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G. Eilenberger

Solitons

Mathematical Methods for Physicists

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With 31 Figures

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Preface

This book was written in connection with a graduate-level course in theoretical physics at the University of Cologne. The required mathematical background is that which is usually required for courses in theoretical physics, namely an elementary knowledge of function theory, differential equations, and operators in Hilbert space.

The main topic covered in the lectures was a fairly detailed introduction to inverse scattering theory as it is applied to one-dimensional systems exhibiting solitons and the new mathematical ideas and methods developed in this connection. These have been treated in a manner and language appropriate for physicists. Thus, not all topics are treated with full mathematical rigor, which might have resulted in smothering the important and interesting new ideas in too many nonessential details.

The aim of the book is more to offer those who might want to investigate applications of the systems treated here a self-contained introduction which would spare them a tedious search of the original literature.

The material presented here is thus formal in nature - new mathematical methods in physics. Practical applications exist in almost every area of physics as well as in related areas, from plasma and solid state physics to elementary particle theory, and from communications technology to meteorology. A comprehensible presentation of all of these would go far beyond the limits of a normal-sized book and could not be presented in a coherent fashion. In fact, only a selected part of the formal mathematical aspects of the theory of one-dimensional solitons is presented. For example, the quantum mechanical treatment of solitons is not touched on at all, although this is a research area of great current interest. Originally, it was planned to include a chapter devoted to topological solitons in several spatial dimensions, since these objects, which were once interesting only in quantum field theory, are becoming increasingly important for the interpretation of phenomena in solids. Unfortunately however, it turned out that this would have doubled both the contents and preparation time of this book and the idea was regretfully abandoned.

This book is organized as follows.
The first chapter provides an introduction to the subject in terms of simple examples and describes some possible applications. After this, the Korteweg-deVries (KdV) equation, as the simplest example of an equation with soliton solutions, is investigated in Chaps. 2 and 3. The inverse scattering transformation and its application are treated in detail in Chap. 3. The techniques developed there are generalized to other soliton systems in Chap. 4 and are applied to a discrete system (of difference equations) - the Toda lattice - in Chap. 7. Chapter 5 is devoted to the discussion of the sine-Gordon equation and its solutions, since this is the most interesting special case (of those developed in Chap. 4) for physicists. Finally, an introduction to the thermodynamics of soliton systems will be given in Chap. 6, using the sine-Gordon equation as example. The questions raised there have only been partially answered and deal with currently interesting research problems.

The results and methods presented in this book come from many sources and are sometimes not readily obtainable from the available literature. Some aspects are quite new. Since the author wanted to provide a self-contained introduction rather than a review article, explicit citations in the text are, for the most part, omitted. An annotated list of literature suitable for further study is given in an appendix.

In conclusion, I hope that this presentation will not only impart new knowledge, but will also provide the reader with the same aesthetic enjoyment which I, as author, had while "discovering" (from the literature) and summarizing this fascinating system of theory and methods.

I am particularly grateful to Dr. E. Borie, Karlsruhe, who translated the original German text into English. The text has been greatly improved by her criticism and her willingness at all times to find appropriate formulations. I also thank numerous colleagues for helpful criticism and for drawing my attention to typographical errors in the formulae and unclear statements in the original version. Finally, I should like to thank Miss Ch. Arnaud for her unfailing patience in typing the numerous versions and corrections of the original German text. I am extremely indebted to Ulrich Kursawe, who traced out and eliminated the abundant typographical errors in the formulas of the first edition and to Dr. Kok of Groningen and his students who kindly provided a list of corrigenda.

Jülich, May 1983 *G. Eilenberger*

Contents

1. Introduction

1.1 Why Study Solitons ?

The last century of physics, which was initiated by Maxwell's completion of the theory of electromagnetism, can, with some justification, be called the era of *linear* physics. With few exceptions, the methods of theoretical physics have been dominated by linear equations (Maxwell, Schrödinger), linear mathematical objects (vector spaces, in particular Hilbert spaces), and linear methods (Fourier transforms, perturbation theory, linear response theory).

Naturally the importance of nonlinearity, beginning with the Navier-Stokes equations and continuing to gravitation theory and the interactions of particles in solids, nuclei, and quantized fields, was recognized. However, it was hardly possible to treat the effects of nonlinearity, except as a perturbation to the basis solutions of the linearized theory.

During the last decade, it has become more widely recognized in many areas of "field physics" that nonlinearity can result in qualitatively new phenomena which cannot be constructed via perturbation theory starting from linearized equations. By "field physics" we mean all those areas of theoretical physics for which the description of physical phenomena leads one to consider field equations, or partial differential equations of the form

$$\phi_t \quad \text{or} \quad \phi_{tt} = F(\phi,\ \phi_x\ \dots) \tag{1.1.1}$$

for one- or many-component "fields" $\phi(t,x,y,\ \dots)$ (or their quantum analogs). These include classical areas, such as hydro- or magnetohydrodynamics, and thus also some areas of meteorology, oceanography, and plasma physics, as well as newer areas such as solid state physics, nonlinear optics, and elementary particle physics.

It has been known for a long time that nonlinearities can result in fundamentally new phenomena. One needs to think only of shock waves in aerodynamics, or of cyclones in meteorology. A more recent characteristic example can be found in the Ginzburg-Landau theory of superconductivity. This is a system of nonlinear coupled differential equations for the vector potential and the wave function of the superfluid condensate. A simple global condition on the solutions, namely the uniqueness of the phase of the wave function, results in the existence of magnetic "flux tubes" with flux hc/(2e). In this expression, it is to be noticed that the coupling constant $g = e/c$ between the vector potential and the current appears in the *denominator*; such a result can never be obtained from perturbation theory, i.e., from a power series in the coupling constant. Superconducting flux tubes are a prototype for "topological" solitons, which are so called because their stability is guaranteed by a topological constraint: invariance under a change of 2π in the phase of the wave function arbitrarily far from the center of the flux tube.

The equation for a scalar field $\phi(x,t)$,

$$\phi_t = \frac{1}{2}\phi_{xx} + (a - \phi)(\phi^2 - 1) \quad , \quad -1 < a < 0 \tag{1.1.2}$$

provides a very different but instructive example. It was considered as a simple model for one aspect of the propagation of nerve pulses. Although the differential part of this equation resembles the diffusion equation, it has a solitary wave as a special solution, the "nerve pulse"

$$\phi = \tanh(x - at) \quad . \tag{1.1.3}$$

This solution of (1.1.2) can be interpreted in a mechanical model as the friction-dominated (hence ϕ_t) motion of an elastic (hence ϕ_{xx}) string which slips out of a potential trough at $\phi = -1$, parallel to the x-axis over a potential maximum at $\phi = a$, into a deeper lying trough at $\phi = +1$. It is also clear that this solution cannot be obtained as a perturbation of the two linearized equations which describe small vibrations about $\phi = \pm 1$.

A characteristic property of such solitary waves is the constance in time of their wave form and velocity; they represent wave packets (or in some cases energy packets) which do not spread. The effect of dispersion is compensated by the effect of nonlinearity. One can construct such solitary waves (which can be regarded as solitons in a broader sense) as solutions of many partial differential equations by means of the ansatz

$$\phi(\underline{x},t) = \phi(\underline{k}\underline{x} - \omega t) \quad , \tag{1.1.4}$$

where $\underline{x} = (x_1, \ldots, x_n)$ and ϕ can have several components.

This aspect is one of the two foundations of the increasingly important soliton concept in field physics, and we will consider it more thoroughly. The other foundation is the topological aspect, which we cannot go into here. If the field equations have particular solutions of the form (1.1.4), it is attractive to regard these as elementary excitations, or "quasiparticles", and to attempt to construct the complete solution to the initial value problem (Cauchy problem) for (1.1.1) from such solutions, insofar as this is possible. Thus one would have taken the nonlinearity into account from the beginning. The practicability of this program depends, of course, on whether one can take into account the interaction between the solitary waves, since the superposition principle is not valid for nonlinear equations.

As one might expect, this is not always possible. However, one of the surprising discoveries of the last decade in mathematical physics is that there exists a rather large number of specific nonlinear evolution equations (NLE), mostly in one spatial dimension, which permit a complete analytical treatment within the framework of the above program. Solutions to these equations exhibit true solitons, in the sense the term is being used by mathematicians. The physics community has not generally accepted this strict terminology but rather talks about solitons whenever "lumps" of the field under consideration move around (in the sense of a) below). These equations and their soliton solutions are characterized by the following properties:

a) They have "particle-like" solutions (solitary waves)

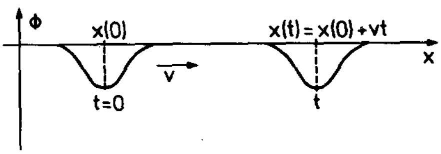

Fig. 1. Solitary wave motion

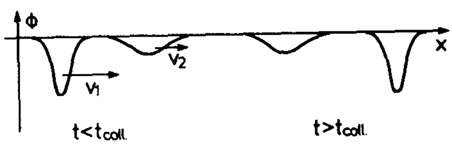

Fig. 2. Interpenetration of solitons with different velocity

b) These solitary waves can penetrate each other without destroying each other in spite of their nonlinear interaction, as schematically illustrated below, even when arbitrarily many such "solitons" are superposed.

c) The initial value problem can be solved analytically as a development in terms of solitons and continuous "radiation", in analogy with Fourier analysis in the case of linear equations.

d) If the system of equations is formulated as a Lagrangian field theory, it has infinitely many constants of the motion other than energy and momentum. This is related to the complete solubility of the inital value problem.

e) If the solutions $\phi(x,t)$ of these equations are used as the potential in a suitable linear eigenvalue equation,

$$D\cdot\psi + M[\phi(x,t)]\cdot\psi = \lambda(t)\psi \quad , \tag{1.1.5}$$

where D is a differential operator in x and the time t is regarded as a fixed parameter, then λ is independent of time. The analytical methods of solution (Chaps. 3, 4, 6) are based on this fact.

For every operator of the form (1.1.5) it is possible to find a denumerable set of nonlinear evolution equations which exhibit solitons in the sense of a)-e). The higher-order members of such a set are extremely complicated; only a few simple systems in each series are physically interesting. The best-known examples are:

the Korteweg-de Vries (KdV) equation	$\phi_t = 6\phi\phi_x - \phi_{xxx}$
the sine-Gordon equation (SGE)	$\phi_{tt} = \phi_{xx} - \sin\phi$
the nonlinear Schrödinger equation	$i\phi_t = -\phi_{xx} \pm \lvert\phi\rvert^2\phi$.

Equations with these properties are certainly mathematically singular in a certain sense, and although no algorithm for determining whether a given nonlinear evolution equation belongs to this category or not exists at the present time, it is reasonable to assume that the majority of field equations which one encounters in the physics of condensed matter are not precisely of this kind.

The justification for the great physical interest of these singular equations in spite of this is due to the fact that many nonlinear systems of equations can be approximated by such soliton exhibiting NLE. These provide a better starting point, or better zeroth approximation, than the linearized equations in many cases, as we have previously indicated. The real system's departures from these special equations, whether they be due to friction, spatial variation of the coefficients, or other homogeneous or inhomogeneous terms in the equations, must be handled subsequently in a perturbative treatment of the soliton degrees of freedom. Celestial mechanics provides a useful analogy. The astronomical Kepler orbits are solutions of Newton's equations of motion for a potential proportional to r^{-1}, which does not exist in unperturbed form. The departures from these orbits are not taken into account by calculating the planetary orbits from the beginning, including all their mutual interactions, but rather by allowing for a slow time dependence of the parameters of the unperturbed Kepler orbits. The use of linearized evolution equations instead of those exhibiting solitons would correspond in this analogy to using force free motion in a straight line rather than the Kepler orbits as a lowest order approximation.

The three previously mentioned NLE which exhibit solitons are particularly important for physical applications (see [1.1 and 1.2]). If one is investigating a physical system which, to a first approximation, is described by the linear wave equation

$$\phi_{tt} - \Delta\phi = 0 \tag{1.1.6}$$

(Δ is the Laplacian), then there are two limiting cases at the next level of approximation which are particularly well suited to take into account the effects of dispersion and the interaction of wave packets.

In the case of small amplitudes and relatively long wavelength, so that dispersion (or more precisely, the departures from linear dispersion, $\omega = \pm k$) and nonlinearity are only important relative to the leading terms (1.1.6) over long time spans, we are dealing with the "hydrodynamic" limit. Then, as will be shown in the next chapter, it is possible to derive the KdV equation for the amplitude under quite general assumptions. This is true for the original theory of shallow water waves (KORTEWEG and deVRIES, 1895: see [1.3]) as well as for the description of ion plasma waves, waves in elastic media, and many others.

In the short wavelength limit, one wishes to discuss a nearly monochromatic wave train, whose envelope is slowly varying (compared with the wavelength and period) in space and time, and for which the nonlinearity is also slowly varying, and thus couples only to the envelope. This situation occurs frequently in nonlinear optics and can be described by the nonlinear Schrödinger equation, again under very general conditions.

The sine-Gordon equation has almost become ubiquitous in the theory of condensed matter, since it is the simplest wave equation in a periodic medium. It is well known to many solid state physicists as the Josephson equation for the propagation of flux quanta in sandwich type superconducting tunnel junctions. It has also been used as a model for domain walls in ferromagnets and ferroelectrics, for the propagation of charge density waves in one-dimensional metals, and for the motion of dislocations in crystals, or for adsorbate molecules on surfaces. The sine-Gordon equation is also familiar in other areas, and it plays a role in the theory of wave propagation in lipid membranes, for the description of self-induced transparency in laser physics, and as a model for elementary particles. This last application has been especially interesting due to the proof that its quantized form (with ϕ regarded as a *boson* field) is equivalent to the Thirring-Luttinger model, which is a model for interacting *fermions* in one spatial dimension [1.4].

The utility of being able to handle such equations should herewith be evident. A complete enumeration of all the known interesting equations or possible applications would be even longer. In addition, as the methods described here become better known, the discovery and application of equations which exhibit solitons will increasingly attract the attention of physicists in all fields of research. We are entering the era of nonlinear physics.

1.2 Basic Concepts Illustrated by Simple Examples

Nonlinear evolution equations, the objects being investigated, are equations of the form (1.1.1)

$$\phi_t \quad \text{or} \quad \phi_{tt} = F(\phi, \phi_x \ldots) \quad , \tag{1.2.1}$$

where ϕ is a one or several component, real or complex function of the time t and spatial variables $x_1, x_2, \ldots$. For the most part, we will discuss only functions of a single spatial variable x, corresponding to the present state of the art. These are called evolution equations because one is interested in the time development of the function ϕ, for which at a given time t_o, the *initial conditions*

$$\phi(t_o,x) \quad \text{and, if necessary, also} \quad \phi_t(t_o,x) \tag{1.2.2}$$

are prescribed. We also require that *boundary conditions*, generally

$$\phi(t,\pm\infty) = 0 \quad \text{or} \quad \phi_x(t,\pm\infty) = 0 \quad , \tag{1.2.3}$$

be satisfied.

We will not be concerned with the question as to the conditions under which this is a mathematically well-defined problem. For the physicist, it is generally clear that a sufficiently smooth initial condition, which approaches the boundary value sufficiently rapidly with increasing |x| permits a unique time development according to (1.2.1), at least over a short time interval. Of course, it would be interesting to know under what circumstances

ϕ develops singularities after a finite time, or when it remains smooth and unique for all times. It is surprising that this remains a largely unsolved problem of the mathematical theory of partial differential equations [1.5].

We remind the reader of some basic terminology with illustrations from well-known equations.

The most important *linear* equation in physics is certainly the wave equation

$$\phi_{tt} - \phi_{xx} = 0 \quad . \tag{1.2.4}$$

As is the case for every linear equation, the superposition principle is valid, and it can be solved by the method of Fourier transforms. One obtains the familiar dispersion law

$$\omega = \pm k \tag{1.2.5}$$

and the

$$\begin{aligned} &\text{phase velocity } v_{ph} = \omega/k = \pm 1 \\ &\text{group velocity } v_{gr} = \partial\omega/\partial k = \pm 1 \quad . \end{aligned} \tag{1.2.6}$$

The fact that the right-hand side of (1.2.6) is constant is described as the absence of dispersion; wave packets do not spread, and the most general solution of (1.2.4) can be simply given in terms of the initial conditions:

$$\begin{aligned} \phi(x,t) &= \phi^{+}(x + t) + \phi^{-}(x - t) \\ \phi_x^{\pm}(x) &= \frac{1}{2} [\phi_x(x,o) \pm \phi_t(x,o)] \quad . \end{aligned} \tag{1.2.7}$$

This is a trivial example of the undistorted interpenetration of two wave packets.

The diffusion equation

$$\phi_t = \phi_{xx} \tag{1.2.8}$$

is also frequently encountered, but has very different properties. Here we obtain (by Fourier transformation) the dispersion law

$$\omega = - ik^2$$

and thus rapid spreading of wave packets. The solution is given by

$$\phi(x,t) = \int \frac{dk}{2\pi} \phi_k(t = 0) \, e^{ikx-k^2t} \quad . \tag{1.2.9}$$

If one reverses the direction of time and considers $\phi_t = -\phi_{xx}$, one obtains a simple example of an evolution equation whose solution can develop singularities within a finite time.

Like every other linear equation with constant coefficients, (1.2.8) has infinitely many *local constants of the motion*. These are defined as follows.

If for two functions N and I with arguments ϕ, ϕ_x, ... and possibly also explicitly x and t, the relation

$$\frac{d}{dt} N(\phi,\phi_x,\phi_t, \ldots) + \frac{d}{dx} I(\phi,\phi_x,\phi_t, \ldots) = 0 \tag{1.2.10}$$

is valid, then it follows that

$$\frac{dQ}{dt} = \frac{d}{dt} \int_{-\infty}^{\infty} N dx = 0 \quad . \tag{1.2.11}$$

That is, Q is a constant of the motion with local density N and corresponding local current I. Clearly, (1.2.8) is already in the form (1.2.10), and thus

$$\frac{d}{dt} \int_{-\infty}^{\infty} \phi dx = 0 \quad , \tag{1.2.12}$$

so that $\int \phi dx$ is a constant of the motion, as is each Fourier component

$$e^{k^2 t} \int_{-\infty}^{\infty} e^{-ikx} \phi dx = \phi_k(t = 0) \quad . \tag{1.2.13}$$

This is the reason for complete solubility according to (1.2.9).

Finally, we consider one instructive nonlinear example, the so-called "Burgers' equation"

$$\phi_t = 2\phi\phi_x + \phi_{xx} \quad , \tag{1.2.14}$$

which is a sort of diffusion equation with a nonlinear term. It can be rewritten in the form of a local conservation law

$$\phi_t - [\phi^2 + \phi_x]_x = 0 \quad . \tag{1.2.15}$$

Such nonlinear evolution equations in one spatial dimension permit linear transformations with seven free parameters:

$$t \to (t - t_0)/\tau \ , \quad x \to (x - x_0)/\xi \ , \quad \phi \to (\phi - \phi_0)/A$$

$$(1.2.15) \to (1.2.15)/B$$

so that it is possible to choose the constants which appear for convenience, without loss of generality. We will frequently make use of this fact without mentioning it again. We remark that, in contrast to the case of linear equations, the constants A and B have different effects.

If one considers the two terms on the right-hand side of (1.2.14) individually, one finds that

$$\phi_t = \phi_{xx}$$

results in the familiar spreading of wave packets, while

$$\phi_t = 2\phi\phi_x$$

has the formal solution (compare $\phi_t = v\phi_x$)

$$\phi = f(x + 2\phi t) \quad , \tag{1.2.16}$$

which can be represented graphically as shown below.

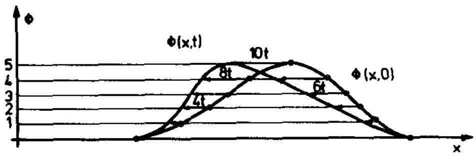

Fig. 3. Evolution of wave packet according to (1.2.16)

One sees that the nonlinear term by itself results in "breakers" and thus, for the proper initial conditions, leads to singularities in the solution after a finite time.

Both terms together result in stable behavior. One can solve (1.2.14) by means of the apparently artificial Hopf-Cole transformation (notice also the Bäcklund transformation of Sect. 4.7),

$$\begin{aligned} \psi_x &= \phi\psi \\ \psi_t &= [\phi^2 + \phi_x]\,\psi \quad , \end{aligned} \tag{1.2.17}$$

which is possible because the integrability condition [i.e., the equality of the mixed derivatives from the first and second of (1.2.17)] is precisely (1.2.14).

Using

$$\psi = \exp(\int^x \phi dx) \quad , \quad \phi = \psi_x/\psi \tag{1.2.18}$$

one eliminates ϕ, and obtains from the second of (1.2.17)

$$\psi_t = \psi_{xx} \quad . \qquad (1.2.19)$$

The initial value problem for Burgers' equation is thus exactly soluble. Solitary waves are obtained from the ansatz

$$\psi(x,t) = \psi(x - ct)$$

from which follows

$$\psi(x) = a\{1 + \exp[-c(x - x_0)]\}$$

and

$$\phi_s(x) = -c/\{1 + \exp[c(x - x_0)]\} \qquad (1.2.20)$$

as is shown below.

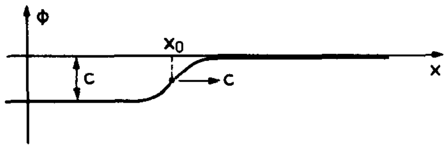

Fig. 4. Solitary wave of Burgers' equation

The exact solubility of Burgers' equation permits one to study the interaction between solitary waves. This is why we have chosen this equation as an example. From the solution of (1.2.19), we obtain two solitary waves: $\psi(x,t) = a\{1 + \exp[-c_1(x - x_1 - c_1t)] + \exp[-c_2(x - x_2 - c_2t)]\}$.

We show in Fig. 5 the result of the collision of two solitary waves propagating in opposite directions and in Fig. 6, in the same direction. The main result is that the faster solitary wave absorbs the slower one; we do not have true solitons in the sense of Sect. 1.1.

Fig. 5. $0 < c_1 < c_2$