L. Sirovich

# Introduction to Applied Mathematics

# 应用数学导论 [英]

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# Introduction to Applied Mathematics

With 133 Illustrations

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# Series Preface

Mathematics is playing an ever more important role in the physical and biological sciences, provoking a blurring of boundaries between scientific disciplines and a resurgence of interest in the modern as well as the classical techniques of applied mathematics. This renewal of interest, both in research and teaching, has led to the establishment of the series: *Texts in Applied Mathematics (TAM).*

The development of new courses is a natural consequence of a high level of excitement on the research frontier as newer techniques, such as numerical and symbolic computer systems, dynamical systems, and chaos, mix with and reinforce the traditional methods of applied mathematics. Thus, the purpose of this textbook series is to meet the current and future needs of these advances and encourage the teaching of new courses.

*TAM* will publish textbooks suitable for use in advanced undergraduate and beginning graduate courses, and will complement the *Applied Mathematics Sciences (AMS)* series which will focus on advanced textbooks and research level monographs.

# Preface

The material in this book is based on notes for a course which I gave several times at Brown University. The target of the course was juniors and seniors majoring in applied mathematics, engineering and other sciences. In actual fact, the students ranged from occasional highly prepared freshmen to graduate students. The last category usually made up one third to one half of the class. Overall, I would say that the students found the contents of the book challenging and exacting.

My basic goal in the course was to teach standard methods, or what I regard as a basic *bag of tricks*. In my opinion the material contained here, for the most part, does not depart widely from traditional subject matter. One such departure is the discussion of discrete linear systems (and this is really just a return to classical material). Besides being interesting in its own right, this topic is included because the treatment of such systems leads naturally to the use of discrete Fourier series, discrete Fourier transforms, and their extension, the $Z$-transform. On making the transition to continuous systems we derive their continuous analogues, viz., Fourier series, Fourier transforms, Fourier integrals and Laplace transforms. A main advantage to the approach taken is that a wide variety of techniques are seen to result from one or two very simple but central ideas. Students appeared both to grasp and to appreciate this consolidation of concepts.

Related to this and a recurrent theme in this text is the idea of transforming a problem to another simpler problem. This in turn leads to the use of eigenfunction methods. Virtually every method developed here is also derived by an eigenfunction approach. Moreover, some weight is laid on this being a natural way to view and analyze problems. This then leads to the geometrical point of view and to the introduction of abstract spaces. Since I felt that this was a very desirable approach I went to some lengths to motivate these ideas and make learning them as painless as possible.

As the remarks thus far imply I have placed emphasis on presenting a variety of approaches and perspectives—as many as I deemed possible. This is in keeping with a general principle which I subscribe to, namely that a deeper understanding of a subject is gained by viewing it from as many aspects as possible.

There are two basic prerequisites for this course: linear algebra and ordinary differential equations. The latter on the level of, for example, the books by Braun and by Boyce and DiPrima. (A list of references appears at the end of the book.) It is also appropriate to mention a word about the first three chapters which cover basic topics in complex variable theory. If one views this as a course in applied complex analysis then the first three chapters are the underpinnings. This portion of the course was taught in roughly five weeks and since a broad range of topics are included some sacrifices were required. Consequently there was no intention of having this course replace the traditional complex variable course. If anything I contend that the standard material in complex variable theory will be better appreciated by the student after a course of this type.

Above all, this course is intended as being one which gives the student a *can-do* frame of mind about mathematics. Too many math courses give the impression that mathematics is a minefield and that unless one is very very careful disasters will befall them. My view and the one that I have tried to present in this book is diametrically opposed to this. Students should be given confidence in using mathematics and not be made fearful of it. Partly with this in mind I have forgone the theorem-proof format for a more informal style. Although I have endeavored to make the mathematics respectable, rigor has not been given a high priority. Finally a concerted effort was made to present an assortment of examples from diverse applications with the hope of attracting the interest of the student, and an equally dedicated effort was made to be kind to the reader.

Only the help of many people made the completion of this book possible. Madeline Brewster and Andria Durk prepared an earlier version and played an essential role in assembling the present version; Kate MacDougall painstakingly and patiently prepared this final version. My colleague and friend Jack Pipkin performed the experiment of teaching this material from an earlier version of the manuscript. His criticism (sometimes severe) often took root. I take pleasure in expressing sincere gratitude to them all. Finally no words can express my deep appreciation to Candace Kent who took the course, corrected my errors, mathematical and otherwise. Her many improvements appear throughout the text. The blemishes, flaws and errors that remain are due to me and are there in spite of the best efforts of all these people. Finally thanks, with mixed feelings, also go to the late Walter Kaufmann-Bühler for sweet-talking me into writing this book.

I dedicate this book to the memory of my mother, Libby, who was my first and best teacher.

L. S.
Saltaire
July, 1988

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# 1

# Complex Numbers

## 1.1 Complex Numbers

The concept of *imaginary* numbers occurs early in the discussion of algebraic equations. For example, the quadratic equation

$$x^2 + 1 = 0$$

has the solutions $x = \pm i$, where $i = \sqrt{-1}$. In general, hybrid forms, called *complex numbers*, are found containing both real and imaginary parts. For example, if

$$x^2 - 2x + 2 = 0,$$

then

$$x = 1 \pm i$$

are the solutions. Complex numbers, which extend the *real* number system, are made necessary by the solution of algebraic equations with real coefficients. It is interesting to note that algebraic equations with complex coefficients have solutions which are complex—no further extension is necessary.

Complex numbers can be viewed as belonging to a two-space called the *complex plane* (see Figure 1.1). According to common convention, a typical complex number is denoted by the letter $z$, with

$$z = x + iy.$$

We also define the real and imaginary parts through

$$x = \operatorname{Re} z, \quad y = \operatorname{Im} z. \tag{1.1}$$

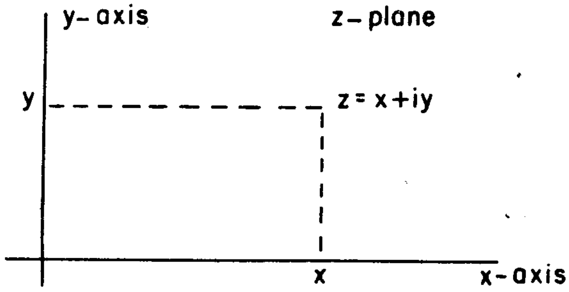


FIGURE 1.1.

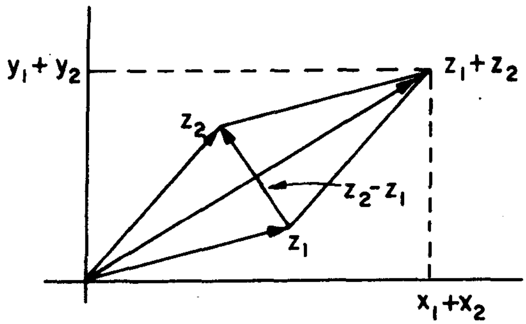


FIGURE 1.2.

The addition of the two complex numbers

$$z_1 = x_1 + iy_1, \quad z_2 = x_2 + iy_2$$

follows the rules of vector addition in two-space:

$$z = z_1 + z_2 = x_1 + x_2 + i(y_1 + y_2) = z_2 + z_1.$$

Both this operation and that of subtraction are indicated in Figure 1.2. The figure is familiar from analytical geometry and further explanation is not deemed necessary.

Complex numbers can be multiplied in the ordinary way and this differs from Cartesian two-vectors. In particular,

$$\begin{aligned} z_1 z_2 &= (x_1 + iy_1)(x_2 + iy_2) \\ &= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1) = z_2 z_1, \end{aligned}$$

where $i^2 = -1$ has been used. Actually, the explicit appearance of $i$ can be avoided by writing

$$z = (x, y)$$

and defining

$$z_1 + z_2 = (x_1 + x_2, y_1 + y_2),$$

$$z_1 z_2 = (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1).$$

Such rules can be used to generate a complex arithmetic for use on a computer.

As usual, division is the operation inverse to that of multiplication. Thus $z$ is called the quotient of $a$ and $b$ if $bz = a$. If we write $a = a_1 + ia_2$, $b = b_1 + ib_2$, and $z = x + iy$, then

$$\begin{aligned} bz &= (b_1 + ib_2)(x + iy) \\ &= (b_1 x - b_2 y) + i(b_2 x + b_1 y) = a_1 + ia_2. \end{aligned}$$

This complex equation can be put into the form of a matrix problem:

$$\begin{pmatrix} b_1 & -b_2 \\ b_2 & b_1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}.$$

Thus solving $bz = a$ is equivalent to solving a $2 \times 2$ linear system, but not all $2 \times 2$ linear systems can be put into this complex form. It follows from the construction of complex numbers that in a complex equation the real and imaginary parts are separately equal. Thus we are led to two linear equations in $x$ and $y$ which, when solved, yields

$$x + iy = \frac{a_1 b_1 + a_2 b_2}{b_1^2 + b_2^2} + i \frac{-a_1 b_2 + a_2 b_1}{b_1^2 + b_2^2}$$

The denominator $b_1^2 + b_2^2$ is the squared distance of the complex number $b_1 + ib_2$ from the origin. More generally, if $z = x + iy$, its distance from the origin is denoted by

$$r = |z| = \text{mod}\, z = (x^2 + y^2)^{1/2} \tag{1.2}$$

(mod, short for modulus). We can also write

$$|z|^2 = x^2 + y^2 = (x + iy)(x - iy) = z\bar{z}.$$

The last expression defines the complex conjugate; i.e., if $z = x + iy$, then $\bar{z} = x - iy$. (In certain instances the complex conjugate of $z$ will also be denoted by $z^*$.) The conjugate is useful in representing the division of two complex numbers in terms of real and imaginary parts:

$$\frac{z_2}{z_1} = \frac{z_2 \cdot \bar{z}_1}{z_1 \cdot \bar{z}_1} = \frac{x_1 x_2 + y_1 y_2}{x_1^2 + y_1^2} + i \frac{-x_2 y_1 + x_1 y_2}{x_1^2 + y_1^2}.$$

We mention also that since $\bar{z}$ can be determined from $z$, $\bar{z}$ is really a function of $z$; i.e., formally $\bar{z} = \bar{z}(z)$. Figure 1.3 indicates the location of these quantities.

### Polar Coordinates

Equation (1.2) defines the modulus of a complex number $z$. To complete the transformation from Cartesian to polar coordinates, we define the positive polar angle $\theta$ as being the angle measured counterclockwise from the positive real axis to the ray to $z$ (see Figure 1.3.). For negative $\theta$, measurement is made in the clockwise direction. The angle $\theta$ is also written as

$$\theta = \arg z,$$

where arg is short for argument.

It is clear that $\theta$ has the property

$$\tan \theta = \frac{y}{x},$$

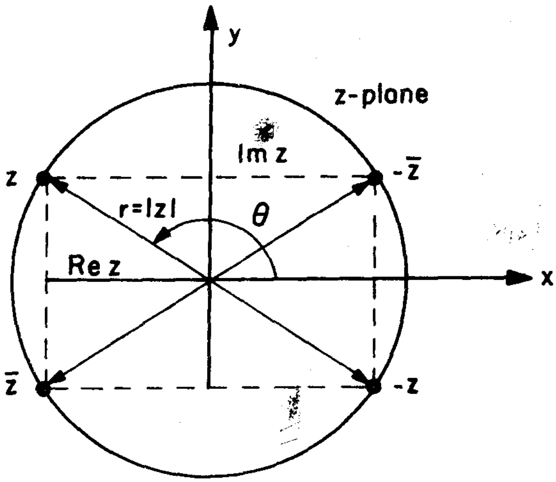


FIGURE 1.3.

which can be formally solved to give

$$\theta = \tan^{-1} \frac{y}{x}. \tag{1.3}$$

However, it should be recalled that the arctangent is customarily defined with its range as the open interval $(-\pi/2, \pi/2)$ (see Figure 1.4); therefore, some *fine print* is required along with (1.3). For example, if we consider the case where $\theta \in [-\pi, \pi]$, then $\theta$ is given by (1.3) for $x > 0$ and $\theta = \tan^{-1}(y/x) + \pi \operatorname{sign} y$ for $x < 0$. (See Exercise 7.)

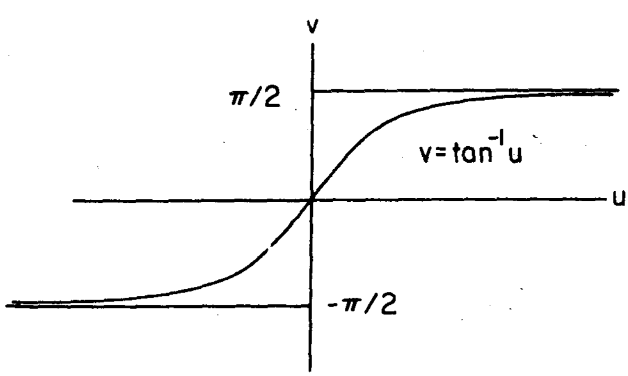


FIGURE 1.4.

## 1.2 Exponential Notation

The exponential function $e^{at}$ for real $a$ and $t$ is defined by the convergent infinite series

$$e^{at} = \sum_{n=0}^{\infty} \frac{(at)^n}{n!} = \exp(at). \tag{1.4}$$

This can be regarded as the solution of the initial value problem

$$\frac{dw}{dt} = aw, \quad w(0) = 1.$$

If $a$ is complex, the meaning of the series is not clear since we have not yet considered the idea of the convergence of a sum in the complex plane. For the moment we accept the above definition of the exponential as being valid for complex numbers (this will be justified in the next section).

If $a = i$ in (1.4), then formally

$$\begin{aligned} e^{it} &= \sum_{n=0}^{\infty} \frac{i^n t^n}{n!} = 1 + it - \frac{t^2}{2} - \frac{it^3}{3!} + \frac{t^4}{4!} + \cdots \\ &= \left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots\right) + i\left(t - \frac{t^3}{3!} + \cdots\right). \end{aligned}$$

We recognize the sum in the first set of parentheses first as $\cos t$ and the second as $\sin t$. Thus we have shown

$$e^{it} = \cos t + i \sin t. \tag{1.5}$$

Further, from the differential equation, if

$$\frac{dw_1}{dt} = a_1 w_1, \quad \frac{dw_2}{dt} = a_2 w_2,$$

then cross-multiplication gives

$$\frac{dw_1 w_2}{dt} = (a_1 + a_2) w_1 w_2,$$

which demonstrates that

$$e^{a_1 t} = e^{a_2 t} = e^{(a_1 + a_2)t}$$

for complex $a_1$ and $a_2$. In particular,

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y).$$

If we return to polar notation using (1.2) and (1.3), then

$$\begin{aligned} z &= x + iy + r \cos \theta + ir \sin \theta \\ &= r(\cos \theta + i \sin \theta), \end{aligned}$$

which from (1.5) states

$$z = x + iy = re^{i\theta}. \tag{1.6}$$

If this notation is applied to the product $z_1 z_2$, where $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, then

$$z_1 z_2 = (r_1 e^{i\theta_1})(r_2 e^{i\theta_2}) = r_1 r_2 e^{i(\theta_1+\theta_2)},$$

from which we have

$$\arg z_1 z_2 = \arg z_1 + \arg z_2,$$

$$\operatorname{mod} z_1 z_2 = |z_1 z_2| = r_1 r_2 = (\operatorname{mod} z_1)(\operatorname{mod} z_2). \tag{1.7}$$

Therefore, *under multiplication, arguments add and moduli multiply.*

## De Moivre's Formula

Consider $z^n$ for integer $n$. If we use (1.6), then

$$z^n = (re^{i\theta})^n = r^n e^{in\theta} = r^n(\cos n\theta + i \sin n\theta);$$

or, if $|z| = r = 1$, then

$$e^{in\theta} = \cos n\theta + i \sin n\theta = (\cos \theta + i \sin \theta)^n, \tag{1.8}$$

which is known as *De Moivre's Formula* (or *Theorem*). If, for example, $n = 2$, this says

$$\begin{aligned} \cos 2\theta + i \sin 2\theta &= e^{2i\theta} = (e^{i\theta})^2 = (\cos \theta + i \sin \theta)^2 \\ &= \cos^2 \theta - \sin^2 \theta + 2i \sin \theta \cos \theta, \end{aligned}$$

the real and imaginary parts of which give the familiar trigonometric relations

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta$$

and

$$\sin 2\theta = 2 \sin \theta \cos \theta.$$

In general, De Moivre's Formula facilitates the demonstration of many trigonometric relations.

## Roots of a Complex Number

If, for complex numbers $w$ and $z$ and integer $n$, we have that

$$w^n = z,$$

then $w$ is said to be an $n$th root of $z$ and is written as $z^{1/n}$. To find $w$, first write

$$w = Re^{i\Theta}, \quad z = re^{i\theta}$$

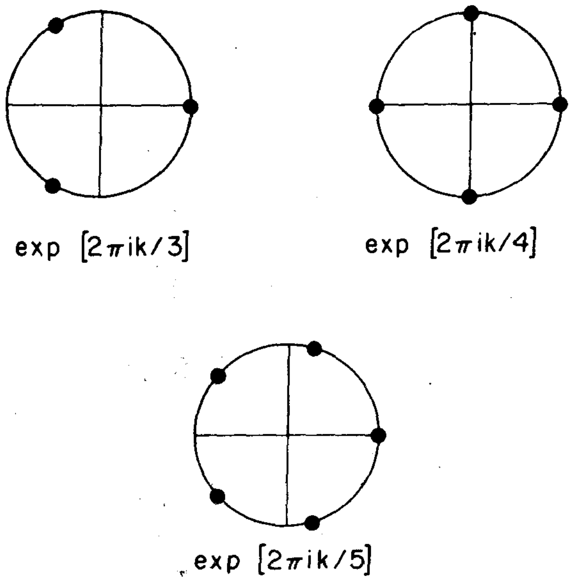


FIGURE 1.5.

so that

$$w^n = R^n e^{in\Theta} = re^{i\theta}.$$

On comparison of moduli and arguments, we obtain

$$R^n = r, \quad n\Theta = \theta + 2\pi N.$$

The term $2\pi N$ with integer $N$ is included since arg is ambiguous up to an integer multiple of $2\pi$; i.e., $\exp(i\theta) = \exp[i(\theta + 2N\pi)]$. If we solve for $R$ and $\Theta$, then

$$R = r^{1/n}$$

and

$$\Theta = \frac{\theta}{n} + 2\pi\frac{N}{n}, \quad N = 0, 1, \ldots, n-1.$$

These form the only choices for $\Theta$ since any other integer choice of $N$ will yield a $\Theta$ which differs from one of the above by a multiple of $2\pi$.

As a particular example, consider the $n$th roots of unity as determined by

$$w^n = 1.$$

From the above discussion, the $n$ different roots are

$$w_k = e^{i2\pi k/n}, \quad k = 0, 1, \ldots, n-1.$$