\|$, enjoying the following properties:

1) $\|x + y\| \leq \|x\| + \|y\| \; \forall x \in X, \forall y \in X$;
2) $\|\lambda x\| = |\lambda| \|x\| \; \forall x \in X$;
3) if $\|x\| = 0$, $x \in X$, then $x = 0$.

Here λ is a real or complex number according to whether we consider a real or a complex linear space.

If we on the set of points of a linear space X give a nonnegative function $\|x\|$ satisfying only 1) and 2), we say that X is a *seminormed* space and that $\|x\|$ is a *seminorm*.

Two linear normed spaces are said to be *isomorphic* if there exists a bijective correspondence between its points which preserves the linear operation and the value of the norm.

Two norms (seminorms) $\|x\|$ and $\|x\|^*$ in a normed space X are said to be *equivalent* if there exist constants $c_1 > 0$ and $c_2 > 0$ such that for all $x \in X$ holds

$$c_1\|x\| \le \|x\|^* \le c_2\|x\|.$$

In a finite dimensional space *all* semi-norms are equivalent but in infinite dimensional spaces there exist also nonequivalent norms. It is an important problem for infinite dimensional spaces, in particular for function spaces, to investigate when various norms are equivalent.

Every normed linear space is a metric space with the metric

$$\rho(x, y) = \|x - y\|, \quad x \in X, y \in X.$$

(Instead of convergence in metric one likewise says *convergence in norm.*) Therefore the notions of open and closed sets, closure of a set, compact set, density of a set in a space, separability and completeness are defined for normed spaces. A complete normed linear space is termed a *Banach space.* Every normed linear space is contained as a dense subspace in a Banach space.

The construction of the completion of a given normed linear space X can be obtained by the same scheme as for the completion of a metric space. Let us denote by X^* the set of all classes x^* of equivalent fundamental sequences $\{x_n\}$ of points of a given normed linear space X. If $x^* \in X^*$, $y^* \in X^*$ and $\{x_n\} \in x^*$, $\{y_n\} \in X^*$ then for any numbers λ and μ we set $\lambda x^* + \mu y^* = \lim_{n\to\infty}(\lambda x_n + \mu y_n)$ and $\|x^*\| = \lim_{n\to\infty}\|x_n\|$. These limits exist, the linear operation $\lambda x^* + \mu y^*$ does not depend on the choice of the sequences $\{x_n\}$, $\{y_n\}$ in respectively the classes x^*, y^*, the value $\|x^*\|$ does not depend on the choice of $\{x_n\} \in x^*$ and is a norm in X^*, which space turns out to be complete and contains a dense subset isomorphic to X, with which this subset therefore can be identified. Thus X^* is the completion of X.

There exist also metric linear spaces where the metric is not generated by a norm. Such is the case for the space of measurable functions on a set E with the metric

$$\rho(f, g) = \int_E \frac{|f(x) - g(x)|}{1 + |f(x) - g(x)|}dx, \quad \mu E < +\infty.$$

Convergence in this metric is equivalent to *convergence in measure*, that is $f_n \to f$ in this metric means that for any $\varepsilon > 0$ holds

$$\lim_{n\to\infty} \mu\{x : |f_n(x) - f(x)| \ge \varepsilon\} = 0.$$