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Nonlinear Functional Analysis and its Applications

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Remarks on the Euler and Navier–Stokes Equations in $\mathbf{R}^2$

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Abstract. We consider the Euler and the Navier–Stokes equations in $\mathbf{R}^2$ for an incompressible fluid. It is shown that the solution $u(t) \in H^s(\mathbf{R}^2)$ exists for all time if $u(0) \in H^s(\mathbf{R}^2)$ with div $u(0) = 0$, where s is any real number such that $s > 2$. Moreover, the Navier–Stokes flow converges in $C([0, T]; H^s)$ to the Euler flow as the viscosity tends to zero for each $T > 0$.

1. Introduction. Consider the Navier–Stokes equation in space domain $\mathbf{R}^2$:

$$\text{(NS)} \qquad \partial_t u - \nu\Delta u + (u \cdot \partial)u + \partial p = 0, \qquad \operatorname{div} u = 0,\ t \geqslant 0,\ x \in \mathbf{R}^2,$$

where $\partial_t = \partial/\partial t$, $\partial = \text{grad} = (\partial/\partial x_1, \partial/\partial x_2)$, $u = u(t, x) = (u_1(t, x), u_2(t, x))$ is the velocity field, $p = p(t, x)$ is the pressure, and $\nu > 0$ is the kinematic viscosity. For simplicity we assume that there is no extremal force, but the following results can be extended to include external forces satisfying appropriate conditions.

In the limit $\nu \to 0$, (NS) formally goes over to the Euler equation

$$\text{(E)} \qquad \partial_t u + (u \cdot \partial)u + \partial p = 0, \qquad \operatorname{div} u = 0.$$

It is well known that these equations have global solutions for appropriate initial velocities. For (NS) it suffices that $u(0) \in L^2$, and for (E) that $u(0) \in C^{1+\epsilon}_*$ with some decay at infinity. (We usually suppress the space variable x and write $u = u(t)$. Also L^2 means $L^2(\mathbf{R}^2)$, etc., unless otherwise indicated.)

In these equations the pressure p is automatically determined (up to a function of t) if u is known; indeed, $\partial p = -(1 - P)(u \cdot \partial)u$, where P is the orthogonal projection of L^2 onto the subspace of solenoidal vectors. For this reason it suffices to consider u only when we talk about the solution of (NS) or (E).

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In what follows we are interested in the conditions under which the solutions $u = u^\nu$ of (NS) are uniformly bounded as $\nu \to 0$ in a certain norm and $u^\nu \to u^0$ holds as $\nu \to 0$, where u^0 is a solution of (E). Our main results are given by the following theorems.

THEOREM I. *Let* $s > 2$ (*not necessarily an integer*), *and let* $a^\nu, a^0 \in H^s$ *with* $a^\nu \to a^0$ *in* H^s *as* $\nu \to 0$. *Then there are unique solutions* u^ν *and* u^0 *to* (NS) *and* (E), *respectively, both in* $C([0, \infty); H^s)$ *satisfying the initial conditions* $u^\nu(0) = a^\nu$, $u^0(0) = a^0$. *Moreover, we have* $u^\nu \to u^0$ *in* $C([0, T]; H^s)$ *as* $\nu \to 0$ *for any* $T > 0$.

COROLLARY. *For any* $T > 0$, *the* u^ν *are uniformly bounded in* $C([0, T]; H^s)$.

The novelty of these results is in the small value of s permitted. For larger s such results have long been known. Golovkin [2] and McGrath [7] deduced similar results assuming, roughly, that $a^\nu = a^0 \in H^4$ and proving the convergence in $C^1([0, T] \times \mathbf{R}^2)$. (These authors were mainly interested in classical solutions.) It is interesting to note that Golovkin first proves uniform estimates, which are very sharp (see Remark 4.1 below), and then proves convergence, while McGrath first proves the global existence of u^0 and then proves convergence (which automatically implies uniform estimates, though he does not mention it explicitly). A recent paper by Beale and Majda [1] contains, among other things, related results in which uniform estimates for u^ν are obtained in H^s-norm with $s \geqslant 4$ and in which convergence is proved in H^{s-2}, for example (but with an explicit rate of convergence).

In the present paper, we are primarily interested in the *persistence* property, i.e. we want to show that the solutions stay in the same space H^s as do the initial values and that (strong) convergence takes place also in the same space.

2. Proof of Theorem I. We follow the line of McGrath, first proving the global existence of u^0. Then we invoke a general theorem on *local* continuous dependence to prove the convergence $u^\nu \to u^0$. In this sense the proof is not constructive so far as the uniform estimates are concerned. In a later section we shall give another, more constructive proof in the case when s is an integer $\geqslant 3$. Unfortunately, the second proof does not seem to work for noninteger values of s.

Thus we first prove the following theorem, which is a part of Theorem I.

THEOREM II. *Let* $s > 2$ *and* $a \in H^s$. *Then there is a unique solution* $u \in C([0, \infty); H^s)$ *of* (E) *with* $u(0) = a$.

In this section we prove Theorem I assuming Theorem II. The proof is a simple application of the local convergence theorem given in [4, 5] according to which there is $T_1 > 0$, depending only on $\|a^0\|_s = \|u^0(0)\|_s$, such that u^0 and u^ν exist on $[0, T_1]$ for sufficiently small ν and $u^\nu \to u^0$ in $C([0, T_1]; H^s)$ as $\nu \to 0$. Then we can apply the same argument starting from the initial time $t = T_1$ and extend the convergence to a larger interval $[0, T_2]$, where $T_2 - T_1$ is determined by $\|u^0(T_1)\|_s$. This process can be continued to cover any finite interval $[0, T]$ in a finite number of steps, since $u^0(t)$ exists for all $t > 0$ by Theorem II so that $\|u^0(t)\|_s$ is bounded on any finite interval of t.

It may be remarked that the argument given above already shows that u^ν exists on a larger and larger time interval as $\nu \to 0$, but it alone cannot prove that u^ν exists for *all* time. The latter fact is well known, however (see e.g. Leray [6]).

3. Proof of Theorem II. Since the existence of local solutions $u(t) \in H^s$ for $u(0) = a \in H^s$ is known (see [4, 5]), Theorem II follows as soon as a global estimate for u is obtained. Thus it suffices to prove

THEOREM III. *For each $s > 2$, there is a monotone increasing map Φ_s: $\mathbf{R}^2_+ \to \mathbf{R}_+$ such that for any $T > 0$ and any solution $u \in C([0, T]; H^s)$ of* (E) *with $u(0) = a$, one has*

$$\|u(t)\|_s \leqslant \Phi_s(T, \|a\|_s) \quad \text{for } 0 \leqslant t \leqslant T. \tag{3.1}$$

The proof of Theorem III is not altogether simple, since the only source of such a global estimate is the conservation of the *vorticity*

$$\zeta = \operatorname{rot} u = \partial_1 u_2 - \partial_2 u_1 \tag{3.2}$$

in L^p-norm. To connect it with the H^s-norm of u requires some amount of work.

We start with several remarks. (For the following notions cf. [3].) First, $u(t) \in H^s$ with $s > 2$ implies $u(t) \in C^1$, so that $u(t)$ is sufficiently smooth for the following computations to be justified. Second, the vorticity ζ satisfies the first-order equation

$$\partial_t \zeta + u \cdot \partial \zeta = 0, \qquad \zeta(0) = b = \operatorname{rot} a. \tag{3.3}$$

Third, the solution of (3.3) is explicitly given by

$$\zeta(t, x) = b(U_{0,t}(x)), \tag{3.4}$$

where $U_{s,t} = U_{t,s}^{-1}$ is the C^1-diffeomorphism of $\mathbf{R}^2$ induced by the velocity field u between times s and t. Fourth, u is (formally) recovered from ζ by

$$\partial u = \partial g * \zeta, \tag{3.5}$$

where $g(x) = (1/2\pi)\operatorname{rot} \log|x|$; note that ∂g is a singular integral operator (plus a delta function for some components). The convolution is well defined because $\zeta \in L^2$.

All the functions u, ζ, U are well behaved on $[0, T]$, but we have to estimate their size in appropriate norms in terms of $\|a\|_s$ and T alone. This will be done first for the case $2 < s < 3$ by bootstrap arguments starting with the initial data

$$a \in H^s, \qquad b \in H^{s-1} \in L^2 \cap L^\infty. \tag{3.6}$$

For this purpose, it is convenient to introduce the following terminology: We say a function $F(t)$ defined for $0 \leqslant t \leqslant T$ has a property "uniformly" if the norm describing that property is estimated in terms of $\|a\|_s$ and T alone. Thus Theorem III will be proved if we show that u is "uniformly" H^s. We shall do this by proving the following propositions successively.

(a) *$\zeta(t)$ is "uniformly" $L^\infty \cap L^2$, and $u(t)$ is "uniformly" L^2*. This is obvious from (3.6), since the L^p-norm of $\zeta(t)$ for $1 \leqslant p \leqslant \infty$ and the L^2-norm of $u(t)$ are known to be conserved.

(b) *$u(t)$ is "uniformly" quasi-Lipschitz continuous*. This means that

$$(3.7)\quad |u(t,x) - u(t,y)| \leqslant K|x-y| \log(1/|x-y|) \quad \text{for } |x-y| \leqslant 1/2,$$

where K depends only on $\|a\|_s$ and T. (3.7) was proved in [3] when the space domain Ω is bounded. It was proved in [7] for $\Omega = \mathbf{R}^2$ under the additional assumption that $b \in L^1$. Actually $b \in L^2$ suffices for this purpose, as we shall prove in Lemma A1 in the Appendix. It should be noted that at this point we cannot yet claim that $u(t)$ is "uniformly" L^∞.

(c) *$U_{0,t}$ is "uniformly" Hölder continuous*. The proof in [3], depending exclusively on (b), applies to the present case without modification. We note that the Hölder exponent may be extremely small if T is large, but it is "uniform" in our sense.

(d) *$\zeta(t)$ is "uniformly" Hölder continuous*. This follows immediately from (3.4) and (c), since $b \in H^{s-1}$ is Hölder continuous.

(e) *$\partial u(t)$ is "uniformly" bounded and "uniformly" Hölder continuous*. This follows from (3.5) and (d), as we shall show in Lemma A2 in the Appendix.

(f) *$\partial U_{0,t}$ is "uniformly" bounded and "uniformly" Hölder continuous*. This follows from (e) and the standard theorems in ordinary differential equations.

(g) *$\zeta(t)$ is "uniformly" $H^{1+\varepsilon}$ for some small $\varepsilon > 0$ depending on T and $\|a\|_s$ only*. Indeed, ζ is the composition of b with $U_{0,t}$, where $b \in H^{s-1}$ and $\partial U_{0,t}$ is bounded and Hölder continuous by (f). Thus the assertion follows from Lemma A5 in the Appendix. Note that $U_{0,t}$ has the Jacobian determinant one.

(h) *$u(t)$ is "uniformly" $H^{2+\varepsilon}$*. This follows directly from (g) and Lemma A4 in the Appendix.

(i) *$\zeta(t)$ is "uniformly" H^{s-1}*. This follows from the theory of *linear* evolution equations applied to (3.3) (see [5]). Since $u(t) \in H^{2+\varepsilon}$ and $\zeta(0) = b \in H^{s-1}$, where $s - 1 < 2 < 2 + \varepsilon$, the solution $\zeta(t)$ stays in H^{s-1} "uniformly".

(j) *$u(t)$ is "uniformly" H^s*. This follows directly from (i) and Lemma A4, and completes the proof of Theorem III in the case $2 < s < 3$.

The proof for larger s is easy. Suppose $3 \leqslant s < 4$. Then the above result shows that u is uniformly H^r for any $r < 3$. Since $b \in H^{s-1}$ and r may be chosen to satisfy $2 \leqslant s - 1 < r$, we see as in (i) above that ζ is "uniformly" H^{s-1}, hence u is "uniformly" H^s as required. A larger s can be handled step by step by the same method.

4. Direct estimates for integers $s \geqslant 3$. If s is an integer $\geqslant 3$, we can prove Theorem I more directly by estimating the solutions of (NS) directly. Let $u = u^\nu$ be a solution of (NS) for $u(0) = a \in H^s$, where s is an integer $\geqslant 3$. Then $u(t)$ exists for all time and is smooth, and it is known that

$$(4.1)\qquad \partial_t \|u(t)\|_s^2 \leqslant c_s \|\partial u(t)\|_{L^\infty} \|u(t)\|_s^2.$$

This can be deduced by a formal computation using (NS), in which the contribution from the viscosity term is negative and the term with the highest derivative in the nonlinear terms is as usual eliminated by integration by parts, while the remaining terms are estimated by the Gagliardo–Nirenberg inequality (for details see e.g. Majda [8]). (Actually (4.1) is true also for the solution of (E), but its justification is not so easy since $u(t)$ is not necessarily in H^{s+1}. In any case, however, we do not need it here.)

On the other hand, Lemma A3 of the Appendix gives

$$\|\partial u(t)\|_{L^\infty} \leqslant K\big(1 + \log_+ \|u(t)\|_3\big), \tag{4.2}$$

with a constant K depending only on $\|b\|_{L^\infty}$ and $\|b\|_{L^2}$, where $b = \operatorname{rot} a$; recall that the L^p-norm of $\zeta = \operatorname{rot} u$ is bounded in time.

It follows from (4.1) and (4.2) that

$$\partial_t \|u(t)\|_3^2 \leqslant K\big(1 + \log_+ \|u(t)\|_3\big)\|u(t)\|_3^2. \tag{4.3}$$

This differential inequality has a global solution if $a \in H^3$, with $\|u(t)\|_3$ and, by (4.2), $\|\partial u(t)\|_{L^\infty}$ bounded on any finite interval of t. Then (4.1) shows that $\|u(t)\|_s$ has the same property too. Since the bound obviously does not depend on ν except via $a = a^\nu = u^\nu(0)$, we have uniform boundedness if the a^ν are uniformly bounded in H^s-norm, since the L^∞- and L^2-norms of $\operatorname{rot} a$ is majorized by $\|a\|_3 \leqslant \|a\|_s$.

The remainder of Theorem I concerning the convergence can easily be reduced to the local convergence theorem proved in [5]. Indeed, the uniform bound is inherited by $u^0 = \lim u^\nu$ as long as $u^0(t)$ exists, and $u^0(t)$ exists as long as it is bounded in H^s-norm.

REMARK 4.1. (a) We do not know whether or not inequality (4.1) is true when s is not a positive integer, but (4.2) is true when 3 is replaced by any real number $s > 2$, as is seen from Lemma A3.

(b) Inequality (4.3) is essentially proved in Golovkin [2].

Appendix. We prove several lemmas which are used in the text. The first four lemmas are concerned with estimating the velocity u in terms of the vorticity $\zeta = \operatorname{rot} u$ (see (3.2)).

LEMMA A1. *Let $u \in L^2$ and let $\zeta \in L^\infty \cap L^2$. Then u is quasi-Lipschitz continuous*:

$$|u(x) - u(y)| \leqslant c|x - y|\big(\|\zeta\|_{L^\infty} \log(1/|x - y|) + \|\zeta\|_{L^2}\big) \tag{A1}$$

for $|x - y| \leqslant 1/2$.

PROOF. We use the formula

$$u(x) - u(y) = \int (g(x - z) - g(y - z))\zeta(z)\, dz, \tag{A2}$$

where $g(x) = (1/2\pi)\operatorname{rot} \log |x|$. We split the integral into two parts v_1 and v_2 according as $|x - z| \leqslant 2r$ or not, where $r = |x - y|$. v_1 is estimated by $cr\|\zeta\|_{L^\infty}$ as in [3]. v_2 is further split into two parts v_3 and v_4 according as $2r \leqslant |x - z| \leqslant 1$ or not. v_3 is estimated by $c\|\zeta\|_{L^\infty} r \log(1/r)$ as in [3], while v_4 can be estimated by $cr\|\zeta\|_{L^2}$. Summing up, we have proved the lemma.

LEMMA A2. *In Lemma* A1 *let ζ be Hölder continuous with exponent δ. Then ∂u is bounded and Hölder continuous with exponent δ.*

PROOF. First we note that ζ is bounded, since it is Hölder continuous and L^2. To estimate ∂u we use formula (3.5), which is essentially identical with (A2), and split the integral into two parts according to the size of the argument x in $g(x)$. The contribution from $|x| \leqslant 1$ can be estimated as in the usual Hölder estimate, while the contribution from $|x| \geqslant 1$ can be easily handled by noting that $\zeta \in L^2$.

LEMMA A3. *In Lemma* A1 *let $\partial\zeta \in L^p$ for some $p > 2$. Then*

$$\|\partial u\|_{L^\infty} \leqslant c\|\zeta\|_{L^\infty} + c\|\zeta\|_{L^2} + c_p\|\zeta\|_{L^\infty} \log\left[1 + \left(\|\partial\zeta\|_{L^p}/\|\zeta\|_{L^\infty}\right)\right]. \tag{A3}$$

If in particular $u \in H^s$ with $s > 2$, $\|\partial\zeta\|_{L^p}$ may be replaced by $\|u\|_s$. Note also that the right member of (A3) *is monotone increasing in $\|\zeta\|_{L^\infty}$.*

PROOF. Again we use (3.5), in which we decompose g "smoothly" into two parts $g = g_1 + g_2$, where g_1 has support in $B_{2r} = \{|x| \leqslant 2r\}$ and g_2 is zero in B_r, with $r > 0$, and write $\partial u = w_1 + w_2$ accordingly. Then we have $w_1 = g_1 * \partial\zeta$, and application of the Hölder inequality gives an estimate

$$\|w_1\|_{L^\infty} \leqslant c_p r^{1-2/p}\|\partial\zeta\|_{L^p} \quad \text{for any } p > 2. \tag{A4}$$

To estimate w_2, we split the integral representing $w_2 = \partial g_2 * \zeta$ into two parts according as the argument in g_2 (which is outside B_r) is inside our outside of B_1. (If $r > 1$, the first part does not exist.) Since $|\partial g_2(x)| \leqslant c|x|^{-2}$, the first part is majorized by $c\|\zeta\|_{L^\infty} \log(1/r)$ and the second part by $c\|\zeta\|_{L^2}$. Summing up, we obtain

$$\|\partial u\|_{L^\infty} \leqslant c\|\zeta\|_{L^\infty} \log(1/r) + c\|\zeta\|_{L^2} + c_p r^{1-2/p}\|\partial\zeta\|_{L^p}, \tag{A5}$$

where the first term does not exist if $r > 1$. Thus we may replace $\log(1/r)$ by $\log(1 + 1/r)$ to make (A5) valid for all $r > 0$. If, then, we determine r by $r^{1-2/p}\|\partial\zeta\|_{L^p} = \|\zeta\|_{L^\infty}$, we obtain (A3).

LEMMA A4. *Let $u \in L^2$. For any $s \geqslant 1$, $u \in H^s$ is equivalent to $\zeta \in H^{s-1}$.*

PROOF. The proof is obvious if one considers the Fourier transform of u.

LEMMA A5. *Let $f \in H^{1+h}(\mathbf{R}^m; \mathbf{R})$. Let g be a C^1-diffeomorphism of $\mathbf{R}^m$ onto itself, with the Jacobian determinant $J_g(x) \geqslant \delta > 0$. Assume, moreover, that ∂g is bounded and Hölder continuous with exponent k, where $0 < h < k < 1$. Then $u = f \circ g \in H^{1+h}(\mathbf{R}^m; \mathbf{R})$.*

PROOF. We have to show that $u \in L^2$ and $v = \partial u \in H^h$.

(a) $u \in L^2$ follows from

$$\|u\|^2 = \int |f(g(x))|^2\, dx \leqslant \delta^{-1} \int |f(g(x))|^2 J_g(x)\, dx = \delta^{-1}\|f\|^2.$$

(b) To show that $v \in H^h$, it suffices, in view of (a), to show that the (homogeneous) H^h-seminorm $\|v\|_{[h]}$ is finite. But

$$\|v\|_{[h]}^2 = c \int |v(x) - v(y)|^2 |x - y|^{-m-2h}\, dx\, dy. \tag{A6}$$

Since $v(x) = \partial f(g(x)) \partial g(x)$ in a slightly symbolic notation, the integrand in (A6) is majorized by

$$\begin{aligned} &|\partial f(g(x))|^2 |\partial g(x) - \partial g(y)|^2 |x - y|^{-m-2h} \\ &\quad + |\partial f(g(x)) - \partial f(g(y))|^2 |\partial g(y)|^2 |x - y|^{-m-2h}. \end{aligned} \tag{A7}$$

In the first term of (A7), $|\partial g(x) - \partial g(y)|^2 |x - y|^{-m-2h}$ can be majorized by $c|x - y|^{2k-2h-m}$ for $|x - y| \leqslant 1$ and by $c|x - y|^{-m-2h}$ for $|x - y| \geqslant 1$ because ∂g is bounded and k-Hölder continuous. Thus the integral of this factor in y gives a finite constant, and the contribution of this part to (A6) is finite due to the facts that ∂f is in $H^h \subset L^2$ and that $J_g \geqslant \delta$.

In the second term in (A7), $|\partial g(y)|^2$ is bounded by a constant, and $|x - y|^{-m-2h}$ may be replaced by a constant times $|g(x) - g(y)|^{-m-2h}$ because g is uniformly Lipschitzian by the boundedness of ∂g. Using again $J_g \geqslant \delta$, the integral is seen to be finite because $\partial f \in H^h$. Summing up, we have shown that $v \in H^h$.

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