Mathematics Monograph Series 1

Finite Element Methods: Accuracy and Improvement

Qun Lin Jiafu Lin

(有限元方法：精度及其改善)

SCIENCE PRESS
Beijing

Mathematics Monograph Series 1

Qun Lin Jiafu Lin

Finite Element Methods: Accuracy and Improvement

(有限元方法：精度及其改善)

SCIENCE PRESS
Beijing

Responsible Editors: Lin Peng   Chen Yuzhuo

Published by Science Press
16 Donghuangchenggen North Street
Beijing 100717, China

Printed in Beijing

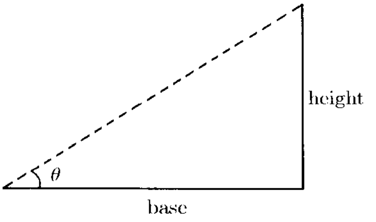
ISBN 7-03-016656-6 (Beijing)

Preface

Mathematics is too hard and too much, except higher is compared with elementary.

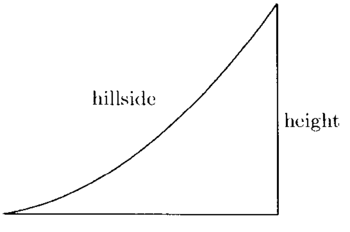
Differential equations and trigonometry measurement

Our target is the differential equations. What is the differential equation? What is the most convictive example? It is the trigonometry measurement, measuring the height (unknown) of a tree from a slope (what we know) of an imaginary hypotenuse:



elementary trigonometry uses a tangent formula: height = slope × base

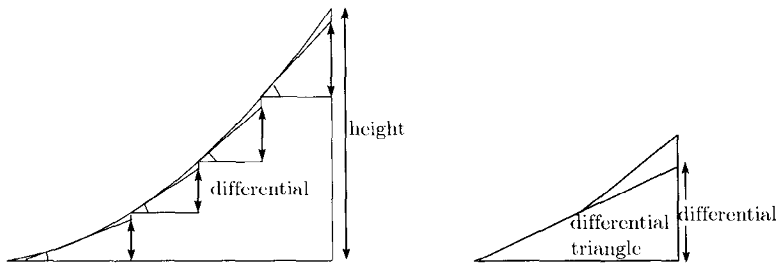
Without trigonometry (or tangent formula) we have to cut or climb the tree! A breakthrough is to change the tree to a hill, measuring the height (unknown) of a hill from variable slopes (what we know) of the curved hillside:



This is called a curved trigonometry to distinguish elementary trigonometry, or read as a differential equation: given slopes and find the height solution.

Solver: height formula

It is still based on the tangent formula, but before using it we first shorten the curve triangle into an infinitesimal straight triangle (called the differential triangle):



curved trigonometry uses tangent formula in every differential triangle which has been appeared in Chinese newspaper[29].

whose height can then be computed by the tangent formula, slope×base, called differential.

Now, the height solution is the sum of differentials:

$$\begin{aligned}\text{total height} &= \text{integral of differentials,}\\ \text{each differential} &= \text{slope} \times \text{base.}\end{aligned}$$

Such a differential, with a quantitative definition, has been stated in Lin[30] and will be represented in section 1.1.2 of Chapter 1 in this book.

However, this differential equation is the simplest one among all the differential equations. Let us observe its variations.

Variations

The simplest differential equation, together with its variations, is widely used. It is not only used to find the areas enclosed by most amazing curves in calculus textbooks but also used, e.g. to predict the recent population in China: in year 2000 there was a census that mobilized the whole nation directly from door to door, spent one year more and gave a number of 12.66 hundred million, while a population prediction can be done by a college student indirectly solving a population differential equation (a variation of the simplest differential equation) in a few minutes, giving a number of 13.45 hundred million (see Section 1.1.5 in Chapter 1). Two answers are about the same but their efficiencies are a world of difference. This is why Newton said: it is necessary to solve differential equations.

So, the differential equation is inevitable extension of the trigonometry measurement, from measure the tree height (from one slope) to measure the area and population (from variable slopes) and, more often, describe all laws in nature. Where can we find such a "lucky mathematics", simple but widely used?

The fundamental formula solves not only the simplest differential equation, but also its variation–a class of differential equations, including the most important differential equation (such as the population equation above) in applied mathematics (see Strang's Calculus, p.242), or more generally: the separable equation, the linear equation and generally the exact equation (as in Braun's Differential Equations), all of which can be put in the form of the simplest one (a class = single one). Thus the simplest differential equation is not simply an isolated equation but plays a fundamental role in a class of differential equations. It is better to focus on the simplest (and fundamental) one than spending on more differential equations.

Approximate solvers

The very sad fact is that we cannot solve all differential equations explicitly.[1] In order that differential equations keep any practical value for us, we must abandon the explicit solver but satisfy with the approximate solvers. They are the Euler tangent line solver (agreeing with the fundamental formula) and the finite element secant line solver (similar to the inscribed polygon approximation of a circle), presented in Chapter 1.

Two aspects, exact and approximate, cannot be neglected.

However, for such approximate methods, we need to know their accuracy as clear as possible (e.g. the sign and size of error). The task of Chapters 3–7 of our book wants to know about the finite element method: accuracy and improvement. For this, we need three axes from analysis, not only

1. integration by parts

a variation of the fundamental formula, and

2. Sobolev inequalities

but also

3. norm equivalence lemmas

(agreeing with Taylor's polynomial) or their variation, expansion lemmas, in Chapter 2. In fact, the first two axes also dominate the theory of partial differential equations, and graduates should be familiar with them.

Other lemmas contained in Chapter 2 are the embodiment of expansion lemmas. They are the cornerstone of Chapter 3–7 and, are the most laborious part of the book.

Sequel: For more details of Chapters 3–7

A quip appears on the first page of conference proceedings edited by Krizek: "you think A, you talk B, you write C." Can we make $C \approx B \approx A$? [2]

π computation

We have mentioned above that the finite element solver contains an

---

1 This is noting strange since a general polynomial equation cannot be solved explicitly.

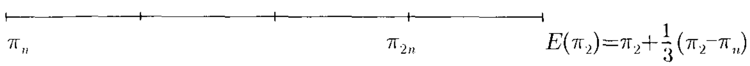
2 Strang said in his Calculus[56] (p.27): "what I write is very close to what I would say."

example–the polygon approximation of a circle. By our methodology, inventing higher knowledge by following elementary, we should invent finite element solver by following this example. The most familiar thing is the polygon approximation, π_n, of circular perimeter π, including

(α) inscribed polygon of n sides;

(β) circumscribed polygon of n sides.

In principle, one can achieve better accuracy by increasing the number of polygon sides. How about its efficiency? It is lower, e.g. to guarantee the polygon method, inscribed or circumscribed, an accuracy of seven decimal places we need $n = 12288$ sides. Can we enhance the efficiency? If we use the two consecutive computations, π_n and π_{2n}, and extrapolate them with a $\frac{1}{3}$ – distance outside the interval (π_n, π_{2n}):



then, the result, extrapolation

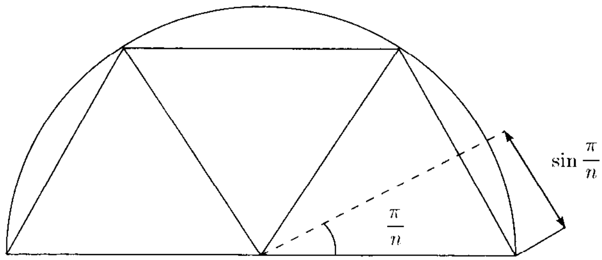
$$E(\pi_n) = \frac{4\pi_{2n} - \pi_n}{3},$$

has much better accuracy (efficiency) than the original polygon method, π_n, e.g. $E(\pi_n)$ has also the same accuracy of seven decimal places but only uses $n = 96$ sides and 192 sides, i.e. a one and a two-hundred-side polygons can together do the same work that a ten-thousand-side polygon can do. Why? The polygon method, (α) or (β), has respectively the explicit expression:

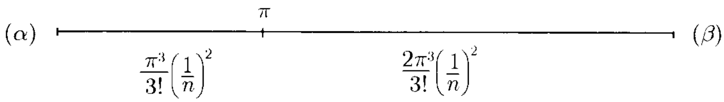
$$\pi_n = \begin{cases} n \sin \dfrac{\pi}{n}, & \text{for } (\alpha) \\[2ex] n \tan \dfrac{\pi}{n}, & \text{for } (\beta) \end{cases}$$

Taylor's formula gives respectively the error expansion:

$$\pi_n - \pi = \begin{cases} -\dfrac{\pi^3}{3!}\left(\dfrac{1}{n}\right)^2 + \dfrac{\pi^5}{5!}\left(\dfrac{1}{n}\right)^4 + \cdots, & \text{for } (\alpha) \\[2ex] \dfrac{2\pi^3}{3!}\left(\dfrac{1}{n}\right)^2 + \dfrac{16\pi^5}{5!}\left(\dfrac{1}{n}\right)^4 + \cdots, & \text{for } (\beta) \end{cases}$$



inscribed method (α)



with respectively the dominant term

$$-\frac{\pi^3}{3!}\left(\frac{1}{n}\right)^2, \quad \frac{2\pi^3}{3!}\left(\frac{1}{n}\right)^2$$

which reveal both the error sign and size. Furthermore, the dominant term can be cancelled by a linear combination, extrapolation:

$$E(\pi_n) - \pi = \begin{cases} -\dfrac{\pi^5}{480}\left(\dfrac{1}{n}\right)^4 + \cdots, & \text{for } (\alpha) \\[2ex] -\dfrac{\pi^5}{30}\left(\dfrac{1}{n}\right)^4 + \cdots, & \text{for } (\beta) \end{cases}$$

two order higher rate than the original polygon method, π_n.

So, extrapolation is a cheap dinner of high efficiency. The polygon method is welcome because it can be extrapolated, without changing the original algorithm, to a higher rate.

Comparing these expansions we conclude that

(i) inscribed method, (α), is a lower approximation while circumscribed method, (β), is an upper approximation;

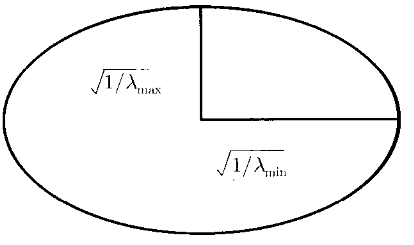
but because of 480 and 30,

(ii) method (α) has better accuracy than (β);

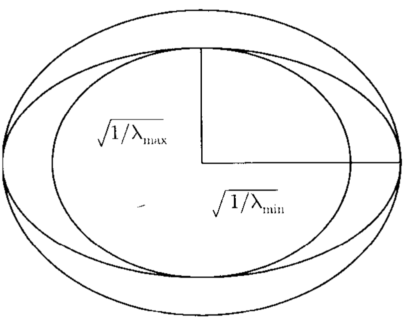
(iii) both of them have the same error order of $\left(\frac{1}{n}\right)^2$ and accompany the same extrapolation rate of $\left(\frac{1}{n}\right)^4$.

So, the error expansion is an ideal tool to determine the error sign and size of an approximation (from the dominant term) and the rate of extrapolation (from the remainder).

Is the story of π true for differential equations, e.g. the "eigenvalue" computation? Such "eigenvalues" have been seen in the minor and major axes of an ellipse



ellipse equation: $\lambda_{\min}x^2 + \lambda_{\max}y^2 = 1$



ellipse equation: $\lambda_{\min}x^2 + \lambda_{\max}y^2 = 1$

(where we recognize higher knowledge again by reviewing elementary). This is our task: exploiting the π-computation to serve for the eigenvalue computation of the differential equation such that the latter, eigenvalues, can be computed well, like computing π.

Let us follow π-computation from word to word. The circular perimeter π can be approximated by the polygonal perimeters π_n from both sides using inscribed and circumscribed methods, each of which has its own expansion. Analogously, can the minor and major axes, or eigenvalues λ_{kl} $(k, l = 1, 2, \cdots)$, be approximated by the "finite element" eigenvalues $\lambda_{kl,h}$ (where $h \approx n^{-1}$) also from both sides using "nonconforming" and "conforming" methods (called (ζ), (η), etc.), each of which has its own expansion? The answer is positive, e.g. for the eigenvalue problem of a typical differential equation and two finite element methods under certain geometrical conditions, we have the expansions

$$\lambda_{kl,h} - \lambda_{kl} = \begin{cases} -\dfrac{2k^2l^2}{12}h^2 + O(h^4), & \text{for } (\zeta) \\ \dfrac{(k^2-l^2)^2}{24}h^2 + O(h^4). & \text{for } (\eta) \end{cases}$$

From the sign and size of the coefficient of h^2 in the dominant term, we conclude that

(iv) method (ζ) is a lower approximation while method (η) is an upper approximation;

(v) method (η) may have better accuracy than (ζ);

(vi) both of them have the same error order of h^2 and accompany the same extrapolation rate of h^4.

Similar eigenvalue expansions can be proved for more nonconforming and conforming methods (under certain geometrical conditions) to judge the error sign and size (from the dominant term) and the extrapolation rate (from the remainder). Even the eigenvalue expansion did not be proved (say, under a general geometrical condition) we can still use the extrapolated eigenvalue $E(\lambda_h)$, to compare with the original eigenvalue, λ_h or $\lambda_{\frac{h}{2}}$, to see if the former has a big difference from the latter.

In short, the expansion method (and extrapolation algorithm) possibly work for the eigenvalue (denoted by λ) approximated by finite elements (denoted by λ_h):

$$\lambda_h = \lambda + ch^2 + o(h^2).$$

To be insatiable, does the expansion method possibly work for the solution, $u(x)$, approximated by a finite element, $u_h(x)$:

$$u_h(x) = u(x) + c(x)h^2 + o(h^2)$$

in L^2 or at almost all points? Indeed, in the literature (1978–1985), such a solution expansion was believed even "proved" to hold for a typical differential equation, and, more seriously, more people in the finite element community still expect it. However, we have an "impossible" result (see Lin-Liu in Appendix 3): such a solution expansion holds at almost no points![1] Such a result avoids the misuse of the method of extrapolation, and presses

1 But at interpolation points.

us to modify the expansion method, e.g. introduce postprocessing of finite element solution, $\bar{u_h}(x)$, such that

$$\bar{u_h}(x) = u(x) + c(x)h^2 + o(h^2).$$

This is the job of second part of our book, Chapters 4–7, where the postprocessing will be the additional work of the solution problem. Such a kind of postprocessing has been used in Shaidurov's books (1995) for the linear finite element, in Lin-Yan's book (1996) for different conforming finite elements including mixed finite elements, and in Lin-Tobiska-Zhou's paper(2001) for nonconforming lowest-order finite elements. In 1999–2001, Dr. Jiafu Lin had a more systematic work in this aspect during his post-doctor research in our group and so was invited to write this part. We must emphasize that the second part is written in a completely independent way[1] such that the readers interested in the solution computation of partial differential equations could read it immediately without reading Chapters 1–3. The second part also connects with the superconvergence theory: when we are unable to establish extrapolation for the solution problem we will try to establish superconvergence, whose general framework has been summarized by Brandts and Krizek (2001). Superconvergence in the generalized finite element method, see Babuska-Banerjee-Osborn(2006).

To end the preface, may we quote a line again from Strang's Calculus: "I could go directly to the formulas but I am really unwilling just to write down formulas and skip over all the ideas". Or in words, mathematics is not only calculations and proofs but needs understanding—our book emphasizes to understand higher knowledge by comparing elementary.

Acknowledgments

Our work is a cooperation of many coworkers, besides J. Liu (1980, 2004, using one-dimensional model to overthrow the solution expansion and propose postprocessing), they are T. Lu (1983, first paper proposing extrapolation to finite element eigenvalues in two dimensions and using piecewise uniform triangulations), T. Lu and S. Shen, and Q. Zhu (1983, and 1984, first paper proposing extrapolation and using the discrete Green's function to the finite element solution, and gradients, on strongly regular triangulations, or piecewisely), J. Wang (1984, using the error expansion to obtain high rate for the linear finite element solution), J. Xu (1985, proposing element by element analysis to the error expansion of the finite element solution on piecewise uniform triangulations), H. Blum and R. Rannacher (1986, establishing sharp estimates and proposing extrapolation to a smooth domain), X. Xie (1986–1989, further development on the error expansion of finite element methods),

1 The first author is sick in recent two years and is not able to unify the two parts.

M. Krizek and Y. Huang (1988, using a combination of linear and bilinear elements to obtain a high rate), C. Chen (1989, proposing a mid-point extrapolation formula), Y. Huang (1990–1994, constructing refinement meshes for polygonal domains), R. Xie and J. Shi (1988, extrapolation for boundary elements). Since 1990, the second period work, my coworkers develop a general framework consisted of exact integral identities and interpolation postprocessing for extrapolation and superconvergence. They are N.Yan and A. Zhou, and J. Zhou (1991, and 1995, first paper using point-line -surface type interpolation postprocessing for high-order rectangle elements, and for high-order triangular elements), J. Li, M. Liu, J. Pan and J. Lin (1991–2002, for mixed finite elements), T. Tobiska and A. Zhou (2002, for nonconforming lowest-order elements), D. Wu (1999, for Carey element), P. Luo (1995, for Adini element), N.Yan and A.Zhou (1995, for coupling elements and for sparse finite element), J. Lin (2001, for discontinuous Galerkin method), Z. Li and N. Yan (1999, for combined methods), N. Yan (1999, for Maxwell equations), X. Liu (1997, for Schrodinger equation), S. Zhang and N. Yan (1998, for Fredholm integral equations), Y. Yang (1991,1992, using postprocessing finite element eigenfunctions to obtain a high rate of finite element eigenvalues), H. Brunner and N. Yan (1996, iterative correction for Volterra integral equation), H. Brunner and Q. Hu (2001, optimal superconvergence at points of a geometric mesh), L. Peng and J. L. Wang(1995, for probability finite elements).

Finally, I want to express a special thank to minister Y. Liu and Bureau chief Z. Lin for their support for the publication. Without their help, the book will be delayed for long time. Our great thanks go also to Drs. Yuzuo Chen and Shanghui Jia for their valuable work in improving and completing the manuscript.

Qun Lin
2004, Beijing

Contents

Second Part

First Part