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EXECUTIVE EDITORS

VOLUME 5

Classical Optimization to Computer Output

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EXECUTIVE EDITORS

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VOLUME 5

Classical Optimization to Computer Output

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CLASSICAL OPTIMIZATION

ORIGINS OF OPTIMIZATION

The concept of the optimum (greatest or least) value of a mathematical function is one that was formulated in a precise way rather late in the history of mathematics. However, its origins are steeped in antiquity.

Virgil tells us of the founding of the city of Carthage by Queen Dido, who was allowed to have the largest area of land that could be surrounded by the hide of a bull. Queen Dido prepared a rope of unspecified thickness (a nontrivial detail omitted in Virgil) and then, perhaps guided by the gods, hit upon the optimal solution. With the sea as a diameter, she arranged the rope of finite length in the form of a semicircle. Indeed, this half-circle has the largest possible area for a fixed perimeter. Archimedes (287–212 B.C.) conjectured, but did not prove, that this was the correct solution. This was not proven until the development of the calculus of variations in the nineteenth century.

A great deal of thought about maximization and minimization is found in some of the work of the ancient Greek geometers. For example, in Book V of his treatise on *Conic Sections*, Appolonius (ca. 262–190 B.C.) deals with the problem on the maximum and minimum lengths that can be drawn from various points to a conic section. In work of great originality, he investigated maximum and minimum length distances for certain points on the major axis of a central conic or on the axis of a parabola. He does the same for points on the minor axis of an ellipse. He also proved that a line from a point within a conic to a point on the conic which is a maximum or minimum is perpendicular to a tangent line through the point, a result of great significance. In short, he proved that the maximum and minimum lines to the point are normal to a tangent at the point of tangency.

Intuitively, the solution to another geometric optimization problem has been known for a long time, viz., that the shortest distance between two points on a plane is a straight line. The early Greeks used this principle in thinking about the behavior of light. Heron of Alexandria thought of light traveling between two points by the shortest path. Later Fermat, in the seventeenth century, formulated a principle of least time which generalized the earlier principle. Virtually all the principles of classical mechanics and optics down to the present day in wave mechanics have been or can be formulated in terms of various minimum principles.

The creation of the calculus, which led to the existence of the subject of this article, was, at least in part, motivated by the problem of finding the maximum or minimum value of a function. If a cannonball is shot from a cannon, the distance it will travel horizontally (the range) depends on the angle of inclination of the cannon to the ground. An early problem was to find the angle that would maximize the range. In the seventeenth century, Galileo determined that the maximum range (in a vacuum) is obtained for an angle of inclination of 45°. He was also able to determine the maximum heights reached by projectiles for various angles. Another early scientific influence on the development of the calculus was the study of the motion of the planets. Problems of optimization were involved in such determinations as the greatest and least distances of a planet from the sun.

Again the pervasive influence of geometry in leading to the development of calculus, one of the two greatest mathematical developments of all time, and hence to classical analysis, should be noted. Kepler made an early and crucial observation in his *Stereometria Doliorum* in 1615. He was interested in the optimal shape of casks for wine. He showed that, of all right parallelepipeds inscribed in a sphere and having square bases, the cube is the largest. He proceeded by calculating the volume for various choices of the dimensions. He then made an exceedingly important observation, viz., that as the maximum volume was approached, the *change* in volume for a *fixed* change in dimensions grew smaller and smaller. In the language of the calculus, the first derivative approached zero.

In a similar vein, Fermat, in his *Methodus ad Disquirendam Maximam and Minimam* in 1637, gave his method for finding maxima and minima by using the following as an example. Given a straight line segment, it is required to find a point on the line segment such that the rectangle contained by the two segments is a maximum. If the length of the whole line segment is L and the point marks off a part of length P, then the rectangle has area $P(L - P) = PL - P^2$. Fermat then replaces P by $P + E$. The remaining part is $L - (P + E)$ and the rectangular area is now $(P + E)(L - P - E)$. He then maintains, indicating complete insight into the principles we now understand, that the two areas should be equated, resulting in

$$PL + EL - P^2 - 2EP - E^2 = PL - P^2 \tag{1}$$

He then subtracts $PL - P^2$ from both sides and divides by E to obtain

$$L = 2P + E \tag{2}$$

He then argues that $E = 0$ at a maximum and so obtains $L = 2P$. Therefore, the rectangle is a square. He also generalizes the argument for any function, but of course does not justify dividing by E and setting E to zero.

It is interesting to note that at least one of the influences that led to the development of the calculus was the problem of finding the maxima and minima of a function. In turn, the calculus became a splendid instrument for examining such problems in a general setting. This is the subject matter of this article.

MATHEMATICAL BACKGROUND

We present here some of the mathematical concepts, either as definitions or theorems, that are required for the subsequent discussion. For proofs of the theorems that are omitted, the reader may consult Apostol's *Mathematical Analysis* [1].

Classical optimization theory is restricted to a consideration of finding maxima and minima of *continuous* and *differentiable* functions. Hence we need to define these terms.

Continuity. A function of n variables $x_1, x_2, \ldots, x_n$ is continuous at the point[1] $\mathbf{x}^0 = (x_1{}^0, x_2{}^0, \ldots, x_n{}^0)$ if for every $\epsilon > 0$ there exists a set of corresponding δ_j, $j = 1, 2, \ldots, n$ such that for $|h_j| < \delta_j$, $j = 1, 2, \ldots, n$ and $\delta_j > 0$, $j = 1, 2, \ldots, n$

$$|f(x_1{}^0 + h_1, x_2{}^0 + h_2, \ldots, x_n{}^0 + h_n) - f(x_1{}^0, x_2{}^0, \ldots, x_n{}^0)| \leq \epsilon \tag{3}$$

In vector notation we would write (3) as:

$$|f(\mathbf{x}^0 + \mathbf{h}) - f(\mathbf{x}^0)| \leq \epsilon \tag{4}$$

That not all functions are continuous should be obvious to the reader. For example, the function

$$f(x) = \begin{cases} 0, & -\infty \leq x \leq 0 \\ x, & 0 \leq x \leq 4 \\ \dfrac{x}{2}, & 4 \leq x \leq \infty \end{cases} \tag{5}$$

is an example of a discontinuous function of one variable.

Differentiability. A function $f(\cdot)$ of n variables is differentiable at a point $\mathbf{x}^0$ if the derivative of $f(\cdot)$ with respect to each of the independent variables exists. The derivative with respect to each variable is defined as:

$$\frac{\partial f(\mathbf{x}^0)}{\partial x_j} = \lim_{h_j \to 0} \frac{f(x_1{}^0, x_2{}^0, \ldots, x_j{}^0 + h_j, \ldots, x_n{}^0) - f(\mathbf{x}^0)}{h_j} \quad (j = 1, 2, \ldots, n) \tag{6}$$

(The notation $\partial f(\mathbf{x}^0)/\partial x_j$ indicates the derivative evaluated at the point $\mathbf{x}^0$. This is sometimes noted as $\partial f(\mathbf{x})/\partial x_j|_{\mathbf{x}=\mathbf{x}^0}$.)

A function may be continuous without being differentiable. For example, $f(x) = x^{2/3}$ has no derivative at the point $x = 0$.

Absolute (Global) Maximum. A function $f(\cdot)$ takes on its absolute (or global) maximum at a point $\mathbf{x}^*$ if $f(\mathbf{x}) \leq f(\mathbf{x}^*)$ for all values of $\mathbf{x}$ over which the function f is defined. We will assume, in order to rule out certain anomalous cases, that the values of $\mathbf{x}$ at which $f(\cdot)$ attains its maximum are actually in the set of values over which $\mathbf{x}$ is defined.

The definition of absolute (or global) minimum can be obtained from the preceding definition by reversing the sense of the inequality between $f(\mathbf{x})$ and $f(\mathbf{x}^*)$. For example, the function $f(x_1, x_2) = 2x_1{}^2 + 3x_2{}^2 - 8x_1 - 12x_2 + 40$ has a

[1] The boldface quantities such as $\mathbf{x}$ are to be regarded as points in an n-dimensional Euclidean space or simply as n-component vectors.

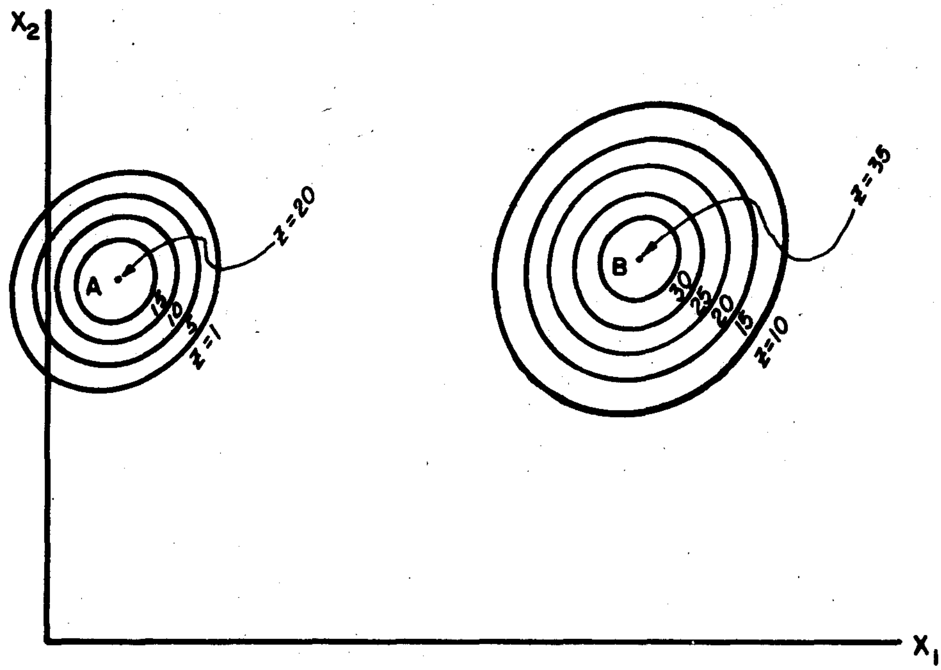

Fig. 1. Local and global maxima.

global minimum at $x^* = (2, 2)$. We will see, subsequently, how this can be established.

Strong Relative (or Local) Maximum. A function $f(\cdot)$ takes on a strong relative (or local) maximum at a point $\mathbf{x}^0$ if there exists an ϵ, $0 < \epsilon < \delta$, such that for all $\mathbf{x}$ satisfying $0 < \|\mathbf{x} - \mathbf{x}^0\| < \epsilon$, it is the case that $f(\mathbf{x}) < f(\mathbf{x}^0)$.

In geometric language, the above definition states that if a function $f(\cdot)$ has a strong local maximum at some point $\mathbf{x}^0$ in an n-dimensional Euclidean space E^n, then there is a hypersphere (neighborhood) about $\mathbf{x}^0$ of radius ϵ, such that for every point $\mathbf{x}$ in the interior of the hypersphere, $f(\mathbf{x})$ is strictly less than $f(\mathbf{x}^0)$. A strong relative minimum is defined by reversing the inequality between $f(\mathbf{x})$ and $f(\mathbf{x}^0)$ in the preceding definition.

Weak Relative (or Local) Maximum. A function $f(\cdot)$ takes on a weak relative (or local) maximum at a point $\mathbf{x}^0$ if there exists an ϵ, $0 < \epsilon < \delta$, such that for all $\mathbf{x}$ satisfying $0 < \|\mathbf{x} - \mathbf{x}^0\| < \epsilon$, it is the case that $f(\mathbf{x}) \leq f(\mathbf{x}^0)$ and there is at least one point $\mathbf{x}$ in the interior of the hypersphere $\|\mathbf{x} - \mathbf{x}^0\| < \epsilon$ such that $f(\mathbf{x}) = f(\mathbf{x}^0)$.

In general, we will not distinguish between strong and weak local maxima. They will simply be called local maxima. It should be clear that a weak relative (or local) minimum is defined by reversing the inequality between $f(\mathbf{x})$ and $f(\mathbf{x}^0)$ in the preceding definition.

In connection with the definition of maxima and minima, it should be noted that if $f(\cdot)$ has an absolute maximum at a point $\mathbf{x}^*$, then $-f(\cdot)$ has an absolute minimum at $\mathbf{x}^*$, and vice versa. Similarly, if $f(\cdot)$ has an absolute maximum at a point $\mathbf{x}^0$, then $-f(\cdot)$ has an absolute *minimum* at $\mathbf{x}^0$, and vice versa.

In Fig. 1 we have shown graphically a function $z = f(\mathbf{x}) = f(x_1, x_2)$. What is plotted are the contours of z in the two-dimensional plane, E^2. At the points marked A and B, two relative maxima are shown. Assuming that $f(x_1, x_2)$ goes to $-\infty$ as $x_1, x_2 \to \infty$, we see that the point B is also the global maximum.

Convex and Concave Functions. A function $f(\cdot)$ is convex over some convex set X in E^n if for any two points $\mathbf{x}_1$ and $\mathbf{x}_2$ in X and for all λ, $0 \leq \lambda \leq 1$, $f[\lambda\mathbf{x}_1 + (1 - \lambda)\mathbf{x}_2] \leq \lambda f(\mathbf{x}_1) + (1 - \lambda)f(\mathbf{x}_2)$. If $f[\lambda\mathbf{x}_1 + (1 - \lambda)\mathbf{x}_2] \geq \lambda f(\mathbf{x}_1) + (1 - \lambda)f(\mathbf{x}_2)$, then the function is concave. If in the aforementioned expressions the inequalities are strict, then the function $f(\cdot)$ is said to be strictly convex in the first case and strictly concave in the second.

Consider the example of Fig. 2. The function $f(\cdot)$ is defined over the convex set X equal to the real line. Any point on the curve of $f(\cdot)$ between x_1 and x_2 is equal to $f[\lambda x_1 + (1 - \lambda)x_2]$ for $0 \leq \lambda \leq 1$, $\lambda f(x_1 + (1 - \lambda)f(x_2)$ will be a point on the straight line segment shown in Fig. 2. Hence a convex function is one that lies on or below a line segment drawn between two points on its curve. In general, a function $z = f(\mathbf{x})$ is a hypersurface in n-dimensional space. It is convex if the line segment which connects any two points $[\mathbf{x}_1, z_1]$ and $[\mathbf{x}_2, z_2]$ on the surface of $f(\mathbf{x})$ lies entirely on or above the hypersurface. The reverse holds true for concave functions.

We will now state without proof some important results which relate to the use of convex and concave functions. Proofs of these theorems can be found in Ref. 2.

Theorem 1. Let the functions $f_k(\cdot)$, $k = 1, 2, \ldots, p$ be convex (concave) functions over some convex set[2] X in E^n. Then the function $f(\mathbf{x}) = \sum_{k=1}^{p} f_k(\mathbf{x})$ is also a convex (concave) function over X.

Theorem 1 says that the sum of convex functions is a convex function and the sum of concave functions is a concave function.

Theorem 2. If $f(\cdot)$ is a convex function over the nonnegative orthant of E^n, then if $W \doteq \{\mathbf{x} \mid f(\mathbf{x}) \leq b, \mathbf{x} \geq \mathbf{0}\}$ is not empty, W is a convex set.

We will now state some general mathematical results from the calculus for purposes of reference subsequently. Proofs of these results can be found in Ref. 1.

[2] A convex set is one such that a straight line between any two points in the set is also in the set. More precisely, for any two points $\mathbf{x}_1$ and $\mathbf{x}_2$ in the set, the convex combination, $\lambda\mathbf{x}_1 + (1 - \lambda)\mathbf{x}_2$, $0 \leq \lambda \leq 1$, is also in the set.

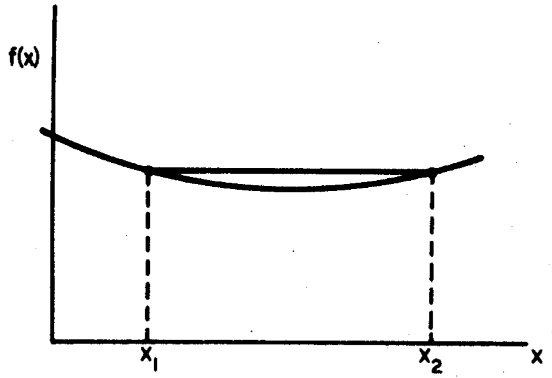

Fig. 2. A convex function.

Theorem 3 **(Mean Value Theorem).** If $f(\cdot)$ is continuous in the closed interval $x_0 \leq x \leq x_0 + h$ and differentiable at every point in the open interval $x_0 < x < x_0 + h$, then there exists at least one point x_m in the open interval specified where

$$\frac{f(x_0 + h) - f(x_0)}{h} = f'(x_m) \tag{7}$$

By $f'(x_m)$ we mean $df(x)/dx \big|_{x=x_m}$

The point x_m is not specified by this theorem. Its existence is merely guaranteed. An alternative way of expressing this theorem is to write:

$$\frac{f(x_0 + h) - f(x_0)}{h} = f'(x_0 + \theta h), \qquad 0 < \theta < 1 \tag{8}$$

This follows from the fact that x_m lies in the open interval $(x_0, x_0 + h)$ and any x in this interval can be expressed as $x = x_0 + \theta h$ where $0 < \theta < 1$. Therefore $x_m = x_0 + \theta h$ from which (8) follows.

A result which is often of interest and use for functions of a single variable but which is not easily generalized for functions of more than one variable is the following. It gives a *necessary and sufficient* condition for a function to have a maximum or minimum at a point and includes the case where the derivative may not exist at the maximal or minimal point—hence its value.

Theorem 4. Given a function $f(\cdot)$ defined on an interval, containing a point x_0 and that $f(\cdot)$ is continuously differentiable everywhere in the interval (with the possible exception of x_0) and further that $f'(\cdot)$ vanishes at a finite number of points. Then $f(\cdot)$ has a maximum or minimum at x_0 *if and only if* the point x_0 divides the interval over which $f(\cdot)$ is defined into two subintervals in which $f'(\cdot)$ has different signs. More precisely, the function has a maximum if the derivative is positive to the left of x_0 and negative to the right. It has a minimum if the reverse holds.

In Fig. 3 we see a function which has a minimum at a point x_0, for which the derivative is not defined at x_0. The theorem, of course, is still true if the derivative is defined everywhere.

One of the most important results in analysis, and of major significance in optimization, is what is known as Taylor's theorem. We shall give the theorem in its multidimensional form. In order to do so we shall require some special notation. Let

$$D^j f(\mathbf{x}) = \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \cdots \sum_{i_j=1}^{n} h_{i_1} h_{i_2} \cdots h_{i_j} \frac{\partial^j f(\mathbf{x})}{\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_j}} \tag{9}$$

$$S_N(\mathbf{x}) = \sum_{j=1}^{n} \frac{1}{j!} D^j f(\mathbf{x}) \tag{10}$$

Further, let $\mathbf{x}_2 = \mathbf{x}_1 + \mathbf{h}$, where $\mathbf{h} = (h_1, h_2, \ldots, h_n)$. Then we define a function, usually called the *remainder term*, as

$$R_N(\mathbf{x} + \theta\mathbf{h}) = \frac{1}{(N+1)!} D^{N+1} f(\mathbf{x} + \theta\mathbf{h}), \qquad 0 < \theta < 1 \tag{11}$$

We now state Taylor's theorem.

Theorem 5 **(Taylor's Theorem).** A function $(\cdot)$ which is continuous and which has continuous partial derivatives of required order may be represented at point $\mathbf{x}_2 = \mathbf{x}_1 + \mathbf{h}$ in terms of its value at a point $\mathbf{x}_1$ by

$$f(\mathbf{x}_2) = f(\mathbf{x}_1) + S_N(\mathbf{x}) + R_N(\mathbf{x}_1 + \theta\mathbf{h}), \tag{12}$$

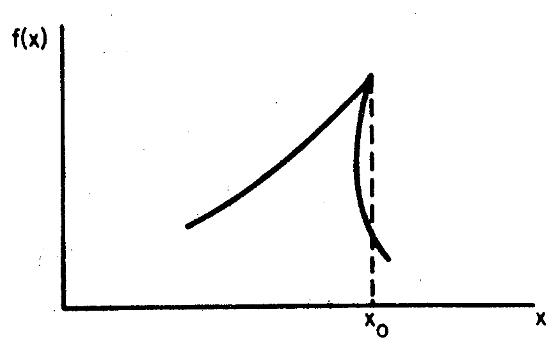

Fig. 3. Nondifferentiable maximum.

In particular, a value of $0 < \theta < 1$ exists to make (12) hold true. In simple language what Taylor's theorem states is that we may approximate any given function $f(\cdot)$ by polynomial of order N if the first $N + 1$ partial derivatives of the function are continuous. The two most commonly used forms of Taylor's theorem, which we shall use subsequently, are as follows.

$$f(\mathbf{x}_2) = f(\mathbf{x}_1) + \nabla f[\theta\mathbf{x}_1 + (1 - \theta)\mathbf{x}_2]\mathbf{h} \tag{13}$$
$$f(\mathbf{x}_2) = f(\mathbf{x}_1) + \nabla f(\mathbf{x}_1)\mathbf{h} + \tfrac{1}{2}\mathbf{h}'H[\theta\mathbf{x}_1 + (1 - \theta)\mathbf{x}_2]\mathbf{h} \tag{14}$$

Equation (13) is a first-order approximation and ∇f is the gradient vector and is defined by

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}\right) \tag{15}$$

The notation $\nabla f[\theta\mathbf{x}_1 + (1 - \theta)\mathbf{x}_2]$ indicates that the gradient vector is to be evaluated at the point $\mathbf{x} = \theta\mathbf{x}_1 + (1 - \theta)\mathbf{x}_2$ in E^n. Taylor's theorem assures us that there exists a θ such that (13) holds.

In Eq. (14) we have written the theorem in terms of second partial derivatives as well as first partial derivatives. H in Eq. (14) is the Hessian matrix of $f(\cdot)$ and is defined as a matrix of the n^2 second partial derivatives of $f(\cdot)$. In other words, it is the $n \times n$ matrix $H = \|\partial^2 f/\partial x_u \partial x_v\|$ where $u, v = 1, 2, \ldots, n$. The notation $H[\theta\mathbf{x}_1 + (1 - \theta)\mathbf{x}_2]$ indicates that the Hessian matrix is evaluated at the point $\mathbf{x} = \theta\mathbf{x}_1 + (1 - \theta)\mathbf{x}_2$.

We require one last result from analysis. It is known as the *implicit function theorem*. Consider a set of m equations in n variables where $m < n$:

$$g_i(\mathbf{x}) = 0, \qquad i = 1, 2, \ldots, m$$
and
$$\mathbf{x} = (x_1, x_2, \ldots, x_n) \tag{16}$$

There may be situations in which we may wish to use these m equations to eliminate some subset m of the n variables in some expression. Suppose we wish to do this at some specific point $\mathbf{x}^0$. Without loss of generality, suppose we wish to eliminate the first m variables. Therefore, what we would like to know is under what circumstances there exist a set of m functions ϕ_i, $i = 1, 2, \ldots, m$, such that

$$x_i = \phi_i(x_{m+1}, x_{m+2}, \ldots, x_n), \qquad i = 1, 2, \ldots, m \tag{17}$$

at $\mathbf{x}^0$ or in a neighborhood of $\mathbf{x}^0$. The following theorem supplies this information.

Theorem 6 **(Implicit Function Theorem).** If the rank of the Jacobian matrix J_m evaluated at the point $\mathbf{x}^0$ is equal to m, then this is a necessary and sufficient condition for the existence of a set of m functions ϕ_i, $i = 1, 2, \ldots, m$, which are unique, continuous, and differentiable in some neighborhood of $\mathbf{x}^0$.

The Jacobian matrix J_m is defined as

$$J_m = \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \cdots & \frac{\partial g_1}{\partial x_m} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \cdots & \frac{\partial g_2}{\partial x_m} \\ \vdots & \vdots & & \vdots \\ \frac{\partial g_m}{\partial x_1} & \frac{\partial g_m}{\partial x_2} & \cdots & \frac{\partial g_m}{\partial x_m} \end{bmatrix} \tag{18}$$

The statement that the Jacobian matrix is evaluated at $\mathbf{x}_0$ means that each of the elements of J_m is evaluated at the point $\mathbf{x}_0$.

For simplicity in the statement of Theorem 6 we assumed that the first m variables were to be expressed in terms of the remaining $n - m$ variables. A more exact statement is as follows. If one selects any m column vectors of the form

$$\begin{bmatrix} \frac{\partial g_1}{\partial x_j} \\ \frac{\partial g_2}{\partial x_j} \\ \vdots \\ \frac{\partial g_m}{\partial x_j} \end{bmatrix}$$

Then, if the resulting Jacobian matrix has rank m, these particular variables may be eliminated, i.e., the required set of m functions ϕ_i exist. There are $\binom{m}{n} = n!/m!(n - m)!$ possible combinations of m columns out of a total set of n and hence that number of Jacobians. Every Jacobian that has rank m represents a set of variables that can be eliminated. An equivalent statement to having rank m is that the Jacobian matrix be nonsingular, i.e., that its determinant is not zero.

UNCONSTRAINED OPTIMIZATION

We shall divide the subject of unconstrained optimization, i.e., maximizing or minimizing a function under conditions where the variables are not constrained in any way, into two major parts. The first is the study of functions of a single variable and the second treats functions of several variables.

There are two basic approaches that one can take to find the optimum of a function. The first approach, which is often called a *direct method*, consists of evaluating the given function $f(\cdot)$ at some point $\mathbf{x}^1$ and then seeking, by some method, to find another point $\mathbf{x}^2$, where $\mathbf{x}^2 = g(\mathbf{x}^1)$, some well-defined function of $\mathbf{x}^1$, such that $f(\mathbf{x}^2) > f(\mathbf{x}^1)$ if we seek a maximum or $f(\mathbf{x}^2) < f(\mathbf{x}^1)$ if we seek a minimum. We repeat this process until no further change is possible. We shall *not* be concerned with this general approach in this article.

The second approach is what is referred to as an *indirect method* and is the approach of calculus and classical optimization. What is involved here is the development and use of necessary and sufficient conditions that an optimal point, a local maximum or minimum, must satisfy. In certain cases it is possible to determine whether or not a global optimum has been found.

We first consider the derivation and proof of a *necessary* condition for the existence of a local or relative optimum of a function. This is provided in the following theorem.

Theorem 7. A necessary condition that a continuous function $f(\cdot)$, whose first derivative is continuous over E^1, has a local minimum or maximum at a point x_0 is that $df(x_0)/dx = 0$.

Proof: We recall from the definition that if $f(\cdot)$ has a relative minimum at a point x_0, then there exists an $\epsilon > 0$ such that for some interval about x_0, $f(x) \geq f(x_0)$ for $|x - x_0| < \epsilon$. Therefore, let us consider points about x_0 of the form

$$x = x_0 + h, \qquad 0 < |h| < \epsilon$$

Then we may write

$$f(x_0 + h) - f(x) \geq 0, \qquad 0 < |h| < \epsilon \tag{19}$$

We may then divide Eq. (19) by h to obtain

$$\frac{f(x_0 + h) - f(x)}{h} \geq 0, \qquad h > 0 \tag{20}$$

$$\frac{f(x_0 + h) - f(x)}{h} \leq 0, \qquad h < 0 \tag{21}$$

If we take the limit of the expressions in (20) and (21), we obtain

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x)}{h} = \frac{df(x_0)}{dx} \geq 0 \tag{22}$$

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x)}{h} = \frac{df(x_0)}{dx} \leq 0 \tag{23}$$

Together, Eqs. (22) and (23) imply that

$$\frac{df(x_0)}{dx} = 0 \tag{24}$$

A corresponding argument with the inequalities reversed can be made for a relative maximum. Hence the theorem is proved.

A point at which the first derivative is zero is called a stationary point. Therefore, we have proven that a *necessary* condition for x_0 to be a local maximum or minimum is that x_0 be a stationary point. Since Theorem 7 provides only a necessary condition, a stationary point may be either a maximum, a minimum, or neither. The following obvious examples illustrate this:

$f_1(x) = 4x^2$	has a minimum
$f_2(x) = -3x^2$	has a maximum
$f_3(x) = 7x^3$	has neither a maximum nor a minimum

In order to discriminate between maxima and minima and other stationary points we require a sufficient condition. This is provided by the following theorem.

Theorem 8. Given a continuous function $f(\cdot)$ whose first two derivatives are continuous at x_0. Then if $f'(x_0) = 0$, a sufficient condition for $f(\cdot)$ to have a minimum at x_0 is that $f''(x_0) > 0$ and a sufficient condition for $f(\cdot)$ to have a maximum at x_0 is that $f''(x_0) < 0$.

Proof: Using Taylor's theorem we may write

$$f(x_0 + h) = f(x_0) + hf'(x_0) + \frac{h^2}{2} f''[\theta x_0 + (1 - \theta)(x_0 + h)], \qquad 0 \le \theta \le 1 \tag{25}$$

If $f(\cdot)$ has a relative minimum at x_0, then we know from Theorem 7 that $f'(x_0) = 0$. Therefore we may rewrite Eq. (25) as

$$f(x_0 + h) - f(x_0) = \frac{h^2}{2} f''[\theta x_0 + (1 - \theta)(x_0 + h)], \qquad 0 \le \theta \le 1 \tag{26}$$

If $f(\cdot)$ is to have a minimum at x_0, then it follows from Eq. (26) that

$$f(x_0 + h) - f(x_0) = \frac{h^2}{2} f''[\theta x_0 + (1 - \theta)(x_0 + h)] > 0, \qquad 0 \le \theta \le 1 \tag{27}$$

Suppose that $f''(x_0) < 0$. Then it follows from the continuity of the second derivative that $f''[\theta x_0 + (1 - \theta)(x_0 + h)] < 0$ and therefore that

$$f(x_0 + h) - f(x_0) = \frac{h^2}{2} f''(\theta x_0 + (1 - \theta)(x_0 + h)] < 0 \tag{28}$$

and therefore x_0 cannot be a minimum point. Conversely, if $f''(x_0) > 0$ at $f'(x_0) = 0$, $f(x_0)$ is clearly a minimum. We can repeat a corresponding argument for the case of a maximum and arrive at the conclusion that if $f''(x_0) < 0$ at $f'(x_0) = 0$, $f(x_0)$ is a maximum.

While Theorem 8 gives sufficient conditions for functions for which the second derivative does not vanish, it is certainly possible that at some point x_0, both first and second derivatives will vanish. The following theorem is more general and gives sufficient conditions for any case.

Theorem 9. Assume that $f(\cdot)$ and its first n derivatives are continuous. Then