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A Course in Mathematical Logic

Translated from the Russian by
Neal Koblitz

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Preface

1. This book is above all addressed to mathematicians. It is intended to be a textbook of mathematical logic on a sophisticated level, presenting the reader with several of the most significant discoveries of the last ten or fifteen years. These include: the independence of the continuum hypothesis, the Diophantine nature of enumerable sets, the impossibility of finding an algorithmic solution for one or two old problems.

All the necessary preliminary material, including predicate logic and the fundamentals of recursive function theory, is presented systematically and with complete proofs. We only assume that the reader is familiar with "naive" set theoretic arguments.

In this book mathematical logic is presented both as a part of mathematics and as the result of its self-perception. Thus, the substance of the book consists of difficult proofs of subtle theorems, and the spirit of the book consists of attempts to explain what these theorems say about the mathematical way of thought.

Foundational problems are for the most part passed over in silence. Most likely, logic is capable of justifying mathematics to no greater extent than biology is capable of justifying life.

2. The first two chapters are devoted to predicate logic. The presentation here is fairly standard, except that semantics occupies a very dominant position, truth is introduced before deducibility, and models of speech in formal languages precede the systematic study of syntax.

The material in the last four sections of Chapter II is not completely traditional. In the first place, we use Smullyan's method to prove Tarski's theorem on the undefinability of truth in arithmetic, long before the introduction of recursive functions. Later, in the seventh chapter, one of the proofs of the incompleteness theorem is based on Tarski's theorem. In the second place, a large section is devoted to the logic of quantum mechanics and to a proof of von Neumann's theorem on the absence of "hidden variables" in the quantum mechanical picture of the world.

The first two chapters together may be considered as a short course in logic apart from the rest of the book. Since the predicate logic has received the widest dissemination outside the realm of professional mathematics, the author has not resisted the temptation to pursue certain aspects of its relation to linguistics, psychology, and common sense. This is all discussed in a series of digressions, which, unfortunately, too often end up trying to explain "the exact meaning of a proverb" (E. Baratynskiĭ[1]). This series of digressions ends with the second chapter.

The third and fourth chapters are optional. They are devoted to complete proofs of the theorems of Gödel and Cohen on the independence of the continuum hypothesis. Cohen forcing is presented in terms of Boolean-valued models; Gödel's constructible sets are introduced as a subclass of von Neumann's universe. The number of omitted formal deductions does not exceed the accepted norm; due respects are paid to syntactic difficulties. This ends the first part of the book: "Provability."

The reader may skip the third and fourth chapters, and proceed immediately to the fifth. Here we present elements of the theory of recursive functions and enumerable sets, formulate Church's thesis, and discuss the notion of algorithmic undecidability.

The basic content of the sixth chapter is a recent result on the Diophantine nature of enumerable sets. We then use this result to prove the existence of versal families, the existence of undecidable enumerable sets, and, in the seventh chapter, Gödel's incompleteness theorem (as based on the definability of provability via an arithmetic formula). Although it is possible to disagree with this method of development, it has several advantages over earlier treatments. In this version the main technical effort is concentrated on proving the basic fact that all enumerable sets are Diophantine, and not on the more specialized and weaker results concerning the set of recursive descriptions or the Gödel numbers of proofs.

[1] Nineteenth century Russian poet (translator's note). The full poem is:

> We diligently observe the world,
> We diligently observe people,
> And we hope to understand their deepest meaning.
> But what is the fruit of long years of study?
> What do the sharp eyes finally detect?
> What does the haughty mind finally learn
> At the height of all experience and thought,
> What?—*the exact meaning of an old proverb.*
>
> 1828

The last section of the sixth chapter stands somewhat apart from the rest. It contains an introduction to the Kolmogorov theory of complexity, which is of considerable general mathematical interest.

The fifth and sixth chapters are independent of the earlier chapters, and together make up a short course in recursive function theory. They form the second part of the book: "Computability."

The third part of the book, "Provability and Computability," relies heavily on the first and second parts. It also consists of two chapters. All of the seventh chapter is devoted to Gödel's incompleteness theorem. The theorem appears later in the text than is customary because of the belief that this central result can only be understood in its true light after a solid grounding both in formal mathematics and in the theory of computability. Hurried expositions, where the proof that provability is definable is entirely omitted and the mathematical content of the theorem is reduced to some version of the "liar paradox," can only create a distorted impression of this remarkable discovery. The proof is considered from several points of view. We pay special attention to properties which do not depend on the choice of Gödel numbering. Separate sections are devoted to Feferman's recent theorem on Gödel formulas as axioms, and to the old but very beautiful result of Gödel on the length of proofs.

The eighth and final chapter is, in a way, removed from the theme of the book. In it we prove Higman's theorem on groups defined by enumerable sets of generators and relations. The study of recursive structures, especially in group theory, has attracted continual attention in recent years, and it seems worthwhile to give an example of a result which is remarkable for its beauty and completeness.

3. This book was written for very personal reasons. After several years or decades of working in mathematics, there almost inevitably arises the need to stand back and look at this research from the side. The study of logic is, to a certain extent, capable of fulfilling this need.

Formal mathematics has more than a slight touch of self-caricature. Its structure parodies the most characteristic, if not the most important, features of our science. The professional topologist or analyst experiences a strange feeling when he recognizes the familiar pattern glaring out at him in stark relief.

This book uses material arrived at through the efforts of many mathematicians. Several of the results and methods have not appeared in monograph form; their sources are given in the text. The author's point of view has formed under the influence of the ideas of Hilbert, Gödel, Cohen, and especially John von Neumann, with his deep interest in the external world, his open-mindedness and spontaneity of thought.

Various parts of the manuscript have been discussed with Yu. V. Matijasevič, G. V. Čudnovskiĭ, and S. G. Gindikin. I am deeply grateful to all of these colleagues for their criticism.

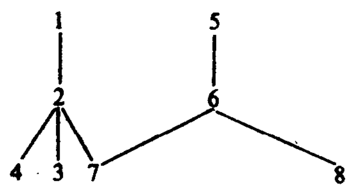
W. D. Goldfarb of Harvard University very kindly agreed to proofread the entire manuscript. For his detailed corrections and laborious rewriting of part of Chapter IV, I owe a special debt of gratitude.

I wish to thank Neal Koblitz for his meticulous translation.

Yu. I. Manin

Moscow, September 1974

Interdependence of Chapters



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Part I

PROVABILITY

[1] "A metaphorical compound word or phrase used especially in Old English and Old Norse poetry, e.g., 'swan-road' for 'ocean' "—Webster's New Collegiate Dictionary (translator's note).

I

PROVABILITY

CHAPTER I

Introduction to formal languages

Gelegentlich ergreifen wir die Feder
Und schreiben Zeichen auf ein weisses Blatt,
Die sagen dies und das, es kennt sie jeder,
Es ist ein Spiel, das seine Regeln hat.

H. Hesse, "Buchstaben"

We now and then take pen in hand
And make some marks on empty paper.
Just what they say, all understand.
It is a game with rules that matter.

H. Hesse, "Alphabet"
(translated by Prof. Richard S. Ellis)

1 General information

1.1. Let A be any abstract set. We call A an *alphabet*. Finite sequences of elements of A are called *expressions* in A. Finite sequences of expressions are called *texts*.

We shall speak of a *language with alphabet A* if certain expressions and texts are distinguished (as being "correctly composed," "meaningful," etc.). Thus, in the Latin alphabet A we may distinguish English word forms and grammatically correct English sentences. The resulting set of expressions and texts is a working approximation to the intuitive notion of the "English language."

The language Algol 60 consists of distinguished expressions and texts in the alphabet $\{\text{Latin letters}\} \cup \{\text{digits}\} \cup \{\text{logical signs}\} \cup \{\text{separators}\}$. *Programs* are among the most important distinguished texts.

In natural languages the set of distinguished expressions and texts usually has unsteady boundaries. The more formal the language, the more rigid these boundaries are.

The rules for forming distinguished expressions and texts make up the *syntax* of the language. The rules which tell how they correspond with reality make up the *semantics* of the language. Syntax and semantics are described in a *metalanguage*.

1.2. "Reality" for the languages of mathematics consists of certain classes of (mathematical) arguments or certain computational processes using (abstract) automata. Corresponding to these designations, the languages are divided into formal and algorithmic languages. (Compare: in natural languages, the declarative versus imperative moods, or—on the level of texts—statement versus command.)

Different formal languages differ from one another, in the first place, by the scope of the formalizable types of arguments—their expressiveness; in the second place, by their orientation toward concrete mathematical theories; and in the third place, by their choice of elementary modes of expression (from which all others are then synthesized) and written forms for them.

In the first part of this book a certain class of formal languages is examined systematically. Algorithmic languages are brought in episodically.

The "language–parole" dichotomy, which goes back to Humboldt and Saussure, is as relevant to formal languages as to natural languages. In §3 of this chapter we give models of "speech" in two concrete languages, based on set theory and arithmetic, respectively; because, as many believe, habits of speech must precede the study of grammar.

The language of set theory is among the richest in expressive means, despite its extreme economy. In principle, a formal text can be written in this language corresponding to almost any segment of modern mathematics—topology, functional analysis, algebra, or logic.

The language of arithmetic is one of the poorest, but its expressive possibilities are sufficient for describing all of elementary arithmetic, and also for demonstrating the effects of self-reference à la Gödel and Tarski.

1.3. As a means of communication, discovery, and codification, no formal language can compete with the mixture of mathematical argot and formulas which is common to every working mathematician.

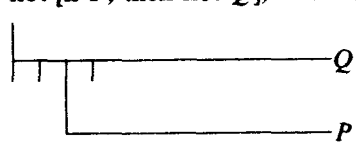
However, because they are so rigidly normalized, formal texts can themselves serve as an object for mathematical investigation. The results of this investigation are themselves *theorems of mathematics*. They arouse great interest (and strong emotions) because they can be interpreted as *theorems about mathematics*. But it is precisely the possibility of these and still broader interpretations that determines the general philosophical and human value of mathematical logic.

1.4. We have agreed that the expressions and texts of a language are elements of certain abstract sets. In order to work with these elements, we must somehow fix them materially. In the modern European tradition (as opposed to the ancient Babylonian tradition, or the latest American tradition, using computer memory), the following notation is customary. The elements of the alphabet are indicated by certain symbols on paper (letters of different kinds of type, digits, additional signs, and also combinations of these). An expression in an alphabet A is written in the form of a sequence of symbols, read from left to right, with hyphens when necessary. A text is written as a sequence of written expressions, with spaces or punctuation marks between them.

1.5. If written down, most of the interesting expressions and texts in a formal language either would be physically extremely long, or else would be psychologically difficult to decipher and learn in an acceptable amount of time, or both.

They are therefore replaced by "abbreviated notation" (which can sometimes turn out to be physically longer). The expression "$xxxxxx$" can be briefly written "$x \cdots x$ (six times)" or "x^6." The expression "$\forall z(z \in x \Leftrightarrow z \in y)$" can be briefly written "$x = y$." Abbreviated notation can also be a way of denoting any expression of a definite type, not only a single such expression; (any expression $101010 \cdots 10$ can be briefly written "the sequence of length $2n$ with ones in odd places and zeros in even places" or "the binary expansion of $\frac{2}{3}(4^n - 1)$.")

Ever since our tradition started, with Vieta, Descartes, and Leibniz, abbreviated notation has served as an inexhaustible source of inspiration and errors. There is no sense in, or possibility of, trying to systematize its devices; they bear the indelible imprint of the fashion and spirit of the times, the artistry and pedantry of the authors. The symbols Σ, $\int$, $\in$ are classical models worthy of imitation. Frege's notation, now forgotten, for "P and Q" (actually "not [if P, then not Q]," whence the asymmetry):



shows what should be avoided. In any case, abbreviated notation permeates mathematics.

The reader should become used to the trinity

formal text

written text——————interpretation of text,

which replaces the unconscious identification of a statement with its form and its sense, as one of the first priorities in his study of logic.

2 First order languages

In this section we describe the most important class of formal languages $\mathcal{L}_1$ —the first order languages—and give two concrete representatives of this class: the Zermelo–Fraenkel language of set theory L_1Set, and the Peano language of arithmetic L_1Ar. Another name for $\mathcal{L}_1$ is *predicate languages*.

2.1. The alphabet of any language in the class $\mathcal{L}_1$ is divided into six disjoint subsets. The following table lists the generic name for the elements in each subset, the standard notation for these elements in the general case, the special notation used in this book for the languages L_1Set and L_1Ar. We then describe the rules for forming distinguished expressions and briefly discuss semantics.

The distinguished expressions of any language L in the class $\mathcal{L}_1$ are divided into two types: *terms* and *formulas*. Both types are defined recursively.

2.2. **Definition.** *Terms* are the elements of the least subset of the expressions of the language which satisfies the two conditions:

(a) Variables and constants are (atomic) terms.
(b) If f is an operation of degree r and $t_1, \ldots, t_r$ are terms, then $f(t_1, \ldots, t_r)$ is a term.

In (a) we identify an element with a sequence of length one. The alphabet does not include commas, which are part of our abbreviated notation: $f(t_1, t_2, t_3)$ means the same as $f(t_1t_2t_3)$. In §1 of Chapter II we

Language Alphabets

Subsets of the Alphabet	Names and Notation		
	General	in L_1Set	in L_1Ar
connectives and quantifiers	$\Leftrightarrow$ (equivalent); $\Rightarrow$ (implies); $\vee$ (inclusive or); $\wedge$ (and); $\neg$ (not); $\forall$ (universal quantifier); $\exists$ (existential quantifier)		
variables	$x, y, z, u, v, \ldots$ with indices		
constants	$c \cdots$ with indices	$\varnothing$ (empty set)	$\bar{0}$ (zero); $\bar{1}$ (one)
operations of degree $1, 2, 3, \ldots$	$f, g, \ldots$ with indices	none	$+$ (addition, degree 2); $\cdot$ (multiplication, degree 2)
relations (predicates) of degree $1, 2, 3, \ldots$	$p, q, \ldots$ with indices	$\in$ (is an element of, degree 2); $=$ (equals, degree 2)	$=$ (equality, degree 2)
parentheses	((left parenthesis);)(right parenthesis)		

explain how a sequence of terms can be uniquely deciphered despite the absence of commas.

If two sets of expressions in the language satisfy conditions (a) and (b), then the intersection of the two sets also satisfies these conditions. Therefore the definition of the set of terms is correct.

2.3. **Definition.** *Formulas* are the elements of the least subset of the expressions of the language which satisfies the two conditions:

(a) If p is a relation of degree r and $t_1, \ldots, t_r$ are terms, then $p(t_1, \ldots, t_r)$ is an (atomic) formula.

(b) If P and Q are formulas (abbreviated notation!), and x is a variable, then the expressions

$$(P)\Leftrightarrow(Q),\quad (P)\Rightarrow(Q),\quad (P)\vee(Q),\quad (P)\wedge(Q),$$
$$\neg(P),\quad \forall x\,(P),\quad \exists x\,(P)$$

are formulas.

It is clear from the definitions that any term is obtained from atomic terms in a finite number of steps, each of which consists in "applying an operation symbol" to the earlier terms. The same is true for formulas. In Chapter II, §1 we make this remark more precise.

The following initial interpretations of terms and formulas are given for the purpose of orientation and belong to the so-called "standard models" (see Chapter II, §2 for the precise definitions).

2.4. Examples and interpretations

(a) The terms stand for (are notation for) the objects of the theory. Atomic terms stand for indeterminate objects (variables) or concrete objects (constants). The term $f(t_1, \ldots, t_r)$ is the notation for the object obtained by applying the operation denoted by f to the objects denoted by $t_1, \ldots, t_r$. Here are some examples from $\mathrm{L_1Ar}$:

$\bar{0}$ denotes zero;

$\bar{1}$ denotes one;

$+(\bar{1}, \bar{1})$ denotes two ($1 + 1 = 2$ in the usual notation);

$+\big(\bar{1} + (\bar{1}, \bar{1})\big)$ denotes three;

$\cdot\big(+ (\bar{1}, \bar{1}) + (\bar{1}, \bar{1})\big)$ denotes four ($2 \times 2 = 4$).

Since this normalized notation is different from what we are used to in arithmetic, in $\mathrm{L_1Ar}$ we shall usually write simply $t_1 + t_2$ instead of $+(t_1, t_2)$ and $t_1 \cdot t_2$ instead of $\cdot(t_1, t_2)$. This convention may be considered as another use of abbreviated notation.

x stands for an indeterminate integer;

$x + \bar{1}$ (or $+(x, \bar{1})$) stands for the next integer.

In the language L_1Set all terms are atomic:

x stands for an indeterminate set;
$\varnothing$ stands for the empty set.

(b) The formulas stand for statements (arguments, propositions, . . .) of the theory. When translated into formal language, a statement may be either true, false, or indeterminate (if it concerns indeterminate objects); see Chapter II for the precise definitions. In the general case the atomic formula $p(t_1, \ldots, t_r)$ has roughly the following meaning: "The ordered r-tuple of objects denoted by $t_1, \ldots, t_r$ has the property denoted by p." Here are some examples of atomic formulas in L_1Ar. Their general structure is $=(t_1, t_2)$, or, in nonnormalized notation, $t_1 = t_2$:

$$\bar{0} = \bar{1}, \qquad x + \bar{1} = y.$$

Here are some examples of formulas which are not atomic:

$$\neg(\bar{0} = \bar{1}),$$
$$(x = \bar{0}) \Leftrightarrow (x + \bar{1} = \bar{1}),$$
$$\forall x\big((x = \bar{0}) \vee (\neg(x \cdot x = \bar{0}))\big).$$

Some atomic formulas in L_1Set:

$$y \in x \qquad (y \text{ is an element of } x),$$

and also $\varnothing \in y$, $x \in \varnothing$, etc. Of course, normalized notation must have the form $\in(xy)$, and so on.

Some nonatomic formulas:

$$\exists x(\forall y(\neg(y \in x))): \qquad \text{there exists an } x \text{ of which no } y \text{ is an element.}$$

Informally this means: "The empty set exists." We once again recall that an informal interpretation presupposes some standard interpretive system, which will be introduced explicitly in Chapter II.

$$\forall y(y \in z \Rightarrow y \in x): \qquad z \text{ is a subset of } x.$$

This is an example of a very useful type of abbreviated notation: four parentheses are omitted in the formula on the left. We shall not specify precisely when parentheses may be omitted; in any case, it must be possible to reinsert them in a way that is unique or is clear from the context without any special effort.

We again emphasize: the abbreviated notation for formulas are only material designations. Abbreviated notation is chosen for the most part with psychological goals in mind: speed of reading (possibly with a loss in formal uniqueness), tendency to encourage useful associations and discourage harmful ones, suitability to the habits of the author and reader,