J. J. CONNOR
C. A. BREBBIA

Finite Element Techniques For Fluid Flow

Finite Element Techniques for Fluid Flow

J. J. CONNOR, SC.D.
Professor of Civil Engineering
Massachusetts Institute of Technology, USA

C. A. BREBBIA, PH.D.
Senior Lecturer in Civil Engineering
University of Southampton, UK

NEWNES–BUTTERWORTHS
LONDON BOSTON
Sydney Wellington Durban Toronto

THE BUTTERWORTH GROUP

UNITED KINGDOM
Butterworth & Co (Publishers) Ltd
London: 88 Kingsway, WC2B 6AB

AUSTRALIA
Butterworths Pty Ltd
Sydney: 586 Pacific Highway, NSW 2067
Also at Melbourne, Brisbane, Adelaide and Perth

CANADA
Butterworth & Co (Canada) Ltd
Toronto: 2265 Midland Avenue, Scarborough, Ontario M1P 4S1

NEW ZEALAND
Butterworths of New Zealand Ltd
Wellington: 26–28 Waring Taylor Street, 1

SOUTH AFRICA
Butterworth & Co (South Africa) (Pty) Ltd
Durban: 152–154 Gale Street

USA
Butterworth (Publishers) Inc
Boston: 19 Cummings Park, Woburn, Mass. 01801

First published in 1976

ISBN 0 408 00176 3

Filmset and printed in England by J. W. Arrowsmith Ltd, Bristol

Bound in England by The Pitman Press, Bath

Preface

Until recently, finite element techniques were almost exclusively used in structural engineering problems but now there is a growing awareness of their potential in other engineering fields, especially in fluid mechanics.

This book presents these recent advances in a simple way. The authors have been mainly concerned with producing a text for teaching which can be easily followed by the self-taught student. The last part will undoubtedly also be of use to research workers.

The reader will be led from the basic principles of Chapter 1, and the simple finite element concepts and models given in Chapters 2 and 3, step by step to more complex applications. A chapter (4) on the governing equations of fluid flow has been included to provide a more complete progression, though this is not intended for those already well versed in fluid dynamics. Chapter 5 is concerned with the solution of potential type problems and Chapter 6 sets out viscous flow problems in porous media; both are topics well suited to finite element solutions and of general interest to the engineer, applied mathematician and physicist.

In the remaining chapters the solutions of more specialised problems are presented. Chapter 7 describes how circulation problems can be tackled using finite elements, Chapter 8 deals with the solution of the mass transfer equation and Chapter 9 discusses ways of solving general transient incompressible flows.

Since this book contains more material than could be used in a standard course, the authors have also indicated in the contents some sections which could be omitted without affecting the general structure. Those students not interested in coastal engineering and transport type problems could in addition leave out Chapters 7 and 9.

Finally, the authors wish to thank all those who made this book possible, especially their research associates Dr. R. Adey, Mr. J. Rodenhuis, Dr. S. Smith and Dr. J. Wang.

The Authors
Southampton 1976

Contents

*Topics which may be omitted in an introductory course without affecting continuity.

1 Weighted Residual and Variational Methods

1.1 BASIC DEFINITIONS

We start by introducing some basic definitions and properties for a sequence of functions such as

$$\phi_1(x), \phi_2(x), \phi_3(x) \ldots \phi_n(x) \tag{1.1}$$

The functions are assumed to satisfy certain given conditions, called admissibility conditions, relating to the boundary conditions and the degree of continuity. We will study them in more detail in the following.

If the functions can be linearly combined, for instance,

$$\phi = \alpha\phi_1 + \beta\phi_2 \tag{1.2}$$

where α and β are numbers, they are called *elements* of a linear space R, and the following properties hold:

$$\begin{aligned} \phi_1 + \phi_2 &= \phi_2 + \phi_1 \\ (\alpha + \beta)\phi &= \alpha\phi + \beta\phi \\ \alpha(\phi_1 + \phi_2) &= \alpha\phi_1 + \alpha\phi_2 \end{aligned} \tag{1.3}$$

The *inner product* of two functions ϕ_1 and ϕ_2 is denoted by

$$\langle \phi_1, \phi_2 \rangle \tag{1.4}$$

and it represents an operation on ϕ_1 and ϕ_2, such as

$$\langle \phi_1, \phi_2 \rangle = \int_{x_1}^{x_2} \phi_1(x)\phi_2(x)\,dx \tag{1.5a}$$

or

$$\langle \phi_1, \phi_2 \rangle_c = \int_0^t \phi_1(t-\tau)\phi_2(\tau)\,d\tau \tag{1.5b}$$

The second definition is called the convolution. We will consider only the first type of product here.

For real functions, the inner product has the following properties:

$$\begin{aligned} \langle \phi_1, \phi_2 \rangle &= \langle \phi_2, \phi_1 \rangle \\ \alpha\langle \phi_1, \phi_2 \rangle &= \langle \alpha\phi_1, \phi_2 \rangle \\ \langle \phi_1, \phi_2 + \phi_3 \rangle &= \langle \phi_1, \phi_2 \rangle + \langle \phi_1, \phi_3 \rangle \\ \langle \phi_1, \phi_1 \rangle &> 0 \quad \text{if } \phi_1 \neq 0 \\ &= 0 \quad \text{if } \phi_1 = 0 \end{aligned} \tag{1.6}$$

where $\phi_1 = 0$ is a 'null' function which exists in the space R.

A measure (*norm*) of the function ϕ can be taken as the square root of the inner product of ϕ by itself and is denoted by $\|\phi\|$.

$$\|\phi\| = \sqrt{\langle \phi, \phi \rangle} \tag{1.7}$$

A sequence of functions such as (1.1) is said to be *linearly independent* if

$$\alpha_1\phi_1 + \alpha_2\phi_2 + \ldots + \alpha_n\phi_n = 0 \tag{1.8}$$

only when all α_i are zero.

A sequence of linearly independent functions is said to be *complete* if a number N and a set of constants α_i can be found such that, given an admissible but otherwise arbitrary function u, we have

$$\left\| u - \sum_{i=1}^{N} \alpha_i\phi_i \right\| < \varepsilon \tag{1.9}$$

where ε is any small quantity.

The functions ϕ_i are called *basis functions* and the coefficients α_i are the *Fourier coefficients.*

If the normalised basis functions are mutually orthogonal,

$$\begin{aligned} \langle \phi_i, \phi_j \rangle &= 0 \quad \text{if } i \neq j \\ \langle \phi_i, \phi_i \rangle &= 1 \end{aligned} \tag{1.10}$$

Each additional term we take in the *linearly independent* and *complete* sequence ϕ_i will introduce a further α_i. For the Nth approximation, we have

$$u^{(N)} = \sum_1^N \alpha_i \phi_i$$

Thus

$$\|u^{(N)}\| \to \|u\| \quad \text{as} \quad N \to \infty \tag{1.11}$$

The norm of $u^{(N)}$ for a mutually orthogonal complete sequence (if the sequence is not orthogonal we will accept that we can always reduce it to an orthogonal one) is

$$\begin{aligned} \|u^{(N)}\| &= \sqrt{\left\{ \left\langle \sum_{i=1}^N \alpha_i \phi_i, \sum_{j=1}^N \alpha_j \phi_j \right\rangle \right\}} \\ &= \sqrt{\left\{ \sum_{i=1}^N \sum_{j=1}^N \alpha_i \alpha_j \langle \phi_i, \phi_j \rangle \right\}} \end{aligned} \tag{1.12}$$

and since $\langle \phi_i, \phi_j \rangle = 0$ if $i \neq j$, we have

$$\|u^{(N)}\| = \sqrt{\left\{ \sum_{i=1}^N \alpha_i^2 \langle \phi_i, \phi_i \rangle \right\}} \tag{1.13}$$

Each term in the summation is positive, thus $\|u^{(N)}\|$ approaches $\|u\|$ from below as N increases.

$$\|u^{(N)}\| \leqslant \|u^{(M)}\| \leqslant \|u\| \quad \text{with } N < M \tag{1.14}$$

An *operator* $\mathscr{L}(\ )$ is defined as a process which, when applied to a given function u, produces another function p:

$$\mathscr{L}(u) = p \tag{1.15}$$

An operator is *linear* if

$$\mathscr{L}(\alpha u_1 + \beta u_2) = \alpha \mathscr{L}(u_1) + \beta \mathscr{L}(u_2) \tag{1.16}$$

This definition is general but we will consider here only differential operators.

Properties analogous to symmetry and positive definiteness for a matrix can also be defined for an operator. Consider a square matrix, $\mathbf{a} = [a_{ij}]$. We say $\mathbf{a}$ is symmetrical when $\mathbf{a}^T = \mathbf{a}$, where $\mathbf{a}^T$ (the transpose of $\mathbf{a}$) is formed by interchanging the rows and columns of $\mathbf{a}$. Symmetry requires $a_{ij} \equiv a_{ji}$. Another way of defining symmetry is to require

$$\langle \mathbf{y}, \mathbf{ax} \rangle \equiv \langle \mathbf{x}, \mathbf{ay} \rangle \tag{a}$$

for arbitrary vectors $\mathbf{x}$ and $\mathbf{y}$. Expanding (a) and noting that $(\mathbf{bc})^T = \mathbf{c}^T\mathbf{b}^T$,

$$\langle \mathbf{y}, \mathbf{ax} \rangle = \mathbf{y}^T\mathbf{ax} \tag{b}$$

$$\langle \mathbf{x}, \mathbf{ay} \rangle = \mathbf{x}^T\mathbf{ay} = \mathbf{y}^T\mathbf{a}^T\mathbf{x} \tag{c}$$

shows that (a) is equivalent to $\mathbf{a}^T = \mathbf{a}$. The latter definition is more convenient for extension to operators. Positive definiteness is defined by

$$\langle \mathbf{x}, \mathbf{ax} \rangle \geqslant 0 \tag{d}$$

for all $\mathbf{x}$ and equals 0 only when $\mathbf{x}$ is a null vector. This property is extremely valuable in establishing solution schemes and also in constructing variational statements.

With this as background, let us consider the problem represented by a set of homogeneous equations in the interior of a domain, V

$$\mathscr{L}(u) = 0, \qquad x \in V \tag{1.17}$$

We form the inner product of $\mathscr{L}(u)$ with another function, say v. The matrix transpose operations in (b), (c) are equivalent here to integration by parts of $\langle \mathscr{L}(u), v \rangle$ until the derivatives of u are eliminated. This leads to the 'transposed' form of the inner product and also to boundary terms. We write the results as

$$\langle \mathscr{L}(u), v \rangle = \langle u, \mathscr{L}^*(v) \rangle + \int_S (F(v)G(u) - F(u)G^*(v))\, dS \tag{1.18}$$

where S is the exterior surface and F, G are differential operators whose forms follow naturally from the integration by parts. By definition, $F(v)$ contains the v terms resulting from the first phase of the partial integration and $G(u)$ contains the corresponding u terms. Some examples which illustrate this operation are included below.

The operator $\mathscr{L}^*$ is called the *adjoint* of $\mathscr{L}$. If $\mathscr{L}^* = \mathscr{L}$, $\mathscr{L}$ is said to be self adjoint. In this case, $G^* = G$ also. Self-adjointness of an operator is analogous to symmetry of a matrix. In addition to

determining whether the operator is self-adjoint, the partial integration also generates two different categories of boundary conditions. The set $F(u)$ prescribed are called the *essential* boundary conditions and $G(u)$ prescribed are the *nonessential* or *natural* boundary conditions. One can specify either type of boundary condition on the surface of a domain. However, the essential boundary conditions must be enforced at some point in order for the solution to be unique. Letting S_1 and S_2 represent complementary portions of the total surface, S, we can state the boundary conditions for the self-adjoint problem ($\mathscr{L}^* = \mathscr{L}$) as:

$$\begin{aligned} &F(u) \text{ prescribed on } S_1 \\ &G(u) \text{ prescribed on } S_2 \\ &S_1 + S_2 = S \end{aligned} \tag{1.19}$$

The positive definite property of a self-adjoint operator is defined by the requirement that

$$\langle \mathscr{L}(u), u \rangle > 0 \tag{1.20}$$

for all nontrivial u which satisfy the homogeneous form of the boundary conditions. One determines whether $\mathscr{L}$ is positive definite by integrating the inner product until it contains only products of derivatives of the same order. This operation is the mid-point in the transformation of $\mathscr{L}$ into $\mathscr{L}^*$ (i.e. equation (1.18)).

Example 1.1

(i) Consider

$$\mathscr{L}(u) = \frac{d^2u}{dx^2} \qquad 0 < x < 1 \tag{a}$$

Forming the inner product and integrating yields

$$\begin{aligned} \int_0^1 v\mathscr{L}(u)\,dx &= \int_0^1 v\frac{d^2u}{dx^2}\,dx \\ &= \left| v\frac{du}{dx}\right|_0^1 - \int_0^1 \frac{dv}{dx}\frac{du}{dx}\,dx \\ &= \left| v\frac{du}{dx} - u\frac{dv}{dx}\right|_0^1 + \int_0^1 u\frac{d^2v}{dx^2}\,dx \end{aligned} \tag{b}$$

Using the notation of (1.18), we have

$$F(v) = v$$
$$G(u) = \mathrm{d}u/\mathrm{d}x \qquad G^*(v) = \mathrm{d}v/\mathrm{d}x \tag{c}$$
$$\mathscr{L}^* = \mathscr{L}$$

and the operator is self-adjoint. The essential boundary condition is u prescribed and the natural boundary condition is $\mathrm{d}u/\mathrm{d}x$ prescribed.

Referring back to (b), if we take $v = u$ and homogeneous boundary conditions, the first partial integration yields

$$\int_0^1 u\mathscr{L}(u)\,\mathrm{d}x = -\int_0^1 \left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 \mathrm{d}x \tag{d}$$

Then $\mathscr{L}(u) = \mathrm{d}^2u/\mathrm{d}x^2$ is *negative definite*.

(ii) We examine next a more general operator,

$$\mathscr{L}(u) = \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(a_1(x)\frac{\mathrm{d}^2u}{\mathrm{d}x^2}\right) + \frac{\mathrm{d}}{\mathrm{d}x}\left(a_2(x)\frac{\mathrm{d}u}{\mathrm{d}x}\right) + a_3(x)u \qquad 0 < x < 1 \tag{a}$$

The first partial integration operation results in

$$\int_0^1 v\mathscr{L}(u)\,\mathrm{d}x = \int_0^1 \left\{a_1\frac{\mathrm{d}^2u}{\mathrm{d}x^2}\frac{\mathrm{d}^2v}{\mathrm{d}x^2} - a_2\frac{\mathrm{d}u}{\mathrm{d}x}\frac{\mathrm{d}v}{\mathrm{d}x} + a_3uv\right\}\mathrm{d}x + \left|v\left\{\frac{\mathrm{d}}{\mathrm{d}x}\left(a_1\frac{\mathrm{d}^2u}{\mathrm{d}x^2}\right) + a_2\frac{\mathrm{d}u}{\mathrm{d}x}\right\} + \frac{\mathrm{d}v}{\mathrm{d}x}\left\{-a_1\frac{\mathrm{d}^2u}{\mathrm{d}x^2}\right\}\right|_0^1 \tag{b}$$

If we continue, we would find that $\mathscr{L}^* = \mathscr{L}$. The boundary terms follow from (b) and are summarised below.

Essential boundary conditions

$$\left.\begin{aligned} F_1(u) &= u \\ F_2(u) &= \frac{\mathrm{d}u}{\mathrm{d}x} \end{aligned}\right\} \text{prescribed} \tag{c}$$

Natural boundary conditions

$$\left.\begin{aligned} G_1(u) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(a_1\frac{\mathrm{d}^2u}{\mathrm{d}x^2}\right) + a_2\frac{\mathrm{d}u}{\mathrm{d}x} \\ G_2(u) &= -a_1\frac{\mathrm{d}^2u}{\mathrm{d}x^2} \end{aligned}\right\} \text{prescribed} \tag{d}$$

Considering (b), we can write

$$\langle u\mathscr{L}(u)\rangle = \int_0^1 \left\{ a_1\left(\frac{d^2u}{dx^2}\right)^2 - a_2\left(\frac{du}{dx}\right)^2 + a_3u^2 \right\} dx \qquad \text{(e)}$$

for u satisfying homogeneous boundary conditions. If a_1, a_3 are > 0 and $a_2 < 0$ in the interval $0 < x < 1$, the operator is obviously *positive definite*.

(iii) Operating on

$$\mathscr{L}(u) = \frac{d^2u}{dx^2} + \frac{du}{dx} + u \qquad \text{(a)}$$

yields

$$\int_0^1 \left(\frac{d^2u}{dx^2} + \frac{du}{dx} + u\right) v\, dx = \int_0^1 \left(\frac{d^2v}{dx^2} - \frac{dv}{dx} + v\right) u\, dx + \left| v\left(\frac{du}{dx} + u\right) - u\left(\frac{dv}{dx}\right)\right|_0^1 \qquad \text{(b)}$$

The operator is not self-adjoint due to the presence of the first derivative term. Odd-order derivatives will lead to skew-symmetric terms in $\mathscr{L}^*$ and G^*. The essential boundary condition is u prescribed. In this case, we take the nonessential (natural) boundary condition as du/dx prescribed.

1.2 WEIGHTED RESIDUAL METHODS

Weighted residual methods are numerical procedures for approximating the solution of a set of differential (or integral) equations of the form

$$\mathscr{L}(u_0) = p \qquad x \in V \qquad (1.21)$$

with boundary conditions

$$\mathscr{S}(u_0) = g \qquad x \in S \qquad (1.22)$$

where x represents the spatial coordinates x_1, x_2 and x_3; S is the external surface of the continuum; and u_0 is the exact solution. The function u_0 is approximated by a set of functions $\phi_k(x)$,

$$u = \sum_{k=1}^{N} \alpha_k \phi_k \qquad (1.23)$$

where α_k are undetermined parameters and ϕ_k are linearly independent functions taken from a complete sequence.

We will initially require that these functions satisfy *all* the boundary conditions of the problem [equation (1.22)] and have the necessary degree of continuity as to make the left-hand side of (1.21) different from zero. A procedure for relaxing the boundary condition requirements is discussed in the next section.

Substitution of (1.23) into (1.21) produces an error function ε, which is called the *residual*:

$$\varepsilon = \mathscr{L}(u) - p \neq 0 \tag{1.24}$$

Note that ε is equal to zero for the exact solution. This error is forced to be zero, in an average sense, by setting weighted integrals of the residual equal to zero:

$$\langle \varepsilon, w_i \rangle = 0, \qquad i = 1, 2 \ldots N \tag{1.25}$$

where w_i is a set of weighting functions. In what follows, we first review a few of the weighted residual methods and then discuss the Galerkin method in greater detail.

(a) THE COLLOCATION METHOD

In this method, we satisfy the differential equations only at a set of chosen points. For a given approximating function

$$u = \sum_{k=1}^{N} \alpha_k \phi_k \tag{1.26}$$

we have

$$\varepsilon = \mathscr{L}(u) - p = \sum_{k=1}^{N} \alpha_k \mathscr{L}(\phi_k) - p \tag{1.27}$$

The parameters are determined by enforcing the condition $\varepsilon = 0$ at N points in the domain.

We can express these conditions in the same form as (1.25) by introducing a Dirac function $\Delta(x_i)$ such that $\Delta(x_i) = 0$ for x outside the interval $x_i \pm c$ and

$$\int_{x_i - c}^{x_i + c} \Delta(x_i)\, \mathrm{d}x = \int_{x_i - c}^{x_i + c} \Delta_i\, \mathrm{d}x = 1 \tag{1.28}$$

where c is a small value (for point collocation $c \to 0$). Then collocation is equivalent to

$$\langle \varepsilon, \Delta_i \rangle = \langle \mathscr{L}(u) - p, \Delta_i \rangle = 0, \qquad i = 1, 2 \ldots N \tag{1.29}$$

Example 1.2

Consider the following second-order equation, which applies in the domain $0 < x < 1$:

$$\mathscr{L}(u) = \frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + u + x = 0 \tag{a}$$

with boundary conditions

$$\begin{aligned} u &= 0 \quad \text{at } x = 0 \\ u &= 0 \quad \text{at } x = 1 \end{aligned} \tag{b}$$

We propose as an approximating function

$$u = x(1 - x)(\alpha_1 + \alpha_2 x + \ldots) \tag{c}$$

which satisfies the boundary conditions for arbitrary α_i.

If only two terms in the approximation are taken,

$$u = x(1 - x)(\alpha_1 + \alpha_2 x) \tag{d}$$

the error is

$$\varepsilon = \mathscr{L}(u) - p = x + (-2 + x - x^2)\alpha_1 + (2 - 6x + x^2 - x^3)\alpha_2 \tag{e}$$

We choose $x = \frac{1}{4}, x = \frac{1}{2}$ as collocation points. This choice requires

$$\begin{bmatrix} \frac{29}{16} & -\frac{35}{64} \\ \frac{7}{4} & \frac{7}{8} \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix} = \begin{Bmatrix} \frac{1}{4} \\ \frac{1}{2} \end{Bmatrix} \tag{f}$$

Solving (f) yields

$$\begin{aligned} \alpha_1 &= \tfrac{6}{31}, \qquad \alpha_2 = \tfrac{40}{217} \\ u &= \frac{x(x-1)}{217}(42 + 40x) \end{aligned} \tag{g}$$

Comparing this result with the exact solution, the following table can be drawn:

x	u_{app}	u_{exact}
0·25	0·045	0·044914
0·50	0·071	0·069747
0·75	0·062	0·060056

$$u_{exact} = \frac{\sin x}{\sin 1} - x \tag{h}$$

(b) THE LEAST-SQUARE METHOD

In this method we take the inner product of the error by itself, and the quantity thus obtained is required to be a minimum. Starting with

$$\varepsilon = \mathscr{L}(u) - p \tag{1.30}$$

we define F as

$$F = \langle \varepsilon, \varepsilon \rangle = \langle \mathscr{L}(u) - p, \mathscr{L}(u) - p \rangle \tag{1.31}$$

If the approximating function is

$$u = \sum_{k=1}^{N} \alpha_k \phi_k \tag{1.32}$$

we minimise F by differentiating with respect to α_i

$$\frac{\partial F}{\partial \alpha_i} = 0 \qquad i = 1, 2 \ldots N \tag{1.33}$$

This yields

$$\frac{\partial F}{\partial \alpha_i} = \frac{\partial}{\partial \alpha_i} \langle \varepsilon, \varepsilon \rangle = \frac{\partial}{\partial \alpha_i} \left\{ \left\langle \mathscr{L}\left(\sum \alpha_k \phi_k\right), \mathscr{L}\left(\sum \alpha_k \phi_k\right) \right\rangle - 2 \left\langle \mathscr{L}\left(\sum \alpha_k \phi_k\right), p \right\rangle + \langle p, p \rangle \right\} \tag{1.34}$$

When $\mathscr{L}$ is a linear operator, the equations simplify to

$$2 \left\langle \mathscr{L}\left(\sum \alpha_k \phi_k\right), \mathscr{L}(\phi_i) \right\rangle - 2 \langle \mathscr{L}(\phi_i), p \rangle = 0 \tag{1.35}$$

which can be written as

$$\left\langle \mathscr{L}\left(\sum \alpha_k \phi_k\right) - p, \mathscr{L}(\phi_i) \right\rangle = 0 \tag{1.36}$$

Example 1.3

Consider the equation treated in Example 1.2. We take the second-order approximation here also.

$$u = x(1 - x)\alpha_1 + x^2(1 - x)\alpha_2 \tag{a}$$

$$\varepsilon = x + (-2 + x - x^2)\alpha_1 + (2 - 6x + x^2 - x^3)\alpha_2 \tag{b}$$

Squaring ε and minimising it with respect to α_1 and α_2 we obtain

$$\int_0^1 \varepsilon(-2 + x - x^2)\, dx = 0$$

$$\int_0^1 \varepsilon(2 - 6x + x^2 - x^3)\, dx = 0 \tag{c}$$