Mathematical Methods in Science & Engineering

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PREFACE

In presenting to students this single-volume text on mathematical methods, containing all the more important mathematical material necessary for the majority of first and second year degree students of science and engineering, the author would draw attention to the following features.

(*a*) It is written for the large numbers of average students who require routine mathematical procedure. Having marked hundreds of examination scripts at this level, the author is painfully aware of the fact that so many students never seem to grasp even routine methods. The main object of the book is to present blackboard material in a concise way and to help students to solve problems.

(*b*) A knowledge at Advanced Level standard is assumed in the calculus, in algebra, co-ordinate geometry and in trigonometry, although many of the fundamental results in these subjects are restated and briefly derived.

(*c*) The exposition has not been unduly condensed in order that all the relevant subject matter should be included in one volume; rather the text has not been expanded unnecessarily. The author has sought to include what *should* be written and not what *could* be written, bearing chiefly in mind the present syllabuses of the University of London for engineering, chemical engineering, ancillary mathematics for special physics and chemistry, and for the part I of the general degree in science. Part II of the general degree and part III of the engineering degree must of necessity be excluded from any one-volume work.

(*d*) Each subject is treated as an entity in itself, being confined to its own individual chapter. This facilitates ease of reference, and also rapid revision when each subject can be surveyed as a whole.

(*e*) An attempt has been made to treat certain subjects with greater clarity than that found in other texts—the chapter on partial differentiation being an example in question.

(*f*) For all subjects, the theory given is concise, thereby making the text suitable for learning, reference and revision. Every topic dealt with is illustrated by means of worked examples, there being some 400 such examples throughout the text.

(*g*) At the end of each chapter sets of examples are provided, all the questions being of the type set in modern examinations. The questions have been carefully graded in order of difficulty, and also in subject matter, the order and grouping being identical with that in the text itself. The author has attempted to make these sets of examples more useful and more complete than other existing sets, and students are advised to use them systematically, not only when learning the subjects originally, but also for exercises in order to retain and revise methods learnt many months previously.

(*h*) Answers are provided at the end of each set of examples, thereby avoiding the usual awkward fumbling of pages at the end of the book.

(*i*) Since the stress is on *methods* throughout, the author has omitted both formal theory which has very little application at these elementary levels, and applications to science and engineering problems which lecturers may discuss according to the taste and interests of their classes. The author does not believe in hiding a simple exposition of fundamental techniques behind actual physical applications, however interesting and important these applications may be. The techniques are grasped in their own context first of all; then the applications rightly follow.

(*j*) The chapters on applied mathematics at the end of the book are limited in order to provide in one text a complete mathematics course according to the revised London syllabuses, which nowadays for engineering students are more restricted in applied mathematics than in former years.

(*k*) The final chapter on statistics is necessary these days, and it is thought that such a chapter is new in omnibus texts for degree students.

(*l*) Finally, the text is not meant for the few mathematical specialists who attend science and engineering courses. Such students require a more advanced treatment of all topics, including more rigour and a greatly extended syllabus. Many excellent texts are already available for such students.

The author would state that a large portion of this text was written when he was Senior Lecturer in mathematics at the West Ham College of Technology, London. He would express his thanks to Mr. D. C. Salinger, M.Sc., also Senior Lecturer in mathematics at that college, who kindly read the text and offered many useful comments and corrections. Very grateful acknowledgment is due to the Syndics of the Cambridge University Press and to the Senate of the University of London for permission to reproduce examination questions selected from papers set in the respective universities; such questions are marked with an asterisk.

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CONTENTS

CHAPTER 1

THE THEORY OF DETERMINANTS

1.1 The solution of two simultaneous linear equations

Determinants are introduced into algebra in order to simplify the manipulation and evaluation of otherwise complicated algebraical expressions. The theory may be developed by considering first of all determinants of the second order.

The two simultaneous equations containing the ratios $x:y:z$

$$\begin{aligned} a_1x + b_1y + c_1z &= 0 \\ a_2x + b_2y + c_2z &= 0 \end{aligned} \tag{1}$$

may be solved immediately by eliminating x and y in turn, yielding

$$\frac{x}{b_1c_2 - b_2c_1} = \frac{-y}{a_1c_2 - a_2c_1} = \frac{z}{a_1b_2 - a_2b_1}. \tag{2}$$

In the three denominators of these expressions, we have maintained the coefficients a, b and c in the order in which they occur in the given equations. These denominators contain expressions similar in form one to another, and in anticipation of the usefulness of the notation, we are led to write such expressions in the form

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \equiv a_1b_2 - a_2b_1. \tag{3}$$

The expression on the left is called a *second order determinant*, while the expression on the right is the *expansion* or value of this determinant. The four symbols a_1, a_2, b_1 and b_2 are called the *elements* of the determinant, while a_1 and b_1 form the first *row*, a_2 and b_2 the second row, a_1 and a_2 the first *column*, b_1 and b_2 the second column. The diagonal from the top left to the bottom right is termed the *leading diagonal.* The value of the determinant is then the product of the two elements occurring in the leading diagonal *minus* the product of the two elements in the remaining diagonal.

In this new notation, the ratios (2) may be written

$$\frac{x}{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}} = \frac{z}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}. \tag{4}$$

The alternating signs $+$, $-$, $+$ occurring in the numerators should be carefully noticed. The reader is recommended to remember the solution (4) in the verbal form:

x divided by the determinant formed from the coefficients by eliminating the coefficients of x

$= -y$ divided by the determinant formed from the coefficients by eliminating the coefficients of y

$=$ z divided by the determinant formed from the coefficients by eliminating the coefficients of z.

In equations (1), it should be noticed that *all* terms must be placed on the left hand side. In particular, if z is placed equal to 1, we evidently have the solution of two linear equations in the two unknowns x and y; if $a_1b_2 - a_2b_1 = 0$, either no solution exists at all, or no unique solution exists.

Example 1 We have the following three expansions:

$$\begin{vmatrix} 5 & 8 \\ 4 & 7 \end{vmatrix} = 5.7 - 4.8 = 3; \quad \begin{vmatrix} 4 & -5 \\ 3 & 6 \end{vmatrix} = 4.6 + 3.5 = 39;$$

$$\begin{vmatrix} -2 & -3 \\ 7 & 5 \end{vmatrix} = -2.5 + 7.3 = 11.$$

Example 2 Find the ratios $x:y:z$ in the equations

$$2x + y - 2z = 0, \qquad 5x - 2y - 8z = 0.$$

We may write the solution immediately as

$$\frac{x}{\begin{vmatrix} 1 & -2 \\ -2 & -8 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 2 & -2 \\ 5 & -8 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 2 & 1 \\ 5 & -2 \end{vmatrix}};$$

that is,

$$x/(-12) = -y/(-6) = z/(-9),$$

or

$$x:y:z = 4:-2:3.$$

Example 3 Solve the equations

$$7x - 9y = -41, \qquad 10x + 3y = -11.$$

The equations should first be rearranged with all terms on the left hand side thus

$$7x - 9y + 41 = 0$$
$$10x + 3y + 11 = 0.$$

Then

$$\frac{x}{\begin{vmatrix} -9 & 41 \\ 3 & 11 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 7 & 41 \\ 10 & 11 \end{vmatrix}} = \frac{1}{\begin{vmatrix} 7 & -9 \\ 10 & 3 \end{vmatrix}},$$

or

$$x/(-222) = -y/(-333) = 1/111.$$

Hence

$$x = -2, \quad y = 3.$$

1.2 Properties of second order determinants

Property i. If the rows and columns of determinant (3) are respectively changed to columns and rows, the rearranged determinant has the same value as the original, for

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - b_1a_2.$$

Property ii. If the two columns (or the two rows) in determinant (3) are interchanged, the sign of the determinant is changed, for

$$\begin{vmatrix} b_1 & a_1 \\ b_2 & a_2 \end{vmatrix} = b_1a_2 - b_2a_1 = -(a_1b_2 - a_2b_1).$$

Property iii. If the two columns (or the two rows) are identical, the value of the determinant is zero, for

$$\begin{vmatrix} a_1 & a_1 \\ a_2 & a_2 \end{vmatrix} = a_1a_2 - a_2a_1 = 0.$$

Property iv. If all the elements in one column (or row) are multiplied by a constant p, the original determinant is merely multiplied by p, for

$$\begin{vmatrix} pa_1 & b_1 \\ pa_2 & b_2 \end{vmatrix} = pa_1b_2 - pa_2b_1 = p\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.$$

Property v. If each element in a given column (or a row) is split into two distinct parts, the determinant may be expressed as the sum of two determinants, for

$$\begin{aligned} \begin{vmatrix} a_1 + c_1 & b_1 \\ a_2 + c_2 & b_2 \end{vmatrix} &= (a_1 + c_1)b_2 - (a_2 + c_2)b_1 \\ &= (a_1b_2 - a_2b_1) + (c_1b_2 - c_2b_1) \\ &= \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} + \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}. \end{aligned}$$

Property vi. If a constant multiple of the elements of one column (or row) is added to the respective elements of the other column (or row), the value of the determinant is unchanged. For example, *if p times the* elements of column 2 are added to the elements of column 1, we have

$$\begin{vmatrix} a_1 + pb_1 & b_1 \\ a_2 + pb_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} + \begin{vmatrix} pb_1 & b_1 \\ pb_2 & b_2 \end{vmatrix} \quad \text{(from property } v\text{)}$$

$$= \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \quad \text{(from properties } iv \text{ and } iii\text{).}$$

This property may be used to simplify the elements of a determinant before it is evaluated numerically, as the following examples show.

Example 4 Evaluate the determinant

$$D = \begin{vmatrix} 213 & 210 \\ 357 & 351 \end{vmatrix}.$$

Although from the original definition it is correct that D equals 213.351 − 357.210, we use property *vi* to simplify the arithmetic. Subtracting column 2 from column 1, we have

$$D = \begin{vmatrix} 213 - 210 & 210 \\ 357 - 351 & 351 \end{vmatrix} = \begin{vmatrix} 3 & 210 \\ 6 & 351 \end{vmatrix} = 3\begin{vmatrix} 1 & 210 \\ 2 & 351 \end{vmatrix}$$

$$= 3(351 - 420) = -207.$$

When property *vi* is used, a brief note should always be made concerning what has been done. The author would use a note such as ($col_1 - col_2$), to indicate to an independent reader the manipulation that has taken place.

Example 5 Evaluate the determinant

$$D = \begin{vmatrix} 433 & 140 \\ 215 & 67 \end{vmatrix}.$$

We have

$$D = \begin{vmatrix} 3 & 6 \\ 215 & 67 \end{vmatrix} \quad (row_1 - 2row_2)$$

$$= \begin{vmatrix} 3 & 0 \\ 215 & -363 \end{vmatrix} \quad (col_2 - 2col_1)$$

$$= -1089.$$

The reader should note that, if a zero element can be obtained by this process, the arithmetic is immediately simplified.

1.3 The definition of a third order determinant

A *third order determinant* consists of nine elements arranged in the form of a square consisting of three rows and three columns thus:

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}. \tag{5}$$

Its expanded form is suggested by the following considerations.

If any one of the nine elements is chosen, and if the row and column containing that element are deleted, there remains a second order determinant, called the *minor* of the chosen element. Thus the minor of a_1 is $(b_2c_3 - b_3c_2)$, being the value of the determinant formed when the first row and the first column are deleted in D. The minor of b_3 is $a_1c_2 - a_2c_1$.

The *cofactor* of an element in D is its minor with a special sign attached, chosen in keeping with the following scheme:

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}.$$

We choose the sign occupying the same position as the element chosen in D. Thus the elements a_1, c_1, b_2, a_3, c_3 require a plus sign to be attached to their minors to yield the corresponding cofactors, while the elements a_2, b_1, b_3, c_2 require minus signs. We shall denote the cofactors of an element by the corresponding capital letters. Thus

$$A_1 = +(b_2c_3 - b_3c_2), \quad B_3 = -(a_1c_2 - a_2c_1), \quad C_2 = -(a_1b_3 - a_3b_1).$$

Let us now add together the three elements of any row (or column) after multiplying these elements by their respective cofactors. Six such sums may be formed; the first row yields

$$a_1A_1 + b_1B_1 + c_1C_1 = a_1(b_2c_2 - b_3c_2) - b_1(a_2c_3 - a_3c_2) \\ + c_1(a_2b_3 - a_3b_2), \quad (6)$$

while the second column yields the sum

$$b_1B_1 + b_2B_2 + c_2C_2 = -b_1(a_2c_3 - a_3c_2) \\ + b_2(a_1c_3 - a_3c_1) - b_3(a_1c_2 - a_2c_1).$$

If the reader inspects these two sums he will find that they are identical. Moreover, if the sums for the second and third rows and for the first and third columns are written down, it will be seen immediately that all the six sums are identical. This sum is taken to be the definition of the value of the third order determinant. Usually considerations of numerical simplicity determine which row or column should be used to expand the determinant, but the first row is often chosen.

Consider now the nine possible sums formed from the three elements of a row (or column) after multiplication of these elements by the respective cofactors of a distinct row (or column). For example, taking the elements of the first row and the cofactors of the second row, we have

$$a_1A_2 + b_1B_2 + c_1C_2 = -a_1(b_1c_3 - b_3c_1) + b_1(a_1c_3 - a_3c_1) - c_1(a_1b_3 - a_3b_1) = 0,$$

since all terms in this expansion cancel out. Similarly, all nine such sums vanish identically.

Example 6 The determinant

$$D = \begin{vmatrix} 4 & 2 & 3 \\ 7 & 5 & 4 \\ 9 & 2 & 6 \end{vmatrix}$$

may be expanded along all three rows and all three columns thus:

Along row 1: $D = 4(5.6 - 2.4) - 2(7.6 - 9.4) + 3(7.2 - 9.5) = -17;$

along row 2: $D = -7(2.6 - 2.3) + 5(4.6 - 9.3) - 4(4.2 - 9.2) = -17;$

along row 3: $D = 9(2.4 - 5.3) - 2(4.4 - 7.3) + 6(4.5 - 7.2) = -17;$

along column 1: $D = 4(5.6 - 2.4) - 7(2.6 - 2.3) + 9(2.4 - 5.3) = -17;$

along column 2: $D = -2(7.6 - 9.4) + 5(4.6 - 9.3) - 2(4.4 - 7.3) = -17;$

along column 3: $D = 3(7.2 - 9.5) - 4(4.2 - 9.2) + 6(4.5 - 7.2) = -17.$

1.4 Properties of third order determinants

The following six properties are identical with those proved in section 1.2 for second order determinants.

Property i. If the rows and columns of the determinant (5) are respectively changed to columns and rows, the rearranged determinant has the same value as the original, for expanding along the first column we have

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1(b_2c_3 - c_2b_3) - b_1(a_2c_3 - c_2a_3) + c_1(a_2b_3 - b_2a_3).$$

The six terms in this expansion are identical with expression (6), which is the expansion of D along its first row.

Property ii. If two columns (or two rows) in determinant (5) are interchanged, the sign of the determinant is changed. For if this new determinant and D are expanded down the column (or along the row) that remains unchanged, the cofactors used in the two cases are merely changed in sign.

Property iii. If two columns (or two rows) are identical, the value of the determinant is zero. For if D is expanded down the remaining column (or along the remaining row), all the cofactors used are zero.

Property iv. If all the elements in one column (or row) are multiplied by a factor p, the original determinant is multiplied by p. Upon expansion down this particular column (or along the row), the factor p occurs in all six terms in the expansion; this factor p may then be removed, leaving the original determinant D in expanded form.

Property v. If the elements in a column (or a row) are split into two distinct parts, the determinant may be expressed as the sum of two determinants. Let

$$D = \begin{vmatrix} a_1 + d_1 & b_1 & c_1 \\ a_2 + d_2 & b_2 & c_2 \\ a_3 + d_3 & b_3 & c_3 \end{vmatrix}.$$

If the cofactors of the three elements down the first column are denoted by A_1, A_2, A_3 respectively, we have

$$\begin{aligned} D &= (a_1 + d_1)A_1 + (a_2 + d_2)A_2 + (a_3 + d_3)A_3 \\ &= (a_1A_1 + a_2A_2 + a_3A_3) + (d_1A_1 + d_2A_2 + d_3A_3) \\ &= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}. \end{aligned}$$

It can be seen that if both columns 1 and 2 are split up, the determinant may be separated into 4 determinants, while if all three columns are each split into two parts, the given determinant may be expressed as the sum of 8 determinants.

Property vi. If a constant multiple of the elements of one column (or row) is added respectively to the elements of a distinct column (or row), the value of the determinant is unchanged. The proof is identical with that given for property *vi*, section 1.2, *mutatis mutandis.*

This property is used to simplify the elements of a complicated determinant before numerical expansion. There is of course no unique method for simplifying such a determinant.

Example 7 Simplify and evaluate the determinant

$$D = \begin{vmatrix} 87 & 42 & 3 \\ 45 & 18 & 7 \\ 50 & 17 & 3 \end{vmatrix}.$$

We have
$$D = \begin{vmatrix} 0 & 42 & 3 \\ 2 & 18 & 7 \\ 13 & 17 & 3 \end{vmatrix} \quad (\text{col}_1 - 2\text{col}_2 - \text{col}_3)$$

$$= 3\begin{vmatrix} 0 & 14 & 1 \\ 2 & 18 & 7 \\ 13 & 17 & 3 \end{vmatrix}$$

$$= 3\begin{vmatrix} 0 & 0 & 1 \\ 2 & -80 & 7 \\ 13 & -25 & 3 \end{vmatrix} \quad (\text{col}_2 - 14\text{col}_3)$$

$$= 6\begin{vmatrix} 1 & -40 \\ 13 & -25 \end{vmatrix} = 30\begin{vmatrix} 1 & -8 \\ 13 & -5 \end{vmatrix}$$

$$= 30(-5 + 104) = 2970.$$

Example 8 Solve the third order determinantal equation

$$D = \begin{vmatrix} x+1 & 2x & 1 \\ x & 3x-2 & 2x \\ 1 & x & x \end{vmatrix} = 0.$$

We have
$$D = \begin{vmatrix} 1-x & 2x & 1 \\ 2-2x & 3x-2 & 2x \\ 1-x & x & x \end{vmatrix} \quad (\text{col}_1 - \text{col}_2)$$

$$= (1-x)\begin{vmatrix} 1 & 2x & 1 \\ 2 & 3x-2 & 2x \\ 1 & x & x \end{vmatrix}$$

$$= (1-x)\begin{vmatrix} 1 & 2x & 1-x \\ 2 & 3x-2 & 0 \\ 1 & x & 0 \end{vmatrix} \quad (\text{col}_3 - x\text{col}_1)$$

$$= (1-x)^2\begin{vmatrix} 2 & 3x-2 \\ 1 & x \end{vmatrix}$$

$$= (1-x)^2(2-x) = 0;$$

hence $x = 1, 1, 2$.

1.5 Fourth order determinants

Fourth and higher order determinants are defined in a manner similar to that given for third order determinants, and they enjoy exactly similar properties. The minor of an element in the determinant

$$D = \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}$$

is the value of the third order determinant formed by deleting the row and column through that element. The cofactor of the element is formed from its minor by attaching the appropriate sign drawn from the extended scheme:

$$\begin{matrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{matrix}.$$

If these cofactors are again denoted by capital letters, the value of D is defined to be the unique sum obtained by expanding the determinant along any one of its four rows or down any one of its four columns. For example:

$$D = a_1A_1 + b_1B_1 + c_1C_1 + d_1D_1 = a_1A_1 + a_2A_2 + a_3A_3 + a_4A_4.$$

Property *vi* may be used to simplify the elements in the determinant before numerical expansion.

Example 9 Evaluate the determinant

$$D = \begin{vmatrix} 4 & 9 & -1 & 3 \\ 3 & -7 & 2 & 5 \\ 2 & 5 & -3 & 7 \\ 1 & -3 & 4 & 9 \end{vmatrix}.$$

Using property *vi* repeatedly, we have

$$D = \begin{vmatrix} 4 & 9 & -1 & 3 \\ -1 & -16 & 3 & 2 \\ -1 & 12 & -5 & 2 \\ -1 & -8 & 7 & 2 \end{vmatrix} \quad (\text{row}_4 - \text{row}_3,\ \text{row}_3 - \text{row}_2,\ \text{row}_2 - \text{row}_1)$$