CONCEPTS OF MATHEMATICAL MODELING

Thou hast ordered all things
in measure and number and weight.
Ch. 11, verse 20, Book of Wisdom

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PREFACE

This book consists of a number of more or less independent sections designed to illustrate the most important principles of the mathematical modeling process. It is intended for an introductory undergraduate-level course. A student with a good knowledge of calculus and a little probability and matrix theory would find all of it accessible. If one wishes to avoid the sections involving probability, there is still a semester's worth of material here. The sections are organized according to a definite point of view, but the text can also be treated as a sampler with success (especially the sections in the first three chapters).

The basic premise of this book is that all mathematics students need an understanding of mathematical modeling, even those who won't become practitioners of the craft. My aim is to show students that mathematical modeling is interesting and useful and to illustrate, without undue technicality or specialization, some of the main themes and methods of the subject. A course based upon this book would be a good springboard to advanced studies in aspects of modeling which are touched upon lightly here: optimization, computer simulation, hands-on experience with substantial projects, statistics, etc.

The first time I taught mathematical modeling, my notes could have been entitled "Models I Happen to Like." It didn't seem sensible to publish anything so purely personal, and so I began thinking harder about how to teach mathematical modeling. One approach is to organize by mathematics: first we study differential equations models in various fields, then we study probability models in various fields, etc. Another approach is to organize by subject matter: first we study population and various mathematical models for it, then we study political science and various mathematical approaches to it, etc. Excellent books of these types have been written; so I have chosen a third approach for this book: to display aspects of the *process* of modeling. There is a lot to this process besides knowing mathematics and the subject matter of the model. There are strategies, attitudes toward data, choices to be made about mathematical tools— in short, a lot of "know-how." Put another way, this book concerns the question: "What does 'modeling' mean in the phrase 'mathematical modeling'?"

By covering sections out of order, an instructor can easily replace my organization of the material with a more mathematical organization. For example, a substantial part of a course could be devoted to applications of difference and differential equations by starting with the long Section 2 in Chapter 5 and adding to it Sections 3 and 4 of Chapter 1 and Section 5 of Chapter 3. Likewise, a course segment on probabilistic models could use the long Section 3 of Chapter 5, together with Section 6 of Chapter 1, Section 2 of Chapter 2, and Section 1 of Chapter 5. One could assemble a minisegment on applications of geometry from Section 1 of Chapter 2 and Sections 6 and 7 of Chapter 3. A unit on optimization could be based on Chapter 4 and Section 7 of Chapter 1.

For a subject matter organization, one could devise a long course segment dealing with population.

As an aid to instructor and student, each section is preceded by an abstract and statement of prerequisites for that section. This should help the instructor make use of the considerable flexibility of the book.

Answers or hints are provided for selected exercises (those marked by dots in the margin).

There is no shortage of good mathematical models to use in writing a textbook. The challenge seems to be in choosing a good set of constraints to guide the selection process. Some of the constraints (besides personal taste) which I set for myself are:

1. The level should be fairly elementary.
2. The sections should be largely independent.
3. There should be variety in subject matter (physical, biological, social sciences, and operations research), mathematical tools, and historical eras from which the applications are drawn.
4. Certain important subjects, namely difference equations and optimization, which often do not find a natural home in the curriculum for mathematical science majors, should appear.
5. I have tried to emphasize "classic" models and to exclude models of narrow or passing interest, no matter how novel they may be. Hopefully, the models treated here will merit inclusion in the curriculum for many years.

I owe special thanks to the following people who reviewed this book when it was in manuscript form: Martin Braun, Queens College; James P. Jarvis, Clemson University; William F. Lucas, Cornell University; Joseph Malkevitch, York College (CUNY); Rochelle Meyer, Nassau Community College; Herman F. Senter, Clemson University; Donald R. Sherbert, University of Illinois at Urbana-Champaign; Robert L. Wilson, Jr., Washington and Lee University; and Matthew Witten, Illinois Institute of Technology. Others who have influenced or assisted me include: James Frauenthal, Bell Labs; Noreen Goldman, Princeton University; T. N. E. Greville, University of Wisconsin; Frank Hoppensteadt, University of Utah; Nathan Keyfitz, Harvard University; Dick Montgomery, Southern Oregon State University; Fred Pohle, Adelphi University; Bill Quirin, Adelphi University;

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Walter J. Meyer

CONTENTS

CHAPTER

ONE

THE SCOPE OF MATHEMATICAL MODELING

1 MODELS, MATHEMATICAL AND OTHERWISE

Abstract This section introduces the concepts of mathematical and nonmathematical models in a nontechnical way. Our examples show that mathematical models are often, but not always, better; that nonmathematical models may evolve into mathematical ones; and that experimental work may be needed to provide data for mathematical models.

Prerequisites None.

No human investigation can claim to be scientific if it doesn't pass the test of mathematical proof.

Leonardo Da Vinci

Mathematical modeling is an attempt to describe some part of the real world in mathematical terms. It is an endeavor as old as antiquity but as modern as tomorrow's newspaper. It has led to some good mathematical models and some bad ones, which are best forgotten. Sometimes mathematical models have been welcomed with great enthusiasm—even when their value was uncertain or negligible; other times good mathematical models have been greeted with indifference, hostility, or ridicule. Mathematical models have been built in the physical, biological, and social sciences. The building blocks for these models have been taken from calculus, algebra, geometry, and nearly every other field within mathematics.

In short, mathematical modeling is a rich and diverse activity with many interesting aspects. The aim of this book is to display by examples some of the many facets of mathematical modeling.

But before we plunge into this, it seems only fair to say something about models of a nonmathematical nature. In ordinary language the word "model" has many meanings. What we will mean by it is this.

Definition

A model is an object or concept that is used to represent something else. It is reality scaled down and converted to a form we can comprehend.

For example, a model airplane, made of wood, plastic, and glue, is a model of a real airplane. Another example is the idea that, in politics, public opinion is like a pendulum because it changes periodically from left- to right-wing ideas then back again in a way which reminds us of a pendulum swinging back and forth. In our terminology we would say that a pendulum is a model for public opinion.

A model aiplane and pendulum are physical objects; so they are not mathematical models.

Definition

A mathematical model is a model whose parts are mathematical concepts, such as constants, variables, functions, equations, inequalities, etc.

Example 1 that follows illustrates the differences between mathematical and nonmathematical models. In this example the mathematical model is, in many ways, superior to its nonmathematical counterpart. The other examples in this section also illustrate the great value of mathematical models. But we shall see that nonmathematical models have value as well. Among other things, they often stimulate the development of mathematical models.

Example 1 Aircraft Flight

To find out how an aircraft will behave in flight, we could make a physical model of the aircraft and test it under various weather conditions. There are a great many things one might want to know: Is the plane stable in the air? How fast can it go? How steeply can it climb? Etc. To focus our discussion, let's consider the question of how great the lift force on the plane is when it takes off.

The lift force is the force pushing up on the wings. This force is largely what determines how steeply the plane can climb. If we did experiments with a physical model, we could find out almost anything we wanted to know about it. For example, we could discover that the lift force was dependent on how fast the plane was moving. By flying the plane at different speeds, we could make a table of values relating lift force to velocity and a graph of this table of values that might look like Figure 1.

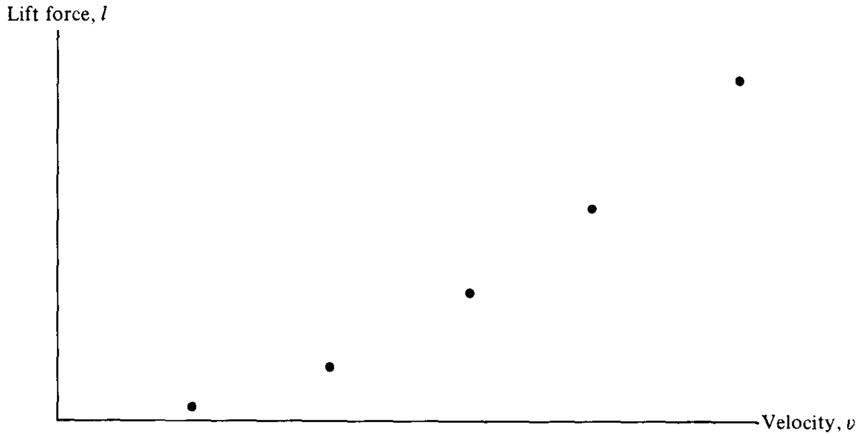


Figure 1 Points are plotted from a table of values obtained from wind-tunnel experiments. The different points represent trials at different speeds.

But there is an entirely different approach to this problem, one based on a mathematical model. This mathematical model consists of a single equation which relates the lift force to other factors. It is

$$l = C_l \frac{\rho}{2} s v^2 \tag{1}$$

where l = lift force
C_l = a certain numerical value called the lift coefficient whose exact value depends on the shape of the plane
ρ = density of the air
v = velocity of the plane
s = total surface area of the tops of the wings

We can estimate s from the blueprints of the plane we propose to build. ρ is a measurement we can make in the atmosphere. (It may differ a little from one airport to another.) C_l is a number which differs from plane to plane and is a little hard to estimate for a plane that has not yet been built and tested. But there are methods that yield reasonable estimates; so let's assume C_l is known. Then the product $C_l(\rho/2)s$ in Equation (1) becomes a known constant. If we call this constant a, then Equation (1) becomes an equation linking only two variables, l and v:

$$l = av^2 \tag{2}$$

Using this equation, we can generate the graph shown in Figure 2 with a moment's worth of calculation and plotting.

Which approach is better, experiments on the physical model or predictions from the mathematical model? Building physical models is time consuming: it might take days to make a good model plane. It's also expensive. In both these

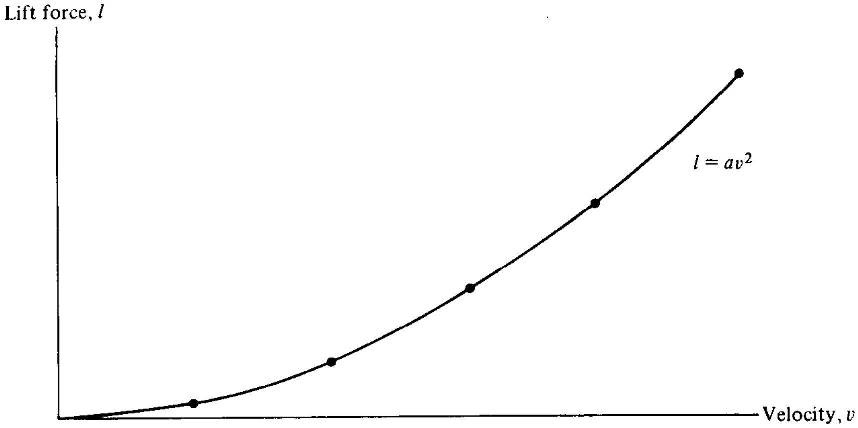


Figure 2 The curve is plotted from the formula $l = av^2$.

respects the mathematical model is superior. But it has another advantage. It tells us things that our physical experiments don't.

For example, suppose we wanted to increase the lift force on the plane. Figure 1, which is based on our experiments, shows that lift increases as velocity increases. A larger velocity will probably do the trick; so we might try outfitting the plane with a more powerful engine. But Figure 1 doesn't tell us exactly what velocity we need to achieve a given lift force. By using Equation (2), we can find exactly what value of v we need to achieve a given value of l. This is merely a matter of solving for v and substituting the value of l desired.

If we go back to Equation (1), we also discover there is another means of accomplishing our goal, namely, to increase s by making the wings bigger. To discover this from our physical experiments would be impossible without building a series of additional physical models with various wing sizes. Thus our mathematical model is cheaper, faster, and more versatile than experimentation on a physical model of a plane.

Of course there is much more to aircraft performance than lift force. There is air drag, stability in flight, and a host of other factors to be considered. In principle, each of these factors *could* be modeled mathematically in an equation. The whole collection of these equations would be a mathematical model for aircraft performance. Such giant models have been built and are being continually improved. Although they are still far from perfect, they are slowly replacing physical models and wind-tunnel experiments in the aircraft industry. In 1976 the American Society of Mechanical Engineers' winter meeting featured a symposium on the question of whether computer-implemented mathematical models of aircraft performance might soon make wind-tunnel experiments obsolete.

One thing one must keep in mind about mathematical models is that they don't arise out of thin air. They need to be discovered first, and this takes time and ingenuity. The present-day availability of good mathematical models for aircraft design is a far cry from the situation that prevailed in the early history of aviation. For example, in 1879 the Aeronautical Society of Great Britain could report:

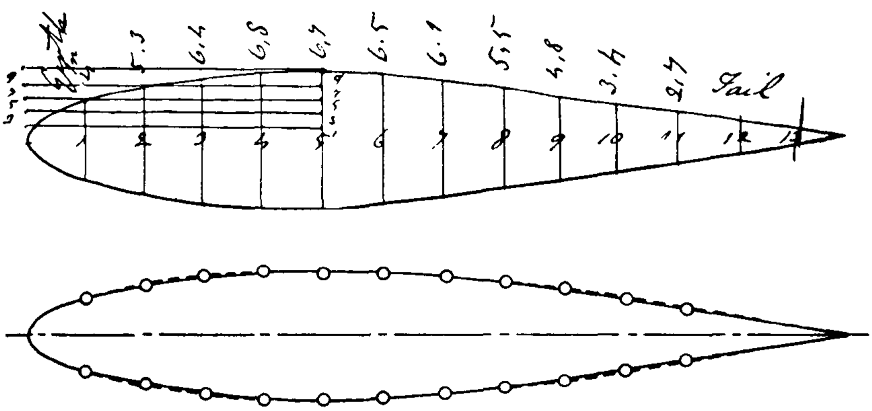


Figure 3 *Above*: Sir George Cayley's sketch of the cross section of a trout. [*From "Aeronautical and Miscellaneous Note-Book (ca. 1790–1826) of Sir George Cayley," Cambridge University Press, 1933.*] *Below*: A comparison of Cayley's trout section with modern low-drag airfoil sections. Circles indicate trout; ——— N.A.C.A. 63A016; ------- LB N-0016. (*Reprinted from T. von Kármán: "Aerodynamics: Selected Topics in the Light of their Historical Development." Copyright 1954 by Cornell University. Used by permission of the publisher, Cornell University Press.*)

"Mathematics up to the present date have been quite useless to us in flying." A cynic might reply that nothing else was much use either. The first sustained flight by the Wright brothers didn't take place until years later in 1903.

Developing a mathematical model requires not only time and ingenuity, but data as well. In the case of aeronautics, a good many of the data required are obtained from wind-tunnel experiments. Thus physical experiments and nonmathematical models play a role in giving birth to mathematical models. Our emphasis on the value of mathematical models does not mean that nonmathematical models have only slight value.

In the case of aircraft flight, there is a striking example of a nonmathematical model that turned out to be beautifully simple and fairly effective. In the early nineteenth century, long before anyone had built a working airplane, Sir George Cayley was concerned about finding a good shape for the cross section of a wing (Figure 3) that would minimize air drag. Lacking a good mathematical model and being unable to carry out the necessary experiments, Cayley hit upon the interesting notion of making the wing cross section in the shape of a trout. His reasoning was that the water drag on the trout swimming through water was analogous to the air drag on a wing traveling through air. A further assumption of Cayley's nonmathematical model was that Mother Nature, through the mechanisms of evolution (the struggle for existence and survival of the fittest) would have already hit upon a good design.

Example 2 Chemistry

In 1786 the philosopher Immanuel Kant asserted that chemistry could never become a science. It would be hard to find a more spectacularly wrong prediction in all of the history of human thought. But we should be kind to Kant because in his time chemistry truly seemed like a random collection of recipes and rules of thumb.

One-hundred years later chemistry was well on its way to becoming a science. Specifically, in 1887 the chemist Clemens Winkler provided dramatic evidence of the value of Dmitri Mendeleev's Periodic Table of the Elements (see Section 5 of Chapter 2). This is an important milestone because it showed that there was order and logic behind the jumble of facts which chemists had collected up until that time.

Yet another hundred years or so brings us to the modern era, in which parts of chemistry are well on their way to becoming a mathematical science in the sense that mathematical models are playing a vital role in the process of chemical discovery. A case in point is the determination of the exact spatial relationships of the atoms making up a molecule by William Nunn Lipscomb, who won the Nobel prize in chemistry in 1976. Lipscomb won his Nobel prize in part for discovering new types of molecules in the borane family. Before many of these molecules were actually found in nature, Lipscomb had predicted their existence with the aid of higher mathematics (and a children's construction toy called *D-stix*).

Lipscomb's own words give testimony to the growing role of mathematics in chemistry:

> To give you an idea of how radically chemistry has changed over the years, I can say that I spend practically no time in a laboratory. At the Gibbs Laboratory we do no chemical analysis and practically no synthesis. But by contrast I probably use more time on Harvard's computers and other computers available to us than any individual—huge chunks of computer time.
>
> It's a far cry from my own antecedents. I became interested in chemistry when my mother gave me an A. C. Gilbert chemistry set. But today the real research is in ideas—even intuition—expressed usually, in the special languages with which mathematicians and computers communicate.

Example 3 Paranoia

One of the most influential models of the twentieth century is also one of the least mathematical: Sigmund Freud's model of the personality as divided into three warring factions—the id, the ego, and the superego. His model contains more than just this three-way division; it also contains a lot of assumptions about how and why these parts of the personality compete for control. None of it is mathematical, and, partly for this reason, it is vague and hard to test but easy to argue about. Nevertheless, even Freud's greatest critics agree that he captured important germs of truth in his model.

Will these germs of truth ever be transformed into a good mathematical model? At the present time there is no satisfactory mathematical model of an entire normal human personality. Some would say that there never can be. After all, would it ever be possible to measure the strength of a feeling such as anxiety? It is worth noting that Freud himself thought of anxiety in quantitative terms.

Although he had no instrument or procedure to measure it, he described it as something which could, in principle, be precisely measured.

Even though there is not yet a satisfactory mathematical model of a whole normal human personality, there is a good model of a small part of a sick human personality. This is a mathematical model, called *PARRY*, devised by K. M. Colby to imitate the conversation of a paranoid personality. The model is in the form of a computer program, which can hold a conversation with an ordinary human being (in written form via computer terminal) about its psychological difficulties. The entire conversation is supposed to resemble a session between a paranoid human being and a therapist.

The model has the capability of "understanding" English well enough that it can devise appropriate responses to most remarks the therapist might make. These responses are not random however. The model has built into it a large repertory of beliefs and feelings, which revolve around betting on horse races, dishonest bookmakers, the Mafia, and other related topics. The model has the capability of generating a very large number of different conversations in response to the various things the human conversational partner might say. The conversations which the model can engage in may be thought of as the fruits of the model (rather like weather reports are fruits of a single mathematical model of the atmosphere).

The model itself is the computer program, i.e., the calculations and instructions that determine how PARRY will respond to a given remark of the therapist. The response at any point depends on two factors: the last few exchanges in the conversation and the emotional state the model has generated for itself at that point. This emotional state is portrayed in the model by numbers which keep track of specific emotions, such as fear, anger, and mistrust. These numbers are constantly updated according to mathematical formulas which take into account the remarks each side makes in the conversation.

How realistic is PARRY? Experiments indicate that it is realistic enough to fool professional therapists. A group of psychiatrists was asked to interview a group of paranoid "patients" via a teletype machine and without personal contact. Although none of them knew it, only some of them were dealing with real patients; the others were conversing with PARRY. None of the psychiatrists who conversed with PARRY suspected that they were not dealing with a human being. In another stage of the experiment, typewritten transcripts of the conversations were sent to other psychiatrists, who were then asked to decide which conversations involved PARRY and which involved human paranoids. These psychiatrists showed no ability to discriminate between PARRY and human paranoids. The number of successful identifications was about what would be expected through chance guessing.

Although PARRY is very successful at what it does, it must be kept in mind that what it does is very limited. It can't say anything intelligent about baseball, mathematics, or other subjects outside the repertory it is programmed for. If you mention the Easter Bunny, it will probably respond that it's all a racket controlled by the Mafia. If you ask it to do a math problem, it might simply refuse. If you tell it a joke, it won't laugh; nor can it tell you one back. It is also very bad at making logical connections. The reason it fools the psychiatrists is that psychotherapy sessions are narrow in exactly the way PARRY is narrow. Psychotherapy sessions do not normally include jokes, tests on arithmetic or logic, or conversation about irrelevant subjects such as the Easter Bunny. In addition, therapists who believe they are talking to paranoids are prepared for the conversations to be slightly weird. Consequently, if the computer program performs poorly now and then, the poor performance can easily be mistaken for mental illness.

Perhaps you are wondering what conversation with PARRY is like. Here is an example. The uppercase letters show the responses of PARRY, and the lowercase letters show the human being's part of the conversation.

Why are you in the hospital?
I SHOULDN'T BE HERE.
Who brought you here?
THE POLICE.
What trouble did you have with the police?
COPS DON'T DO THEIR JOB.
What do they do instead?
COPS ARREST THE WRONG PEOPLE.
They arrest the right ones sometimes.
THE LAW SHOULD GET THOSE ITALIAN CROOKS.
Are you Italian?
MY BACKGROUND IS BRITISH, BUT I WAS BORN IN THIS COUNTRY.
Do you know any Italian crooks?
I TRY TO AVOID THE UNDERWORLD.
Are you afraid of the underworld?
THE MAFIA ARE THE WORST GANGSTERS.
Are you connected with the Mafia?
YOU KNOW, THEY KNOW ME.
How do they know you?
THEY SPY ON ME.
Why do they?
THEY WANT TO GET FALSE EVIDENCE TO FRAME ME.
What do they have against you?
THEY ARE OUT TO GET ME.
What for?
I HAVE INCRIMINATING EVIDENCE AGAINST THE MAFIA.
Are you sure of all this?
YOU DON'T BELIEVE ME?
I am doing my best.
(SILENCE ON THE "PATIENT'S" PART)
(Silence on the psychiatrist's part)
CAN'T YOU THINK OF A DECENT QUESTION TO ASK?

You seem hostile to me.
SOME OF YOUR REMARKS ANNOY ME.
I am sorry.
FORGET THE EXCUSES, YOU GUYS ARE ALL ALIKE.

Example 4 Shortest Paths

Suppose you have a network of roads and intersections, as shown in Figure 4. The numbers on the line segments represent the lengths, in miles, of the road segments. This picture is an example of a weighted graph. This weighted graph may be thought of as a mathematical model of the real network of roads. We might use this model to find the shortest path from one intersection to another.

The weighted graph is not the only way to model the road network mathematically. Indeed, for some purposes it may not be a very good model. For example, if we wanted to feed this information to a computer, we couldn't just show the computer the weighted graph because most computers can't see. They have nothing analogous to the human eye which they could use for visual processing. What we could do instead is to describe the road network with a distance matrix.

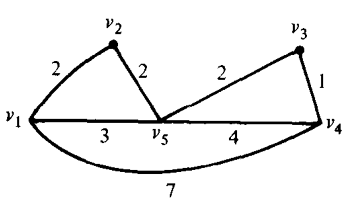
Definition

The distance matrix of a road network is a square matrix with as many rows and columns as intersections. If we denote the ith-row, jth-column entry by a_{ij}, then the following rule shows how a_{ij} is determined:

$$a_{ij} = \begin{cases} \text{the length of the segment joining intersections } i \text{ and } j \\ \infty \text{ if there is no segment joining intersections } i \text{ and } j \\ 0 \text{ if } i = j \end{cases}$$

Finally, we could build a nonmathematical model of the road network out of buttons and string: let the buttons represent the intersections and the strings represent the road segments. Furthermore, cut each string to a length proportional to the length of the road segment it represents. (For example, we might let each mile be represented by 1 inch of string.)

A common mathematical problem is to find the shortest path from one intersection to another in a road network. Which of our three models is best for working this out? In the example of Figure 4 trial and error on the weighted



$$\begin{bmatrix} 0 & 2 & \infty & 7 & 3 \\ 2 & 0 & \infty & \infty & 2 \\ \infty & \infty & 0 & 1 & 2 \\ 7 & \infty & 1 & 0 & 4 \\ 3 & 2 & 2 & 4 & 0 \end{bmatrix}$$

Figure 4

graph seems easy enough. But suppose we had a network with 10, 20, or 50 intersections? For graphs of this size the eye becomes confused and trial and error is not efficient. It is possible to get the answer from our distance-matrix model by doing some numerical manipulations on the matrix (relying on some theorems in matrix algebra which we won't mention). But the string-and-button model also gives a quick, easy to understand, and foolproof solution. Grasp the "start" and "end" buttons in opposite hands and pull in opposite directions until some set of string segments tightens up into a straight line leading from the start button to the end button. Thus, each of the three models can be useful in its own way.

Example 5

Figure 5 shows a schematic diagram of the human eye, which is meant to illustrate how the eye muscles move the eyeball around in its socket. Two of the muscles are shown as shaded areas. Within these shaded areas are springs and dashpots (shock absorbers). Naturally the eye muscles don't really have springs or shock absorbers inside them. These mechanisms are used to represent, or to model, the

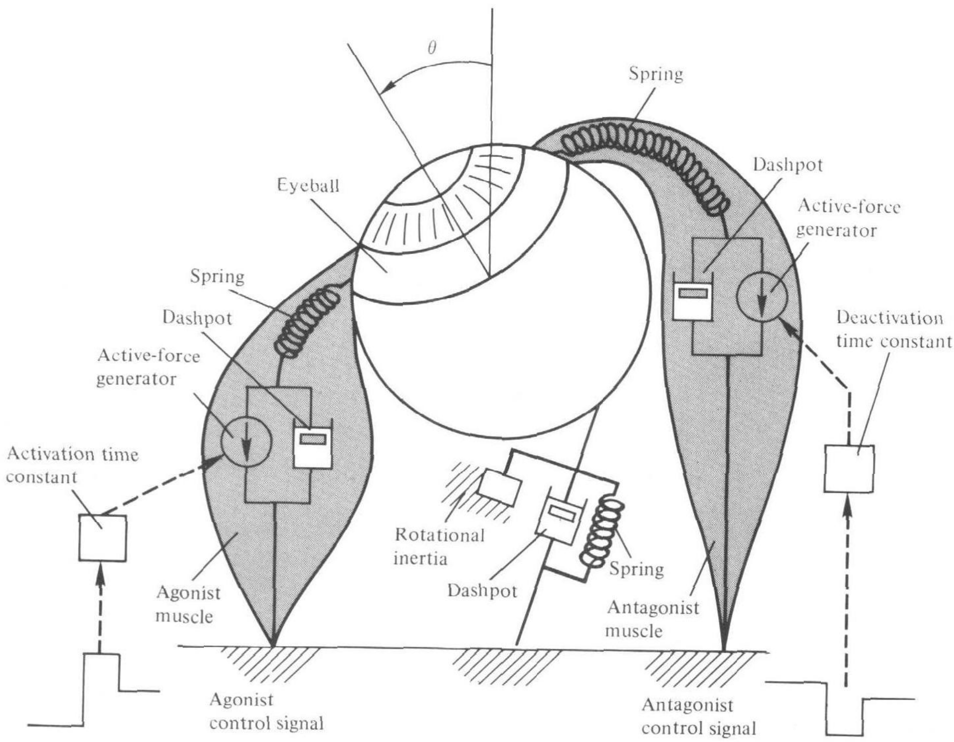


Figure 5 Schematic diagram of the human eye. (*From "The Trajectories of Saccadic Eye Movement" by A. Bahill and L. Stark. Copyright © 1979 by Scientific American, Inc. All rights reserved.*)