Mathematics and Its Applications

Vadim Komkov

Variational Principles of Continuum Mechanics with Engineering Applications

Volume 2: Introduction to Optimal Design Theory

D. Reidel Publishing Company

Vadim Komkov

Air Force Institute of Technology,
Wright-Patterson Air Force Base, Ohio, U.S.A.

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SERIES EDITOR'S PREFACE

Approach your problems from the right end and begin with the answers. Then one day, perhaps you will find the final question.

'The Hermit Clad in Crane Feathers' in R. van Gulik's *The Chinese Maze Murders*.

It isn't that they can't see the solution. It is that they can't see the problem.

G.K. Chesterton. *The Scandal of Father Brown* 'The point of a Pin'.

Growing specialization and diversification have brought a host of monographs and textbooks on increasingly specialized topics. However, the "tree" of knowledge of mathematics and related fields does not grow only by putting forth new branches. It also happens, quite often in fact, that branches which were thought to be completely disparate are suddenly seen to be related.

Further, the kind and level of sophistication of mathematics applied in various sciences has changed drastically in recent years: measure theory is used (non-trivially) in regional and theoretical economics; algebraic geometry interacts with physics; the Minkowsky lemma, coding theory and the structure of water meet one another in packing and covering theory; quantum fields, crystal defects and mathematical programming profit from homotopy theory; Lie algebras are relevant to filtering; and prediction and electrical engineering can use Stein spaces. And in addition to this there are such new emerging subdisciplines as "experimental mathematics", "CFD", "completely integrable systems", "chaos, synergetics and large-scale order", which are almost impossible to fit into the existing classification schemes. They draw upon widely different sections of mathematics. This programme, Mathematics and Its Applications, is devoted to new emerging (sub)disciplines and to such (new) interrelations as exempla gratia:

- a central concept which plays an important role in several different mathematical and/or scientific specialized areas;
- new applications of the results and ideas from one area of scientific endeavour into another;
- influences which the results, problems and concepts of one field of enquiry have and have had on the development of another.

The Mathematics and Its Applications programme tries to make available a careful selection of books which fit the philosophy outlined above. With such books, which are stimulating rather than definitive, intriguing rather than encyclopaedic, we hope to contribute something towards better communication among the practitioners in diversified fields.

In the series editor's preface to the first volume of this two volume work I wrote that the time-gap between theoretical developments and applications of these was fast disappearing in many fields, including the part of mechanical engineering involving continuum mechanics. (Though that is definitely not the only part of mechanical engineering where this is the case; stochastics mechanics, for example, is another (cf. the book by P. Krée and Chr. Soize, Mathematics of random phenomena, in this series) and so is the part centering around questions related to identification and filtering).

Variational problems and optimization questions in continuum mechanics tend to involve a functional, a domain, exterior forces (or controls) and a PDE for the function for which an optimum (extremum) is sought. The more classical problems ask for the optimizing function. Other more modern questions ask for optimal exterior forces in some sense, or optimal shape of the domain involved, or such questions as how much of the boundary must be available (for control through boundary conditions) to be able, say, to control the vibrations of a satellite.

This book is mainly concerned with optimal shape and optimal exterior forces type problems.

All such problems tend to involve abstract differentiation in all kinds of infinite-dimensional (function) spaces and it is definitely not true that 'straightforward' Fréchet or Gateaux differentiation will lead to the right kind of numerical algorithms. A lot of modern mathematics is needed including substantial amounts from that again flowering area of research: the effective use of symmetry properties. (For the matter, also such things as nonstandard analysis have their applications to problems of optimal shape as the author has shown.)

Thus the mechanical engineer is faced with the problem that there are many sophisticated mathematical tools ready to be applied and the mathematician is confronted with the fact there are many important unsolved problems coming out of continuum mechanics. I expect this book will be most useful for both.

The unreasonable effectiveness of mathematics in science ...

Eugene Wigner

Well, if you know of a better 'ole, go to it.

Bruce Bairnsfather

What is now proved was once only imagined.

William Blake

As long as algebra and geometry proceeded along separate paths, their advance was slow and their applications limited.

But when these sciences joined company they drew from each other fresh vitality and thenceforward marched on at a rapid pace towards perfection.

Joseph Louis Lagrange.

Bussum, November 1987 Michiel Hazewinkel

CONTENTS

Introduction to Volume 2

VARIATIONAL PRINCIPLES OF CONTINUUM MECHANICS, INTRODUCTION TO OPTIMAL DESIGN THEORY

The study of variational problems usually starts with the optimization of some functional $J(f)$, where f belongs to a class of admissible functions defined in a fixed region Ω of a Euclidean space and obeys certain a-priori specified constraints. The problem of minimizing $J(f(\underset{\sim}{x}))$, $x \in \Omega$, is equivalent to finding a solution $\hat{f}(x)$ to an equation $L(f) = q$ where L is, generally a differential operator and q is a known function. More frequently problems modeled by some differential equations are formulated in a "variational form". Instead of "solving" a differential system one can attempt to find a function that assigns an extremal value to a functional.

Problems of this type restated in the form: minimize a functional $J(f)$: $H \to \mathbb{R}$, where f is allowed to vary in a class of admissible functions H, belong to calculus of variations.

The restatement of laws of continuum mechanics in a variational form, which was the principal topic of Volume 1 of this work, has been the subject of intensive investigation for almost two centuries, with some of most illustrious names in mathematics and physics associated with it. However, the classical formulation of problems in the calculus of variations is only one of the possible problems arising in the optimization of functionals. Let us consider a specific example. The Saint Venant's problem of pure torsion is modeled by the equation

$$\Delta\,\Phi = f(x,y),\; x,\; y \in \Omega \subset \mathbb{R}^2, \Delta \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2,$$

where Ω is the two-dimensional compact region occupied by the shaft's cross-sectional area. $\phi(x,y)$ is the stress function, such that

$$\frac{\partial \Phi}{\partial x} = \tau_{yz} , \quad \frac{\partial \Phi}{\partial y} = - \tau_{xz}$$

(τ_{yz}, τ_{xz} are the shear stress components, in the usual engineering notation),with all other stress components assumed to be identically equal to zero . Φ vanishes on the boundary, i.e. $\Phi|_\Gamma \equiv 0$.

In the usual formulation of the Saint Venant's problem $f(x,y) \equiv$ constant, the constant is chosen to be (-2) for convenience. The same problem arises in the modeling of the static deflection of a membrane. If Ω denotes the region occupied by the membrane in the x,y - plane, $f(x,y)$-the pressure, $\rho(x,y)$- the mass per unit area, the potential energy assumes the form

$$(0.2) \quad V = \int_\Omega \rho(x,y) \cdot | \text{ grad } \Phi(x,y) |^2 dxdy - \int_\Omega (f\Phi) dxdy,$$

where $\Phi(x,y)$ denotes the deflection of the membrane in the z-direction (i.e. perpendicular to the x,y - plane).

The corresponding Euler-Lagrange system is

$$\begin{cases} -\text{grad } (\rho \cdot \text{grad } \Phi) = f , \\ \Phi|_\Gamma \equiv 0. \end{cases}$$

We see that a number of problems can be formulated.

a) The classical problem of the calculus of variations:
Find a function Φ in the Sobolëv space

$H^1_0(\Omega)$ that minimizes the functional (0.2)

for a given distribution of pressure $f(x,y)$ and a given shape of the simply connected domain Ω (therefore for a given boundary Γ).

b) Subject to some constraints, such as

$\int_{\Omega} f(x,y)\, dxdy = 1$, and $f_1 < |f| < f_2$, find

$f(x,y)$ in some admissible class F, such that $f(x,y)$ assigns an optimal value to the functional $V(\Phi)$. Here Ω and $\rho(x,y)$ are given.

c) Given $f(x,y)$, and the domain Ω find $\rho(x,y)$ such that $V(\Phi(\rho))$ assumes an optimal value.

(d) Given $\rho(x,y)$, $f(x,y)$ in some region C, with measure(C) $\geq$ 1, find $\Omega \subset C$, with

$\int_{C-\Omega} dxdy = 1$, such that $V(\Phi,\Omega)$ assumes

optimal value.

We have been deliberately vague about the optimality requirements. "Optimal" could mean minimal, maximal, "close" to some specific value. $V(\Phi,\Omega)$ does not have to be the energy, and its optimum could mean that Φ approximates in some sense a given deflection, or that the deflection at some specific points assumes preassigned values, or a number of other "optimality" criteria.

The problem (a) was designated as a classical problem. It is the problem discussed extensively in almost any Calculus of Variations text.

Problem (b) involves optimization of the forcing (or control) term. What external forces do we need to apply to the membrane to optimize the cost functional?

If we replace the static problem by a dynamic one (vibration of a membrane) we can find an abundance of modern literature (i.e. after 1945) dealing with this class of problems. It is one of the basic problems of modern control theory.

Problem (c) deals with design of the thickness or mass distribution of the membrane. Some authors refer to such problems as "control of the coefficients." Rapid progress in understanding the problem (c) was made in the late 70-s and the 80-s. The continuous dependence of energy on coefficients regarded as vectors in a Hilbert space turned out to be true in most cases and false in some. Thus, a straightforward Fréchet or, Gateaux differentiation sometimes could produce a convergent numerical algorithm, and at other times could lead to obviously incorrect designs. Problem (d) is probably the least investigated and beast amenable to purely heuristic manipulations. Recently it became a "red hot" item of research, with the "French school" taking the lead in advancing the 1910 idea of Hadamard. It involved variation of the Green's function for the problem, due to a small perturbation of the domain in a manner similar to the vanishing of the first variation in problem (a).

The problems of the type (c) and (d) are the primary subject of this volume. All problems (a)-(d) are usually attacked by techniques that can be regarded as abstract differentiation . The value of the functional is extreme if either the variable quantity lies on the boundary of the admissible region in appropriately chosen space, or else if some abstract first derivative vanishes. These two possibilities are exactly mirrored in some maximal principles (Pontryagin's theory), bang-bang principles in the former case or else in the Gauss-Hertz principle, or the zero sensitivity postulate, or in Hadamard's formulation for the Green's function in the latter case. Thus, we are able to unify several seemingly disconnected ideas and to reexamine critically the corresponding numerical schemes. Moreover, some common features of all of the problems labelled (a) - (d) become quite clear, and the difficulties also appear to have some common origins and are generally related to the lack of smoothness or to the poor choice of what is "admissible".

Chapter 1

Changes of Coordinates and Variation of the Coefficients

1.1 The state space.

Problems of engineering design involve a "cost functional", constraints, and admissibility considerations.

In principle, at least, we wish to design some mechanical or structural project as cheaply as possible, minimizing "a cost functional". We must obey some rules, that is constraints, some laws of physics, which we have no power to alter, and some manufacturing limitations. Also we need to comply with some specifications. For example the bridge must be able to withstand reasonable traffic and wind loads, a machine must be able to operate for a reasonable period of time without excessive wear, a circuit must be able to withstand an unexpected surge of current. Thus, we have a number of constraints imposed on the optimization problem.

Finally, we must decide the admissibility of the mathematical model. But a first step in any modeling must be the decision regarding the choice of coordinates and the mathematical description of the "state of the system".

The underlying frame of reference is commonly based on Euclidean space R^3 with Cartesian coordinates $\underset{\sim}{x} = \{x,y,z\}$ or $\underset{\sim}{x} = \{x_1, x_2, x_3\}$ with the undefined concept of points in that space. A mechanical system (or a continuum) is said to have n-degrees of freedom if its configuration or state is completely defined by n- independent coordinates $\{q^i\}$, $i = 1,2,\ldots n$. We assume that "admissible" variables of system $\{q^i\}$ form a local coordinate cover. At each point $\tilde{q}$ in the configuration space there is a neighborhood of $\tilde{q}$ spanned by $\{q^i\}$, which is locally Euclidean. This means that there is a mapping from the neighborhood $N_{\tilde{q}}$ of $\tilde{q}$ into some neighborhood of zero in the Euclidean space that is an isomorphism (it is one to one with a unique inverse). The configuration space with n-degrees of freedom, or the state space is a set $S \subset R^n$ with a local coordinate cover $\{q^i\}$, $i = 1,2,...n$; (In fact S is a manifold.)

The Kinematic event space is a subset of $R \times S$, that is an ordered pair $\{t, q^i(t)\}$, with t interpreted as time and $\{q^i(t)\}$ as the state of the system at time t.

We assume Newton's rather than Einstein's or even more recent interpretation of the concept of time. All events are well ordered with respect to time axis, which is an isomorph of the real line. Two separate events are always universally ordered with respect to all observers. Either event one preceeds event two, or they are simultaneous, or else event two preceeds event one. Time and state space are independent. Motion of a particle (that is of a single point) is described by a parameterized path : $\underset{\sim}{x} = \underset{\sim}{x}(t)$, $t \geq t_o$, or a map $t(\in R) \to R^n$. We insist that this map is defined for either a discrete system (with n-degrees of freedom) or for a continuum.

1.2 A change of coordinates.

The study of motion of "points" in a continuum, may consist in effect of taking a ride on point particles of the continuum in its motion.

Let $\underset{\sim}{\xi}(t_o)$ be the position of some arbitrary point of the continuum at some (call it initial) time t_o. We regard the collection of particle paths as a system evolving from its initial state at t_o. The position at time t is given by the relation

(1.1) $\underset{\sim}{x} = \underset{\sim}{x}(\underset{\sim}{\xi}, t)$ with $\underset{\sim}{x}(\underset{\sim}{\xi}, t_o) = \underset{\sim}{\xi}$.

$\underset{\sim}{x}$ is regarded as a function of $\underset{\sim}{\xi}$ and t, while the time t is indentified with an infinite ray of the real line.

The state space S is a subset of $\mathbf{R}^n$, or of a Hilbert space $\mathbf{H}$. Without any loss of generality we can assume $t_o = 0$, that is identify the time ray with $\mathbf{R}_+$.

We insist that the map $t \to S \subset \mathbf{R}^n$ or $t \to S \subset \mathbf{H}$ is defined.

For each instant $t \in \mathbf{R}_+$ we have only one possible configuration $\{\underset{\sim}{q}\} \in S$ of the system. We refer to this uniqueness property by calling the system deterministic. Knowing $\underset{\sim}{q}(0)$ we can determine (at least in principle) the unique state of the system at any future time $t > t_o$. The motion of each "point" or "particle" $\underset{\sim}{x}(t) \in \mathbf{R}^3$ follows that point according to an equation

(1.1) $\underset{\sim}{x} = \underset{\sim}{x}(\underset{\sim}{\xi}, t)$, $\underset{\sim}{x}(\underset{\sim}{\xi}, 0) = \underset{\sim}{\xi}$

If each particle's path can be "retracted", that is if we can uniquely solve for $\underset{\sim}{\xi}$, given the relation $\underset{\sim}{x}(\underset{\sim}{\xi}, t) = \hat{\underset{\sim}{x}}(t)$ about the position $\hat{\underset{\sim}{x}}$ at time t, we call this property the <u>solenoidal</u> property, or the <u>impenetrability</u>. We assert that two paths cannot cross each other, and only one particle can occupy some point $\underset{\sim}{x}$ in the Euclidean space at a given time.

We can regard $\underset{\sim}{x}$, t as space variables $\underset{\sim}{\xi}$, t as material variables. It is unfortunate that we perpetuate a confused notation in which $\underset{\sim}{x}$ sometimes denotes a coordinate system and sometimes a function of time, when we identify $\underset{\sim}{x}$

with the position of a particle in the coordinate system designated by the same symbol, and sometimes an ordered n-tuple corresponding to a specific evaluation of the function $t \to \underset{\sim}{x}(t)$.

We discuss possible ways of clarifying this confusing notation in an appendix. For the time-being we shall perpetuate the mess created by our predecessors, where the usual notation fails to distinguish between a function and its range.

For a point-particle we specify the mass density by introducing the Dirac delta measure. (Read [1], [2], or [3]).

$$(1.2)\begin{cases} \rho(\underset{\sim}{x}, t) = \rho_o \cdot \delta(\underset{\sim}{x}(t) - \underset{\sim}{x}_o) \\ \rho(\underset{\sim}{x}(\underset{\sim}{\xi}), t) = \rho_o \cdot \delta(\underset{\sim}{x}(\underset{\sim}{\xi}) - \underset{\sim}{x}_o(\xi)) \end{cases}$$

Here $\{\underset{\sim}{x}, t\}$ - are spatial independent variables

$\{\underset{\sim}{\xi}, t\}$ - are "material" coordinates, or material independent variables.

ρ_o is some constant associated with the mass of the particle in a suitably chosen system of (physical) units. The "point in space" and "position of a particle" are interchangable concepts, corresponding to the coordinate transformation (1.1):

$$\underset{\sim}{x} = \underset{\sim}{x}(\underset{\sim}{\xi}, t).$$

The density of a mass at a point, or the mass density function is given

$$\rho(\underset{\sim}{x}(\underset{\sim}{\xi}, t)) = \mu(\underset{\sim}{\xi}, t)$$

that could be a "genuine" (that is locally integrable) function, or it could be a generalized function (for example Dirac delta, or its derivative). Two concepts are of primary importance in continuum mechanics: the position and all of its pertinent derivatives, and the mass density and its derivatives.

We can regard the position of a particle at

$\underset{\sim}{x}$ and its velocity $\underset{\sim}{V}(\underset{\sim}{x},t)$ as basic independent quantities. The validity of spatial-material coordinate transformation implies

(1.4) $\underset{\sim}{v}(\underset{\sim}{x},t) = \underset{\sim}{v}(\underset{\sim}{x}(\underset{\sim}{\xi}, t), t) = \mathbf{V}(\underset{\sim}{\xi}, t).$

The chain rule implies that

(1.5) $$\frac{\partial \mu(\underset{\sim}{\xi}, t)}{\partial t} = \sum_{i=1}^{3} \frac{\partial \rho}{\partial x^i} \cdot \frac{\partial x^i}{\partial t}\Big|_{x^i = x^i(\underset{\sim}{\xi},t)}$$

$$= \sum \frac{\partial \rho(\underset{\sim}{x}, t)}{\partial x^i} \cdot V^i(\underset{\sim}{\xi}, t)$$

On the other hand

(1.6) $$\frac{\partial \mu(\xi, t)}{\partial t}\Big|_{\underset{\sim}{\xi} = \underset{\sim}{\xi}(\underset{\sim}{x}, t)} = \sum \frac{\partial \rho(\underset{\sim}{x}, t)}{\partial x^i} \cdot V^i(\underset{\sim}{x},t)$$

$$+ \frac{\partial \rho(\underset{\sim}{x}, t)}{\partial t} = \frac{D\rho}{Dt}(\underset{\sim}{x}, t), \text{ or}$$

(1.7) $$\frac{\partial \mu(\xi, t)}{\partial t}\Big|_{\underset{\sim}{\xi} = \underset{\sim}{\xi}(\underset{\sim}{x}, t)} \equiv \frac{D\rho(\underset{\sim}{x}, t)}{Dt}$$

$$= \frac{\partial \rho(\underset{\sim}{x},t)}{\partial t} + (\underset{\sim}{v}(\underset{\sim}{x},t) \cdot \nabla \rho(\underset{\sim}{x},t)),$$

where $\cdot$ denotes the usual dot product in three dimensions.

1.3 Conservation of mass, conservation of energy.

If in a deformation process we observe the evolution of some bounded region Ω_o occupied by the material, that is bounded at time $t_o = 0$, so that tracing the motion of each point of we associate bounded open regions Ω_t with each time instant t. The mass conservation property is expressed by the equation

$$(1.8)\quad \frac{d}{dt}\left\{ \iiint_{\Omega_t} \rho(\underset{\sim}{x},\ t)\cdot d\underset{\sim}{x} \right\} = 0.$$

In fact any "conservation" property assumes an identical form. If, for example, $e(\underset{\sim}{x},\ t)$ is some form of energy density, then the energy conservation law is given by

$$(1.9)\quad \frac{d}{dt}\left\{ \iiint_{\Omega_t} e(\underset{\sim}{x},\ t)\ d\underset{\sim}{x} \right\} = 0.$$

The mapping $t \overset{\mathcal{D}}{\rightarrow} \Omega_t$ ($\mathbb{R} \rightarrow \mathbb{R}^3$), $\Omega(t) = \Omega_t$, $\Omega(0) = \Omega_o$ is assumed to be continuous with respect to the norm $\|\cdot\|$, where $\|\Omega_1 - \Omega_2\| = \sup \|\underset{\sim}{x} - \underset{\sim}{y}\|$, $\underset{\sim}{x} \in \Omega_1$, $\underset{\sim}{y} \in \Omega_2$. It is easy to check that it is a norm. Since $\|\Omega\| = \sup_{\underset{\sim}{x}\in\Omega} \|\underset{\sim}{x}\|$, obviously $\|c\ \Omega\| = |c|\ \|\Omega\|$ for any $c \in \mathbb{R}$, and $\|\Omega\| > 0$ if and only if $\Omega \neq \emptyset$, and the triangular inequality is satisfied. Therefore, continuity of the map $t \rightarrow \Omega_t$ is defined. (Given $\varepsilon > 0$ there exists $\delta > 0$ such that $\|\Omega_{t_1} - \Omega_{t_2}\| < \varepsilon$ whenever $|t_1 - t_2| < \delta$.)

We refer to a functional $I(\Omega_t)$ as the invariant of motion if $\frac{dI(\Omega_t)}{dt} \equiv 0.$ For example, the total mass contained in the region Ω_t is invariant if

$$(1.10)\quad \frac{d}{dt} \iiint_{\Omega_t} \rho(\underset{\sim}{x},t)\, d\underset{\sim}{x} = \frac{d}{dt}\, I(\Omega_t) \equiv 0,$$

for all $t \in \mathbb{R}_+$.

Refering the deformation process to the material coordinates we derive

$$(1.11)\quad \frac{d\, I\,(\Omega_t)}{dt} = \iiint_{\Omega_o} \left[\frac{\partial \mu(\xi,t)}{\partial t} J(x/\xi,(t)) + \mu(\xi,t)\frac{\partial J(\underset{\sim}{x}/\underset{\sim}{\xi},\ (t))}{\partial t}\right] d\,\underset{\sim}{\xi}\ ,\quad \text{where}$$

$J(\underset{\sim}{x}/\underset{\sim}{\xi},\ (t))$ is the time-dependent Jacobian

$$J(x/\xi\ ,(t)) = \left|\frac{\partial x_i(\underset{\sim}{\xi})}{\partial \xi_j}\right| .$$

An intuitive meaning of the Jacobian $J(x/\xi)$ is as follows: $J(x/\xi)$ is a function of time representing the ratio of volume of an infinitesimal element that occupies the volume $d\underset{\sim}{x}$ compared to the initial volume (at time $t_o = 0$) at position ξ. The Euler's formula (sometimes called Euler's expansion formula) relates the rate of change of the Jacobian $J(\underset{\sim}{x}/\underset{\sim}{\xi}\ ,(t))$ to the divergence of the velocity vector.

$$\frac{\partial J(x/\xi)}{\partial t} = [\nabla_{\underset{\sim}{x}} \cdot \underset{\sim}{v}\ (\underset{\sim}{x})]\cdot J\ (x/\xi),$$

where $\nabla_x \cdot \underset{\sim}{v}(\underset{\sim}{x}) = \dfrac{\partial v_1(\underset{\sim}{x})}{\partial x_1} + \dfrac{\partial v_2(\underset{\sim}{x})}{\partial x_2} + \dfrac{\partial v_3(\underset{\sim}{x})}{\partial x_3}$.

Two so called "fundamental theorems of calculus of variations" were stated by du Bois-Raymond. Let us restate them in a slightly more modern terminology.

<u>Theorem 1.1</u> If in a given domain Ω, the functional relation: $\langle f,g\rangle_\Omega = \iiint_\Omega [f(\underset{\sim}{x})\cdot g(\underset{\sim}{x})]\, d\underset{\sim}{x} = 0$

is true for any $g(x) \in L_2(\Omega)$, (or only for any

$g(\underset{\sim}{x})$ in a set of functions G that is complete in Ω), then $f(\underset{\sim}{x}) = 0$ almost everywhere in Ω.

Note: a set $G \subseteq L_2(\Omega)$ is complete in $L_2(\Omega)$ if for any $g \in G$, $\langle h,g \rangle = 0$ implies that $h \in G$. That is, any function orthogonal to g is in G.

Theorem 1.2. If $\iiint_{\Omega_i} f(\underset{\sim}{x})\, d\underset{\sim}{x} = 0$, $f \in L_2(\Omega)$ for any Lebesgue measurable subset $\Omega_i \subset \Omega \subset R^3$, then $f(\underset{\sim}{x}) \equiv 0$ almost everywhere in Ω. Theorems 1.1 and 1.2 can be simplified if we can assume that $f(\underset{\sim}{x})$ is a continuous function of $\underset{\sim}{x}$ in Ω. Specifically, we can replace "almost everywhere in Ω" by "everywhere in Ω".

A simple consequence of theorem 1.2 combined with Euler's expansion formula is the fundamental mass conservation equation of Bernoulli and Euler.

$$(1.12) \quad \frac{D\rho(\underset{\sim}{x})}{Dt} + \rho(\nabla_x \cdot \underset{\sim}{v}) = \frac{\partial \rho(\underset{\sim}{x}(t))}{\partial t} + \nabla_x \cdot (\rho \underset{\sim}{v}) \equiv 0.$$

That is a restatement of equations (1.10) and (1.11).

One can also establish a transformation between the acceleration components expressed in spatial and material coordinates

$$\left.\frac{\partial\, \underset{\sim}{V}(\underset{\sim}{\xi},t)}{\partial t}\right|_{\underset{\sim}{\xi} = \underset{\sim}{\xi}(\underset{\sim}{x},t)} = \frac{D\, \underset{\sim}{v}(\underset{\sim}{x},t)}{Dt} =$$

$\frac{\partial \underset{\sim}{v}}{\partial t} + (\underset{\sim}{v} \cdot \nabla)\, \underset{\sim}{v}$. Let us suppose that the